Midterm I Exam

15-317/657 Constructive Logic
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Instructions

- **Throughout this exam, explain whenever there are notable steps or choices or subtleties and justify the rationale for your particular choice!**
- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 8 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
Eliminators, Eliminationists, and Eliminatu	50	
Rules, the More the Merrier	20	
Conceptual Gadgets	60	
Natural Gadgets	20	
Total:	150	

Please keep in mind that this is a sample solution, *not a model solution. Problems admit multiple correct answers, and the answer the instructor thought of may not necessarily be the best or most elegant.*

1 Eliminators, Eliminationists, and Eliminatu (50 points)

So far, we took the view: "The meaning of a proposition is determined by [...] what counts as a verification of it." In this question, the eliminationist's get their say, who believe: "the meaning of a proposition should be determined by what counts as an elimination or a use of it." They came up with the elimination rules for a new connective $\lhd$:

$$\dfrac{A \lhd B\ \textit{true}}{B\ \textit{true}} \lhd E_{\gg} \qquad\qquad \dfrac{A \lhd B\ \textit{true} \quad B\ \textit{true}}{A\ \textit{true}} \lhd E_{\ll}$$

10 **Task 1** Give the introduction rule(s) that (harmoniously) fit to $\lhd E_{\gg}$ and $\lhd E_{\ll}$:

Solution:

$$\dfrac{B\ \textit{true} \quad \begin{array}{c}\overline{B\ \textit{true}}^{\,u}\\ \vdots\\ A\ \textit{true}\end{array}}{A \lhd B\ \textit{true}} \lhd I^u$$

10 **Task 2** Prove local soundness for the $\lhd$ connective.

Solution:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ B\ \textit{true}\end{array} \quad \begin{array}{c}\overline{B\ \textit{true}}^{\,u}\\ \mathcal{E}\\ A\ \textit{true}\end{array}}{A \lhd B\ \textit{true}} \lhd I^u}{B\ \textit{true}} \lhd E_{\gg} \quad \Rightarrow_R \quad \begin{array}{c}\mathcal{D}\\ B\ \textit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ B\ \textit{true}\end{array} \quad \begin{array}{c}\overline{B\ \textit{true}}^{\,u}\\ \mathcal{E}\\ A\ \textit{true}\end{array}}{A \lhd B\ \textit{true}} \lhd I^u \quad \begin{array}{c}\mathcal{F}\\ B\ \textit{true}\end{array}}{A\ \textit{true}} \lhd E_{\ll} \quad \Rightarrow_R \quad \begin{array}{c}\mathcal{F}\\ \overline{B\ \textit{true}}^{\,u}\\ \mathcal{E}\\ A\ \textit{true}\end{array}$$

In the second reduction, there is no reason to believe the two proofs of B *true* would be the same, hence $\mathcal{D}$ could be a different proof than $\mathcal{F}$.

10 **Task 3** Prove local completeness for the $\lhd$ connective.

Solution:

$$\begin{array}{c}\mathcal{D}\\ A \lhd B\ \textit{true}\end{array} \Rightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \lhd B\ \textit{true}\end{array}}{B\ \textit{true}} \lhd E_{\gg} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \lhd B\ \textit{true}\end{array} \quad \overline{B\ \textit{true}}^{\,u}}{A\ \textit{true}} \lhd E_{\ll}}{A \lhd B\ \textit{true}} \lhd I^u$$

10 **Task 4** Give rules for verifications and uses of $A \lhd B$.

Solution:

$$\frac{A \lhd B \downarrow}{B \downarrow} \lhd E_{\gg} \qquad \frac{A \lhd B \downarrow \quad B \uparrow}{A \downarrow} \lhd E_{\ll} \qquad \frac{B \uparrow \quad \begin{array}{c} \overline{B \downarrow}^{\,u} \\ \vdots \\ A \uparrow \end{array}}{A \lhd B \uparrow} \lhd I^u$$

10 **Task 5** Consider this proof term assignment for $\lhd E_{\ll}$ and $\lhd E_{\gg}$:

$$\frac{M : A \lhd B}{\gg(M) : B} \lhd E_{\gg} \qquad \frac{M : A \lhd B \quad N : B}{\ll(M, N) : A} \lhd E_{\ll}$$

Propose a proof term assignment for the introduction rule(s) and write your local reductions using only the proof terms.

Solution:

$$\frac{M{:}B \quad \begin{array}{c} \overline{u{:}B}^{\,u} \\ \vdots \\ N{:}A \end{array}}{c(M, u.N){:}A \lhd B} \lhd I^u$$

$$\gg(c(M, u.N)) \Rightarrow_r M$$
$$\ll(c(M, u.N), O) \Rightarrow_r [O/u]N$$

2 Rules, the More the Merrier (20 points)

In this question, we consider suggestions for new and improved proof rules that honorable Captain Blackbeard came up with. *Either* show the proof rules to be derived from other natural deduction rules considered in the course. *Or* show that they can be used to prove a formula that we cannot prove soundly and *explain briefly* why that formula should not be proved.

10 **Task 1**

$$\frac{A \lor B\ \textit{true} \qquad A\ \textit{true}}{\neg B\ \textit{true}}\ \lor E_T$$

Solution: Choose $\top$ for B, whose negation allows a proof of $\bot$, which should never be introduced, with $\lor I_L$ on the left and with, e.g., provable tautology $A \supset A$ for A.

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \textit{true}}\ u}{A \supset A\ \textit{true}}\ \supset I^u}{(A \supset A) \lor \top\ \textit{true}}\ \lor I_L \qquad \dfrac{\dfrac{}{A\ \textit{true}}\ u}{A \supset A\ \textit{true}}\ \supset I^u}{\neg \top\ \textit{true}}\ \lor E_T \qquad \dfrac{}{\top\ \textit{true}}\ \top I}{\bot\ \textit{true}}\ \supset E$$

10 **Task 2**

$$\begin{array}{c} \dfrac{}{A\ \textit{true}}\ u \\ \vdots \\ \dfrac{\bot\ \textit{true}}{A\ \textit{true}}\ PBQ^u \end{array}$$

Solution: Choose $\bot$ for A, which should never be introduced and is not even true classically.

$$\frac{\dfrac{}{\bot\ \textit{true}}\ u}{\bot\ \textit{true}}\ PBQ^u$$

3 Conceptual Gadgets (60 points)

Inspector Gadget, has defined the binary connective ! as follows:

$$\dfrac{A\ \mathit{true} \qquad \begin{array}{c}\overline{B\ \mathit{true}}\ u \\ \vdots \\ \bot\ \mathit{true}\end{array}}{A!B\ \mathit{true}}\ !I^u$$

10 **Task 1** Define one or more elimination rules that are in harmony with the above rule. You do not need to prove harmony, but points will be deducted if the two are not in harmony.

Solution:

$$\dfrac{A!B\ \mathit{true}}{A\ \mathit{true}}\ !E_L \qquad \dfrac{A!B\ \mathit{true} \quad B\ \mathit{true}}{\bot\ \mathit{true}}\ !E_R$$

10 **Task 2** Inspector Gadget likes local soundness but doesn't much care for local completeness. According to this view, he has proposed a system for verifying the safety of air traffic control systems. What bad thing might happen when he tries to use his system to verify the safety of a system?

Solution: Inspector Gadget might get stuck not being able to prove further even though it is true. He might not even be able to assemble facts since he can't get them again by decomposing other operators.

10 **Task 3** After reading your answer above, Inspector Gadget has decided to change his ways. He now likes local completeness but, instead, got his system locally unsound. He tries to use this system, again to verify safety of an air traffic control system. What bad thing might happen now?

Solution: Now the aircraft might crash even though Inspector Gadget has a so-called ”proof” since he works with an incorrect proof system.

5 **Task 4** Inspector Gadget has read your answers and now believes in local soundness and local completeness. He has become a strict verificationist and now eats only applesauce and Pop Tarts. When he defines new connectives, he has to write both introduction and elimination rules. Which of those serve to define the *meaning* of each connective?

> **Solution:** introductions.
> “ The meaning of a proposition is determined by [. . .] what counts as a verification of it.” (Per Martin-Löf)

10 **Task 5** Inspector Gadget is really trying hard. He’s come up with a new connective:

$$\frac{A\ \textit{true} \quad B \supset \bot\ \textit{true}}{A \heartsuit B\ \textit{true}}\ \heartsuit I$$

However, when he presents his work, people are not impressed. Explain (informally, in English) what’s wrong with it and why that’s bad. Please write a new introduction rule that fixes the problems but has the same meaning.

> **Solution:** The problem is that it defines the connective $\heartsuit$ in terms of other connectives, which causes potential cycles since the introduction rules are used to give meaning in the verificationist tradition. The analogous rule that does not do that is the following suspiciously analoge to $!I$:
>
> $$\frac{A\ \textit{true} \quad \begin{array}{c} \overline{B\ \textit{true}}\ u \\ \vdots \\ \bot\ \textit{true} \end{array}}{A \heartsuit B\ \textit{true}}\ \heartsuit I^u$$

15 **Task 6** While you’re at it, write corresponding elimination rule(s) that are in harmony with the introduction rule you present. Prove that your new introduction rule is equivalent to $\heartsuit I$ by showing that if something can be proved with one rule, it can also be proved by the other from the same evidence.

> **Solution:** Elimination rules are as for the analogue !, in detail:
>
> $$\frac{A \heartsuit B\ \textit{true}}{A\ \textit{true}}\ \heartsuit E_L \qquad \frac{A \heartsuit B\ \textit{true} \quad B\ \textit{true}}{\bot\ \textit{true}}\ \heartsuit E_R$$
>
> $$\frac{\begin{array}{c} \mathcal{D} \\ A\ \textit{true} \end{array} \quad \begin{array}{c} \overline{B\ \textit{true}}\ u \\ \mathcal{E} \\ \bot\ \textit{true} \end{array}}{A \heartsuit B\ \textit{true}}\ \heartsuit I^u \quad \Rightarrow \quad \frac{\begin{array}{c} \mathcal{D} \\ A\ \textit{true} \end{array} \quad \dfrac{\begin{array}{c} \overline{B\ \textit{true}}\ u \\ \mathcal{E} \\ \bot\ \textit{true} \end{array}}{B \supset \bot\ \textit{true}}\ \supset I}{A \heartsuit B\ \textit{true}}\ \heartsuit I$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B \supset \bot\ \mathit{true}\end{array}}{A \heartsuit B\ \mathit{true}}\ \heartsuit I \quad \Rightarrow \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ B \supset \bot\ \mathit{true}\end{array} \quad \dfrac{}{B\ \mathit{true}}\ u}{\bot\ \mathit{true}}\ {\supset}E}{A \heartsuit B\ \mathit{true}}\ \heartsuit I^u$$

4 Natural Gadgets (20 points)

Inspector Gadget finally got the hang of natural numbers from natural deductions. He went straight ahead to implement the corresponding reductions in his new proof checker Toughch. Toughch uses "nut" as a better name for the natural numbers

$$\frac{}{\mathsf{0} : \mathsf{nut}}\ \mathrm{nut}I_0 \qquad \frac{n : \mathsf{nut}}{\mathsf{s}\, n : \mathsf{nut}}\ \mathrm{nut}I_s$$

Inspector Gadget randomly decided upon assigning the following new and improved proof terms:

$$\frac{n : \mathsf{nut} \quad N_0 : C(\mathsf{0})\ \textit{true} \quad \begin{array}{c} \overline{x : \mathsf{nut}} \quad \overline{u : C(x)\ \textit{true}}^{\,u} \\ \vdots \\ N_s : C(x)\ \textit{true} \end{array}}{G(n, N_0, x.\,u.\,N_s) : C(n)\ \textit{true}}\ \mathrm{nut}E^{x,u}$$

10 **Task 1** Read off the local reductions on proof terms that the proof term assignment from nut$E^{x,u}$ would induce.

Solution: The reductions are broken in the sense that not all of them actually reduce:

$$\begin{array}{lll} G(\mathsf{0}, N_0, x.\,u.\,N_s) & \Longrightarrow_R & N_0 \\ G(\mathsf{s}\, n', N_0, x.\,u.\,N_s) & \Longrightarrow_R & [G(\mathsf{s}\, n', N_0, x.\,u.\,N_s)/u][\mathsf{s}\, n'/x]\, N_s \end{array}$$

10 **Task 2** Toughch uses these reductions to compute with nutural numbers. What behavior will Inspector Gadget observe. Explain why.

Solution: Infinite recursion without progress, because the reduction from n down to n (as opposed to from $\mathsf{s}n$ to n as in nat$E^{x,u}$), can be done infinitely often without any progress whatsoever. It is not well-founded.