Final Exam

15-317: Constructive Logic

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Name: Andrew ID:

Instructions

- This exam is closed Internet. Notes are permitted.
- There are four problems. Not all problems are the same size or difficulty, so it may help to read through the whole exam first. You have three hours to complete the exam.
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Good luck!

	Problem 1	Problem 2	Problem 3	Problem 4	**Total**
Score					
Max	30	20	40	30	120

1 Invertibility and modal logic

Below are the four rules governing the $\Box$ and $\Diamond$ connectives in modal logic.

$$\frac{\Delta; \cdot \vdash A\ \mathsf{true}}{\Delta; \Gamma \vdash \Box A\ \mathsf{true}}\ \Box I \qquad \frac{\Delta; \Gamma \vdash \Box A\ \mathsf{true} \quad (\Delta, A\ \mathsf{valid}); \Gamma \vdash C\ \mathsf{true}}{\Delta; \Gamma \vdash C\ \mathsf{true}}\ \Box E$$

$$\frac{\Delta; \Gamma \vdash A\ \mathsf{poss}}{\Delta; \Gamma \vdash \Diamond A\ \mathsf{true}}\ \Diamond I \qquad \frac{\Delta; \Gamma \vdash \Diamond A\ \mathsf{true} \quad \Delta; A\ \mathsf{true} \vdash C\ \mathsf{poss}}{\Delta; \Gamma \vdash C\ \mathsf{poss}}\ \Diamond E$$

The other rules of modal logic that are relevant are:

$$\frac{}{(\Delta, A\ \mathsf{valid}); \Gamma \vdash A\ \mathsf{true}}\ hyp^* \qquad \frac{}{\Delta; (\Gamma, A\ \mathsf{true}) \vdash A\ \mathsf{true}}\ hyp \qquad \frac{\Delta; \Gamma \vdash A\ \mathsf{true}}{\Delta; \Gamma \vdash A\ \mathsf{poss}}\ poss$$

Of the four rules ($\Box I$, $\Box E$, $\Diamond I$, and $\Diamond E$), which are invertible, and which are non-invertible? Explain.

2 The Game Show problem

Consider the following scenario: You are in the audience of a game show. The game show host selects one member of the audience to play the game. If you are not chosen, you get nothing. If you are chosen, you are presented with three doors: A, B, and C. You select one of the three doors. The host then chooses one of the two remaining doors, and offers you $100 to switch doors. You have the choice to open your door, or to take the money and open the offered door. You win whatever is behind the door you open.

For example, suppose you are selected to play. You might choose door A. Suppose you do. Then the host can choose door B or door C. Suppose he chooses door B. Then you may choose between door A (your original door), and door B (the host's door) plus $100.

In this problem, you will use linear logic to express the options you face. You should express the entire scenario as a single linear logic proposition, as we did with the French menu. For example, if the scenario offered a simple choice between pizza and beer, you would write $Pizza \,\&\, Beer$. **Do not** model intermediate states and transitions, as we did in Blocks World.

Problem 2a: Use linear logic to express the scenario, from the viewpoint of the audience member. Please denote the prize behind door A by simply "A", and likewise for B and C. Denote $100 by "D".

Problem 2b: To be part of the game show's audience, you need to turn over a ticket. You can do this (*i.e.*, exchange a ticket for being part of the audience) as many times as you like.

Use linear logic to express this scenario, from the viewpoint of a person who is not yet in the audience, but might choose to be. Write "Z" for the answer to the previous problem, and "T" for a ticket.

3 Linear logic

Below are four judgements in linear logic. Assume P is atomic. Three of them are deriviable, and one is not. For the ones that are derivable, give a derivation. For the one that is not derivable, prove that it is not derivable. You may assume the completeness of linear sequent calculus.

Problem 3a: $\Vdash 1 \multimap (1 \otimes 1)\ \mathsf{true}$

Problem 3b: $\Vdash \top \multimap (\top \otimes \top)\ \mathsf{true}$

Problem 3c: $\Vdash 0 \multimap (0 \otimes 0)$ true

Problem 3d: $\Vdash P \multimap (P \otimes P)$ true

4 Focusing

Give **all** the derivations of the following judgements in focused linear logic. (There is at least one.) Assume P and Q are atomic.

Problem 4a: $P \mathbin{\&} Q \longrightarrow (\downarrow P) \oplus 0$

Problem 4b: $(\uparrow\downarrow P) \mathbin{\&} Q \longrightarrow (\downarrow P) \oplus 0$

Problem 4c: The two judgements above are almost the same. Why did the second have more derivations than the first?

A Linear logic

$$\frac{}{A\ \mathsf{true} \Vdash A\ \mathsf{true}}\ \mathsf{hyp}$$

$$\frac{\Gamma_1 \Vdash A\ \mathsf{true} \quad \Gamma_2 \Vdash B\ \mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash A \otimes B\ \mathsf{true}}\ \otimes\mathsf{I} \qquad \frac{\Gamma_1 \Vdash A \otimes B\ \mathsf{true} \quad \Gamma_2, A\ \mathsf{true}, B\ \mathsf{true} \Vdash C\ \mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash C\ \mathsf{true}}\ \otimes\mathsf{E}$$

$$\frac{\Gamma, A\ \mathsf{true} \Vdash B\ \mathsf{true}}{\Gamma \Vdash A \multimap B\ \mathsf{true}}\ {\multimap}\mathsf{I} \qquad \frac{\Gamma_1 \Vdash A \multimap B\ \mathsf{true} \quad \Gamma_2 \Vdash A\ \mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash B\ \mathsf{true}}\ {\multimap}\mathsf{E}$$

$$\frac{}{\cdot \Vdash 1\ \mathsf{true}}\ 1\mathsf{I} \qquad \frac{\Gamma_1 \Vdash 1\ \mathsf{true} \quad \Gamma_2 \Vdash C\ \mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash C\ \mathsf{true}}\ 1\mathsf{E}$$

$$\frac{\Gamma \Vdash A\ \mathsf{true} \quad \Gamma \Vdash B\ \mathsf{true}}{\Gamma \Vdash A \mathbin{\&} B\ \mathsf{true}}\ \&\mathsf{I} \qquad \frac{\Gamma \Vdash A \mathbin{\&} B\ \mathsf{true}}{\Gamma \Vdash A\ \mathsf{true}}\ \&\mathsf{E1} \qquad \frac{\Gamma \Vdash A \mathbin{\&} B\ \mathsf{true}}{\Gamma \Vdash B\ \mathsf{true}}\ \&\mathsf{E2}$$

$$\frac{\Gamma \Vdash A\ \mathsf{true}}{\Gamma \Vdash A \oplus B\ \mathsf{true}}\ \oplus\mathsf{I1} \qquad \frac{\Gamma \Vdash B\ \mathsf{true}}{\Gamma \Vdash A \oplus B\ \mathsf{true}}\ \oplus\mathsf{I2}$$

$$\frac{\Gamma_1 \Vdash A \oplus B\ \mathsf{true} \quad \Gamma_2, A\ \mathsf{true} \Vdash C\ \mathsf{true} \quad \Gamma_2, B\ \mathsf{true} \Vdash C\ \mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash C\ \mathsf{true}}\ \oplus\mathsf{E}$$

$$\frac{}{\Gamma \Vdash \top\ \mathsf{true}}\ \top\mathsf{I} \qquad \frac{\Gamma_1 \Vdash 0\ \mathsf{true}}{\Gamma_1, \Gamma_2 \Vdash C\ \mathsf{true}}\ 0\mathsf{E}$$

B Linear sequent calculus

$$\frac{P \text{ atomic}}{\Delta; P \longrightarrow P}\ \mathsf{init}$$

$$\frac{\Delta \longrightarrow A \quad \Delta'; B \longrightarrow C}{\Delta, \Delta'; A \multimap B \longrightarrow C}\ {\multimap}\mathsf{L} \qquad \frac{\Delta, A \longrightarrow B}{\Delta \longrightarrow A \multimap B}\ {\multimap}\mathsf{R}$$

$$\frac{\Delta; A \longrightarrow C}{\Delta; A \mathbin{\&} B \longrightarrow C}\ \&\mathsf{L1} \qquad \frac{\Delta; B \longrightarrow C}{\Delta; A \mathbin{\&} B \longrightarrow C}\ \&\mathsf{L2} \qquad \frac{\Delta \longrightarrow A \quad \Delta \longrightarrow B}{\Delta \longrightarrow A \mathbin{\&} B}\ \&\mathsf{R}$$

$$\frac{\Delta, A, B \longrightarrow C}{\Delta, A \otimes B \longrightarrow C}\ {\otimes}\mathsf{L} \qquad \frac{\Delta \longrightarrow A \quad \Delta' \longrightarrow B}{\Delta, \Delta' \longrightarrow A \otimes B}\ {\otimes}\mathsf{R}$$

$$\frac{\Delta, A \longrightarrow C \quad \Delta, B \longrightarrow C}{\Delta, A \oplus B \longrightarrow C}\ {\oplus}\mathsf{L} \qquad \frac{\Delta \longrightarrow A}{\Delta \longrightarrow A \oplus B}\ {\oplus}\mathsf{R1} \qquad \frac{\Delta \longrightarrow B}{\Delta \longrightarrow A \oplus B}\ {\oplus}\mathsf{R2}$$

$$\frac{\Delta \longrightarrow C}{\Delta, 1 \longrightarrow C}\ \mathsf{1L} \qquad \frac{}{\cdot \longrightarrow 1}\ \mathsf{1R} \qquad \frac{}{\Delta \longrightarrow \top}\ \top\mathsf{R} \qquad \frac{}{\Delta, 0 \longrightarrow C}\ \mathsf{0L}$$

C Focused linear logic

$$\begin{array}{lcl} A^- & ::= & P \mid A^+ \multimap A^- \mid A^- \,\&\, A^- \mid \top \mid {\uparrow}A^+ \\ A^+ & ::= & A^+ \otimes A^+ \mid A^+ \oplus A^+ \mid 1 \mid 0 \mid {\downarrow}A^- \\ U & ::= & A^+ \mid A^- \end{array}$$

$$\frac{P \text{ atomic}}{P \text{ stable}} \qquad \frac{}{A^+ \text{ stable}}$$

Inversion

$$\frac{\Delta, A^+ \longrightarrow B^-}{\Delta \longrightarrow A^+ \multimap B^-} \multimap\mathsf{R} \qquad \frac{\Delta \longrightarrow A^- \quad \Delta \longrightarrow B^-}{\Delta \longrightarrow A^- \,\&\, B^-} \&\mathsf{R} \qquad \frac{}{\Delta \longrightarrow \top} \top\mathsf{R} \qquad \frac{\Delta \longrightarrow A^+}{\Delta \longrightarrow {\uparrow}A^+} {\uparrow}\mathsf{R}$$

$$\frac{\Delta, A^+, B^+ \longrightarrow U}{\Delta, A^+ \otimes B^+ \longrightarrow U} \otimes\mathsf{L} \qquad \frac{\Delta, A^+ \longrightarrow U \quad \Delta, B^+ \longrightarrow U}{\Delta, A^+ \oplus B^+ \longrightarrow U} \oplus\mathsf{L}$$

$$\frac{\Delta \longrightarrow U}{\Delta, 1 \longrightarrow U} 1\mathsf{L} \qquad \frac{}{\Delta, 0 \longrightarrow U} 0\mathsf{L} \qquad \frac{\Delta, A^- \longrightarrow U}{\Delta, {\downarrow}A^- \longrightarrow U} {\downarrow}\mathsf{L}$$

Taking focus

$$\frac{\Delta \text{ all negative} \quad U \text{ stable} \quad \Delta; [A^-] \longrightarrow U}{\Delta, A^- \longrightarrow U} \mathsf{lfocus} \qquad \frac{\Delta \text{ all negative} \quad \Delta \longrightarrow [A^+]}{\Delta \longrightarrow A^+} \mathsf{rfocus}$$

Left focus

$$\frac{P \text{ atomic}}{\Delta; [P] \longrightarrow P} \mathsf{init} \qquad \frac{\Delta \longrightarrow [A^+] \quad \Delta'; [B^-] \longrightarrow U}{\Delta, \Delta'; [A^+ \multimap B^-] \longrightarrow U} \multimap\mathsf{L}$$

$$\frac{\Delta; [A^-] \longrightarrow U}{\Delta; [A^- \,\&\, B^-] \longrightarrow U} \&\mathsf{L1} \qquad \frac{\Delta; [B^-] \longrightarrow U}{\Delta; [A^- \,\&\, B^-] \longrightarrow U} \&\mathsf{L2} \qquad \frac{\Delta, A^+ \longrightarrow U}{\Delta; [{\uparrow}A^+] \longrightarrow U} {\uparrow}\mathsf{L}$$

Right focus

$$\frac{\Delta \longrightarrow [A^+] \quad \Delta' \longrightarrow [B^+]}{\Delta, \Delta' \longrightarrow [A^+ \otimes B^+]} \otimes\mathsf{R} \qquad \frac{\Delta \longrightarrow [A^+]}{\Delta \longrightarrow [A^+ \oplus B^+]} \oplus\mathsf{R1} \qquad \frac{\Delta \longrightarrow [B^+]}{\Delta \longrightarrow [A^+ \oplus B^+]} \oplus\mathsf{R2}$$

$$\frac{\Delta = \cdot}{\Delta \longrightarrow [1]} 1\mathsf{R} \qquad \frac{\Delta \longrightarrow A^-}{\Delta \longrightarrow [{\downarrow}A^-]} {\downarrow}\mathsf{R}$$