Midterm II Exam

15-317/657 Constructive Logic
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Name: ____________________

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Instructions

- **Throughout this exam, explain whenever there are notable steps or choices or subtleties and justify the rationale for your particular choice!**
- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 9 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
Sequent Calculus	70	
Prolog	25	
More Miraculous Sequent Rules	30	
Substitutions	25	
Total:	150	

1 Sequent Calculus (70 points)

This question considers the sequent calculus with cut, weakening, and identity.

10 **Task 1** Prove the following theorem:
Theorem (Connection Property): *If* $\Longrightarrow A \wedge (B \vee C)$ *then* $\Longrightarrow A$ *and either* $\Longrightarrow B$ *or* $\Longrightarrow C$.

10 **Task 2** If $A \wedge (B \vee C)$ has a verification in the natural deduction calculus, then is A provable? And is it also the case that either B or C are provable, too? Briefly justify your answer.

10 **Task 3** Prove that the contraction rule is admissible

$$\frac{\Gamma, A, A \Longrightarrow C}{\Gamma, A \Longrightarrow C}$$

20 **Task 4** To justify cuts, prove the principal cut case when the cut A is of the form $A_1 \wedge A_2$. Assume

$$\mathcal{D} = \cfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma \Longrightarrow A_1 & \Gamma \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2} \wedge R \qquad \text{and} \qquad \mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow C\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow C} \wedge L_1$$

and show

$$\Gamma \Longrightarrow C$$

Explicitly justify all invocations of the induction hypothesis and explain what is smaller.

$\boxed{10}$ **Task 5** How can you change the rules of the sequent calculus to make it *unsound*? Specify the full change of the calculus and justify why it renders the calculus unsound.

$\boxed{10}$ **Task 6** How can you change the rules of the sequent calculus to make it *incomplete*? Specify the full change of the calculus and justify why it renders the calculus incomplete.

2 Prolog (25 points)

You can use these list predicates from lecture: `member/2`, `append/3`, `length/2`. And you can also define auxiliary predicates if you need them. A pair in Prolog is represented by constructor `pair(X,Y)`.

Hint: Use the predicates from the other tasks.

10 **Task 1** Define a predicate `unzip(list,list,list)` which holds if the first list is a list of pairs, the second list is a list of the first elements, while the third list is a list of the second elements.

5 **Task 2** Define a predicate `zip(list,list,list)` which holds if the last list is the list of pairs of the corresponding elements from the first and the second list. You can assume the first and second list to be of the same length.

10 **Task 3** Define a predicate `rotate(list,list)` which holds if the second list is the first list rotated by one as, e.g., in `rotate([a, b, c, d], [b, c, d, a])`.

3 More Miraculous Sequent Rules (30 points)

In this question, we consider suggestions for new and improved proof rules that fierce Captain Toughch came up with. *Either* show the proof rules to be sound by deriving them or proving them to be admissible. *Or* show that they can be used to prove a formula that we cannot prove soundly and *explain briefly* why that formula should not be proved.

$\boxed{10}$ **Task 1**

$$\frac{\Gamma, B \Longrightarrow A \vee C}{\Gamma, A \supset B \Longrightarrow C}\ R1$$

$\boxed{10}$ **Task 2**

$$\frac{\Gamma, A \Longrightarrow D \supset E \quad A, \Gamma \Longrightarrow B}{\Gamma, A \wedge ((B \vee C) \supset D) \Longrightarrow E \wedge (A \vee C)}\ R2$$

$\boxed{10}$ **Task 3** Consider untyped first-order logic with only one type "object", where quantifiers are written $\forall x \,.\, A(x)$ without types.

$$\frac{\Gamma \Longrightarrow A(x)}{\Gamma \Longrightarrow \forall x \,.\, A(x)}\ R3$$

4 Substitutions (25 points)

Recall that a *substitution* is a function σ from terms to terms that satisfies

$$f(t_1, \ldots, t_n)\sigma = f(t_1\sigma, \ldots, t_n\sigma) \qquad \text{for all function symbols } f \text{ and terms } t_i$$

and has a finite domain $\mathrm{dom}(\sigma) = \{x : x\sigma \neq x\}$ of variables. Recall that a *representation ℓ for a substitution* is of the form, e.g.:

$$\ell = (r_1/x_1, r_2/x_2, \ldots, r_n/x_n)$$

$\boxed{5}$ **Task 1** Define *all* the conditions required for a list ℓ to qualify as a representation of a substitution.

$\boxed{10}$ **Task 2** For any such representation ℓ of a substitution, let $\hat{\ell}$ denote the substitution belonging to that representation ℓ.

Construct a representation of the substitution that is a composition $\hat{k}\hat{\ell}$ of $\hat{\ell}$ after $\hat{k}$ from the representation k of $\hat{k}$ and the representation ℓ of $\hat{\ell}$.

You can assume ℓ and k to meet your requirements from Task 1 but should clearly mark if you make any additional assumptions on ℓ and k.

10 **Task 3** Some substitutions can also be applied to formulas. Define how for the cases of $\neg A$, $A \vee B$, and $\forall x.A$. If your substitution is undefined in some cases, briefly explain why.