

Midterm I Exam

15-317/657 Constructive Logic
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Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 6 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
New Connections	50	
Harmonic Series	20	
Using Verifications	40	
Recurse Primitively	40	
Total:	150	

Please keep in mind that this is a sample solution, not a model solution. Problems admit multiple correct answers, and the answer the instructor thought of may not necessarily be the best or most elegant.

1 New Connections (50 points)

Consider the new connective $\Box(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$\frac{\begin{array}{c} \overline{\quad}^u \quad \overline{\quad}^w \\ B \text{ true} \quad C \text{ true} \\ \vdots \quad \quad \quad \vdots \\ A \text{ true} \quad A \text{ true} \end{array}}{\Box(A, B, C) \text{ true}} \Box I^{u,w}$$

10 **Task 1** Give the elimination rule(s) that harmoniously fit to $\Box I$:

Solution:

$$\frac{\Box(A, B, C) \text{ true} \quad B \text{ true}}{A \text{ true}} \Box E_1 \qquad \frac{\Box(A, B, C) \text{ true} \quad C \text{ true}}{A \text{ true}} \Box E_2$$

10 **Task 2** Prove local soundness for the \Box connective.

Solution:

$$\frac{\frac{\frac{\overline{\quad}^u \quad \overline{\quad}^w}{B \text{ true} \quad C \text{ true}}{\mathcal{D} \quad \mathcal{E}}}{A \text{ true} \quad A \text{ true}} \Box I^{u,w}}{\Box(A, B, C) \text{ true}} \quad \frac{\mathcal{F}}{B \text{ true}} \Box E_1}{A \text{ true}} \Rightarrow_R \frac{\mathcal{F}}{B \text{ true} \quad \mathcal{D}}^u \quad A \text{ true}$$

$$\frac{\frac{\frac{\overline{\quad}^u \quad \overline{\quad}^w}{B \text{ true} \quad C \text{ true}}{\mathcal{D} \quad \mathcal{E}}}{A \text{ true} \quad A \text{ true}} \Box I^{u,w}}{\Box(A, B, C) \text{ true}} \quad \frac{\mathcal{F}}{C \text{ true}} \Box E_2}{A \text{ true}} \Rightarrow_R \frac{\mathcal{F}}{C \text{ true} \quad \mathcal{E}}^u \quad A \text{ true}$$

10 **Task 3** Prove local completeness for the \Box connective.

Solution:

$$\frac{\mathcal{D} \quad \Box(A, B, C) \text{ true} \quad \Rightarrow_E}{\Box(A, B, C) \text{ true}} \quad \frac{\frac{\frac{\mathcal{D} \quad \Box(A, B, C) \text{ true} \quad \overline{B \text{ true}}^u}{\Box E_1} \quad \frac{\frac{\mathcal{D} \quad \Box(A, B, C) \text{ true} \quad \overline{C \text{ true}}^w}{\Box E_2} \quad A \text{ true}}{\Box I^{u,w}}}{A \text{ true}} \quad \Box(A, B, C) \text{ true}}$$

10 **Task 4** Propose a proof term assignment for all rules of the \Box connective.

Solution:

$$\frac{\frac{\frac{\overline{u : B}^u \quad \overline{w : C}^w}{\vdots} \quad \frac{M : A \quad N : A}{\Box I^{u,w}}}{b(u.M, w.N) : \Box(A, B, C)}}{\frac{M : \Box(A, B, C) \quad N : B}{l(M, N) : A} \Box E_1} \quad \frac{M : \Box(A, B, C) \quad N : C}{l(M, N) : A} \Box E_2$$

10 **Task 5** Provide local reduction rules for the proof terms of the \Box connective.

Solution:

$$l(b(u.M, w.N), L) \Rightarrow_R [L/u]M$$

$$r(b(u.M, w.N), R) \Rightarrow_R [R/w]M$$

3 Using Verifications (40 points)

The lectures studied natural deduction with proof rules for the truth judgment of the form $A \text{ true}$ as well as verification rules for verifications of the form $A \uparrow$.

- 20 **Task 1** Give a proof of $A \supset A$ in natural deduction that is not also a verification (meaning replacing all $C \text{ true}$ by one of $C \uparrow$ or $C \downarrow$) and briefly indicate why it is not a verification. Or briefly explain why that is impossible.

Solution: Proof of $A \supset A \text{ true}$ using an extra irrelevant redundant proof of $A \supset A \text{ true}$.

$$\frac{\frac{\frac{}{A \text{ true}} w}{A \supset A \text{ true}} \supset I^w \quad \frac{}{A \text{ true}} u}{A \text{ true}} \supset E}{A \supset A \text{ true}} \supset I^u$$

This proof does not correspond to a verification, because there is no license to use $A \supset A \downarrow$ at the indicated step.

- 20 **Task 2** Give a verification of $A \supset A$ that is not also a natural deduction proof (meaning replacing all $C \uparrow$ or $C \downarrow$ by $C \text{ true}$) and briefly indicate why it is not a proof of truth in natural deduction. Or briefly explain why that is impossible.

Solution: That is impossible since verification implies truth, since all verification and uses rule have the same shape as the corresponding rule for truth. So eliding \uparrow and \downarrow and replacing them with the judgment true leads to a proof after skipping the $\uparrow\downarrow$ rule for atomic propositions.

4 Recurse Primitively (40 points)

Your homework showed that multiplication $\text{mult} : \text{nat} \rightarrow (\text{nat} \rightarrow \text{nat})$ on natural numbers is primitively recursive:

$$\begin{aligned}\text{mult}(0) &= \lambda y. 0 \\ \text{mult}(sn) &= \lambda y. (\text{plus} ((\text{mult } n) y)) y\end{aligned}$$

Are the following function definitions primitively recursive? If so, give an equivalent proof term using $R(n, t_0, x.r.t_s)$. Otherwise explain why the definition is not primitively recursive.

20 Task 1

$$\begin{aligned}p(0) &= \lambda y. s0 \\ p(sn) &= \lambda y. (\text{mult}((p\ n)\ y)) y\end{aligned}$$

Solution: yes

$$p = \lambda n. R\left(n, \lambda y. s0, x.r.\lambda y. (\text{mult}(r\ y)) y\right)$$

20 Task 2

$$\begin{aligned}q(0) &= 0 \\ q(sn) &= (\text{mult}(ss(q\ n))) (s(q\ n))\end{aligned}$$

Solution: yes even if it mentions the recursion twice it's still the same recursive value r

$$q = \lambda n. R\left(n, 0, x.r.(\text{mult}(ss\ r)) (s\ r)\right)$$