

Midterm 1

15-317: Constructive Logic

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Instructions

- This exam is closed-book, but one two-sided sheet of notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are 4 problems on 13 pages. Not all problems are the same size or difficulty. You have 80 minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$). Be sure to use the correct notation when substituting proofs for assumptions (e.g., in reductions). When writing natural deduction proofs, you may omit the judgment *true*.
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Stay cool and good luck!

	Max	Score
1	75	
2	40	
3	20	
4	15	
Total:	150	

Please keep in mind that this is a sample solution, not a model solution. Problems admit multiple correct answers, and the answer the instructor thought of may not necessarily be the best or most elegant.

Problems

1. A “new” connective

In this section, we consider the \times connective, which is defined by the following natural deduction rules.

$$\frac{A \text{ true} \quad B \text{ true}}{A \times B \text{ true}} \times I \qquad \frac{A \times B \text{ true} \quad \begin{array}{c} \overline{C \text{ true}} \\ \vdots \\ \overline{u} \quad \overline{v} \end{array}}{C \text{ true}} \times E^{u,v}$$

(a) (10 points) Show local soundness for the \times connective by demonstrating a local reduction.

(b) (10 points) Show local completeness for the \times connective by demonstrating a local expansion.

(c) (5 points) Give sequent calculus rules for the \times connective in the same style as the sequent calculus system we’ve been studying (the rules are listed in Figure 1). There should be one right rule ($\times R$) and one left rule ($\times L$).

Using our natural deduction notation with explicit contexts, we have these natural deduction rules for the \times connective:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B} \times I \qquad \frac{\Gamma \vdash A \times B \quad \Gamma, A, B \vdash C}{\Gamma \vdash C} \times E$$

(d) (10 points) Recall the soundness theorem for sequent calculus, which says that if $\Gamma \Rightarrow A$, then $\Gamma \vdash A$. Give the case of soundness corresponding to the $\times L$ case. As in your homework, you may use the following lemmas:

- **Weakening** For all Γ, A, C , if $\Gamma \vdash C$, then $\Gamma, A \vdash C$.
- **Substitution** For all Γ, A, C , if $\Gamma \vdash A$ and $\Gamma, A \vdash C$, then $\Gamma \vdash C$.

(e) (10 points) Recall the completeness theorem for sequent calculus, which says that if $\Gamma \vdash A$, then $\Gamma \Rightarrow A$. Give the case of completeness corresponding to the $\times E$ case. As in your homework, you may use the following lemmas:

- **Weakening** For all Γ, A, C , if $\Gamma \Rightarrow C$, then $\Gamma, A \Rightarrow C$.
- **Identity** For all Γ, A : $\Gamma, A \Rightarrow A$.
- **Cut** For all Γ, A, C , if $\Gamma \Rightarrow A$ and $\Gamma, A \Rightarrow C$, then $\Gamma \Rightarrow C$.

You may have noticed that this connective is “remarkably similar” to \wedge . It can be shown to be equivalent.

(g) (10 points) Show that $A \wedge B \vdash A \times B$.

- (h) (10 points) Come up with proof terms for introducing and eliminating \times . Present versions of the natural deduction rules (without explicit contexts) $\times E$ and $\times I$ that assign your proof terms, and show the reduction rule for these proof terms.

2. Natural Numbers and Proof Terms

Recall the rules for natural number arithmetic and induction with proof terms (recapped in Figure 2; we have added the rule of reflexivity for your convenience).

- (a) (25 points) Give a natural deduction proof of $\forall n. \neg n = 0 \supset \exists m. n = sm$. Omit typing annotations in quantifications (as we just did) for brevity's sake.

- (b) (15 points) Give the corresponding proof term. Represent proofs of equality by “_”, since they have no computational meaning.

- (c) **For bonus credit** With one more proof step you can prove $\neg 0 = 0 \supset \exists m. 0 = sm$. Show the resulting proof term (don't reduce!), then show the reduction steps that get you to a term that can no longer be reduced. (Reduction rules are in Figure 4.)

3. Classical Logic

Recall the rules for classical logic (recapped in Figure 5).

Here are some familiar tautologies of classical propositional logic. Prove each one using *constructive* natural deduction if possible. If it's not possible, prove it using classical reasoning, that is, using any of the rules *proof by contradiction*, *law of excluded middle*, *double-negation elimination*, or *Peirce's law*. Your classical proof(s) may use either the familiar definition of $\neg A$ as $A \supset \perp$ or the judgmental definition of Figure 5.

(a) (10 points) $(A \vee B) \supset (\neg A \supset B)$

(b) (10 points) $(\neg A \supset B) \supset (A \vee B)$

4. (15 points) **Mistakes Were Made**

Consider the following purported proof:

$$\begin{array}{c}
 \frac{\frac{\frac{}{(\forall x : \tau.A(x)) \supset \exists y : \tau.B(y)}{u} \quad \frac{\frac{\frac{}{A(a)}{v}}{\forall x : \tau.A(x)} \forall I^a \quad \frac{\frac{}{a : \tau}}{a : \tau} \quad \frac{\frac{}{B(a)}{w}}{A(a) \supset B(a)} \supset I^v}{\exists z : \tau.A(z) \supset B(z)} \exists I}{\exists y : \tau.B(y)} \supset E \quad \frac{}{\exists z : \tau.A(z) \supset B(z)} \exists E^{a,w}}{\frac{}{((\forall x : \tau.A(x)) \supset \exists y : \tau.B(y)) \supset \exists z : \tau.A(z) \supset B(z)} \supset I^u}
 \end{array}$$

This proof is incorrect. Circle the label(s) of the rule(s) that are applied incorrectly. Explain what is wrong with each.

Useful Rules

$$\begin{array}{c}
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L_2 \\
\frac{}{\Gamma \Rightarrow \top} \top R \\
\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L \\
\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Rightarrow C \quad \Gamma, A \vee B, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \vee L \\
\frac{}{\Gamma, \perp \Rightarrow C} \perp L \\
\frac{}{\Gamma, P \Rightarrow P} \text{init } (P \text{ atomic})
\end{array}$$

Figure 1: Rules for the sequent calculus.

$$\begin{array}{c}
\frac{}{x : \text{nat}} \quad \frac{}{u : C(x) \text{ true}}^u \\
\vdots \\
\frac{n : \text{nat} \quad M_0 : C(0) \text{ true} \quad M_s : C(sx) \text{ true}}{R(n, M_0, x. u. M_s) : C(n) \text{ true}} \text{natE}^{x,u} \\
\\
\frac{}{0 = 0} =_N I_0 \quad \frac{n = m}{s\ n = s\ m} =_N I_s \quad \frac{0 = s\ n}{J} =_N E_{0s} \quad \frac{s\ n = 0}{J} =_N E_{s0} \quad \frac{s\ m = s\ n}{m = n} =_N E_{ss} \quad \frac{n : \text{nat}}{n = n} \text{refl}
\end{array}$$

Figure 2: Rules for natural numbers and induction.

Constructors

$$\frac{M : A \quad N : B}{\langle M, N \rangle : A \wedge B} \wedge I$$

$$\frac{}{\langle \rangle : \top} \top I$$

$$\frac{\begin{array}{c} \overline{u : A} \quad u \\ \vdots \\ M : B \end{array}}{\lambda u : A. M : A \supset B} \supset I^u$$

$$\frac{M : A}{\mathbf{inl}^B M : A \vee B} \vee I_L$$

$$\frac{N : B}{\mathbf{inr}^A N : A \vee B} \vee I_R$$

no constructor for \perp

$$\frac{\begin{array}{c} \overline{a : \tau} \\ \vdots \\ M : A(a) \end{array}}{\lambda a : \tau. M : \forall x : \tau. A(x)} \forall I^a$$

$$\frac{t : \tau \quad M : A(t)}{\langle t, M \rangle : \exists x : \tau. A(x)} \exists I$$

Destructors

$$\frac{M : A \wedge B}{\mathbf{fst} M : A} \wedge E_L$$

$$\frac{M : A \wedge B}{\mathbf{snd} M : B} \wedge E_R$$

no destructor for \top

$$\frac{M : A \supset B \quad N : A}{MN : B} \supset E$$

$$\frac{\begin{array}{cc} \overline{u : A} \quad u & \overline{w : B} \quad w \\ \vdots & \vdots \\ M : A \vee B & N : C \quad O : C \end{array}}{\mathbf{case} M \mathbf{of} \mathbf{inl} u \Rightarrow N \mid \mathbf{inr} w \Rightarrow O : C} \vee E^{u,w}$$

$$\frac{M : \perp}{\mathbf{abort}^C M : C} \perp E$$

$$\frac{M : \forall x : \tau. A(x) \quad t : \tau}{M t : A(t)} \forall E$$

$$\frac{\begin{array}{cc} \overline{a : \tau} & \overline{u : A(a)} \quad u \\ \vdots & \\ M : \exists x : \tau. A(x) & N : C \end{array}}{\mathbf{let} \langle a, u \rangle = M \mathbf{in} N : C} \exists E^{a,u}$$

Figure 3: Proof terms for natural deduction.

fst $\langle M, N \rangle$	$\Rightarrow_R M$
snd $\langle M, N \rangle$	$\Rightarrow_R N$
$(\lambda u:A. M) N$	$\Rightarrow_R [N/u]M$
case inl ^B M of inl $u \Rightarrow N$ inr $w \Rightarrow O$	$\Rightarrow_R [M/u]N$
case inr ^A M of inl $u \Rightarrow N$ inr $w \Rightarrow O$	$\Rightarrow_R [M/w]O$
$R(0, M_0, x. u. M_s)$	$\Rightarrow_R M_0$
$R(s\ n', M_0, x. u. M_s)$	$\Rightarrow_R [R(n', M_0, x. u. M_s)/u][n'/x] M_s$

Figure 4: Rules for term reduction.

$A \text{ false} := \begin{array}{c} A \text{ true} \\ \vdots \\ \# \end{array}$	$\frac{A \text{ false} \quad A \text{ true}}{J} \text{ contra}$	$\frac{\begin{array}{c} \overline{A \text{ false}}^k \\ \vdots \\ \# \end{array}}{A \text{ true}} PBC^k$
$\frac{\begin{array}{c} \overline{A \text{ true}}^u \\ \vdots \\ \# \end{array}}{\neg A \text{ true}} \neg I^u$	$\frac{\neg A \text{ true} \quad \begin{array}{c} \overline{A \text{ false}}^k \\ \vdots \\ J \end{array}}{J} \neg E^k$	
$\frac{\neg \neg A}{A} DNE$	$\frac{}{A \vee \neg A} LEM$	$\frac{}{((A \supset B) \supset A) \supset A} PL$

Figure 5: Rules (including derived rules) for classical natural deduction.