Midterm I Exam

15-317/657 Constructive Logic
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September 29, 2016

Name: ____________________________________________________________

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Instructions

• This exam is closed-book with one sheet of notes permitted.

• You have 80 minutes to complete the exam.

• There are 4 problems on 6 pages.

• Read each problem carefully before attempting to solve it.

• Do not spend too much time on any one problem.

• Consider if you might want to skip a problem on a first pass and return to it later.

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1 New Connections (50 points)
Consider the new connective \( \square(A, B, C) \) that your friendly verificationists gave meaning to by the following introduction rule:

\[
\begin{array}{c}
B \text{ true}^u \\
\vdots \\
A \text{ true} \\
\hline
\square(A, B, C) \text{ true}^I_{u,w}
\end{array}
\]

10 Task 1 Give the elimination rule(s) that harmoniously fit to \( \square I \):

10 Task 2 Prove local soundness for the \( \square \) connective.
Task 3 Prove local completeness for the □ connective.

Task 4 Propose a proof term assignment for all rules of the □ connective.

Task 5 Provide local reduction rules for the proof terms of the □ connective.
**Harmonic Series (20 points)**

Detective Chase McCase is hunting a group of disharmonious crooks, who wrote down random introduction and elimination rules. Help him sort it out by marking connectives as:

- for harmonious connectives and provide a local reduction on proofs or proof terms.
- for unharmonious connectives and explain in one sentence one case that fails and why.

Note: for harmonious connectives, you do not need to write down the local completeness argument (but make sure it is locally sound and locally complete).

**Task 1**

\[
\frac{\vdash}{u : A^u} \\
\vdots \\
\frac{M : B}{m(u : A, M) : A \rightarrow B} \quad \frac{M : A \rightarrow B}{\rightarrow I^u} \\
\frac{r(M) : B}{\rightarrow E}
\]

**Task 2**

\[
\frac{M : A \quad N : B}{s(M, N) : A \Diamond B} \quad \Diamond I \\
\frac{M : A \Diamond B}{f(M) : A} \quad \Diamond E
\]
3 Using Verifications (40 points)

The lectures studied natural deduction with proof rules for the truth judgment of the form $A \text{ true}$ as well as verification rules for verifications of the form $A \uparrow$.

**Task 1** Give a proof of $A \supset A$ in natural deduction that is not also a verification (meaning replacing all $C \text{ true}$ by one of $C \uparrow$ or $C \downarrow$) and briefly indicate why it is not a verification. Or briefly explain why that is impossible.

**Task 2** Give a verification of $A \supset A$ that is not also a natural deduction proof (meaning replacing all $C \uparrow$ or $C \downarrow$ by $C \text{ true}$) and briefly indicate why it is not a proof of truth in natural deduction. Or briefly explain why that is impossible.
4 Recurse Primitively (40 points)
Your homework showed that multiplication $\text{mult} : \text{nat} \rightarrow (\text{nat} \rightarrow \text{nat})$ on natural numbers is primitively recursive:
\[
\begin{align*}
\text{mult}(0) & = \lambda y. 0 \\
\text{mult}(s\ n) & = \lambda y. (\text{plus} \ (\text{mult} \ n) \ y) \ y
\end{align*}
\]

Are the following function definitions primitively recursive? If so, give an equivalent proof term using $R(n, t_0, x.r.t_s)$. Otherwise explain why the definition is not primitively recursive.

20 Task 1
\[
\begin{align*}
p(0) & = \lambda y. s \ 0 \\
p(s\ n) & = \lambda y. (\text{mult} \ ((p \ n) \ y)) \ y
\end{align*}
\]

20 Task 2
\[
\begin{align*}
q(0) & = 0 \\
q(s\ n) & = (\text{mult} \ (s\ s\ (q\ n))) \ (s\ (q\ n))
\end{align*}
\]