15-819M: Data, Code, Decisions

08: First-Order Logic

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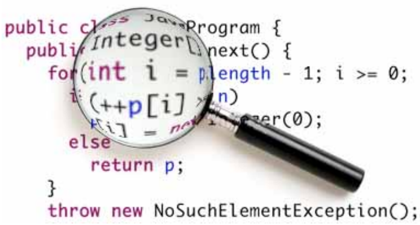

Outline

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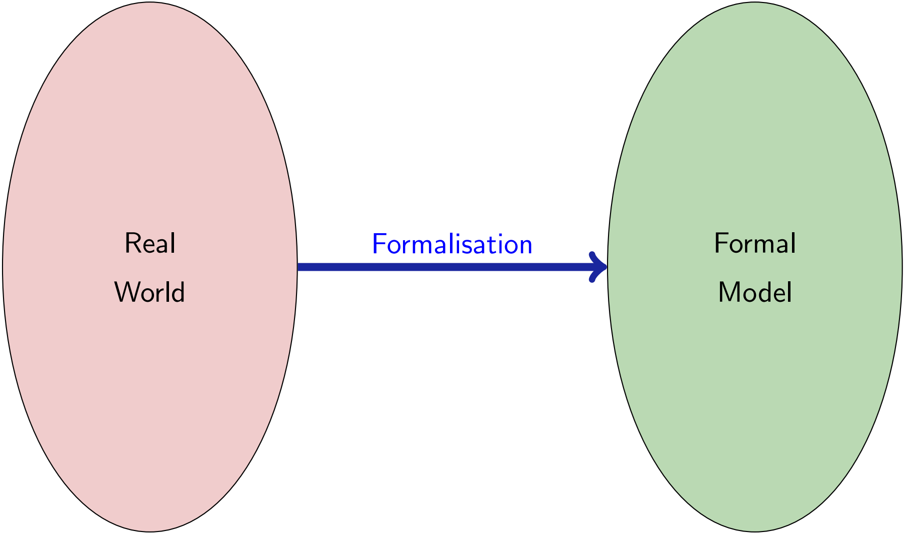
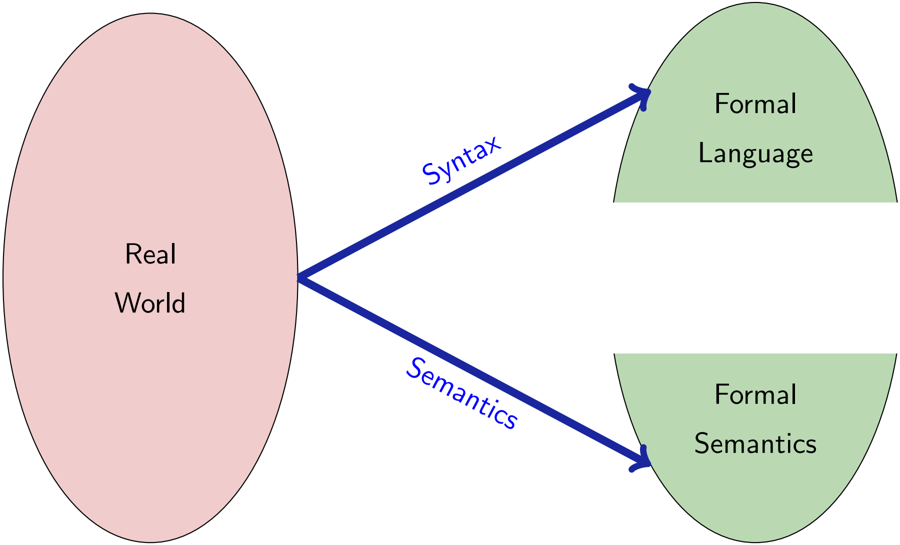
Formalisation



Formalisation: Syntax, Semantics



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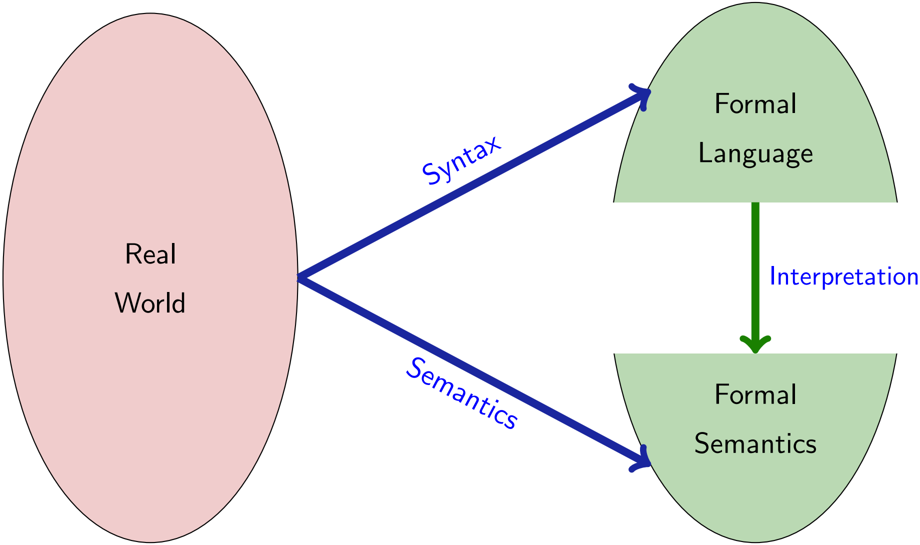

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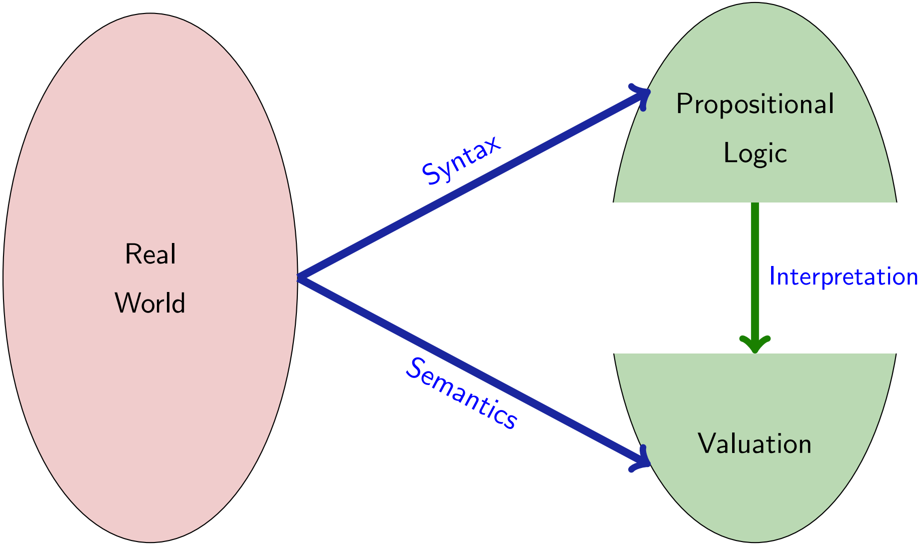

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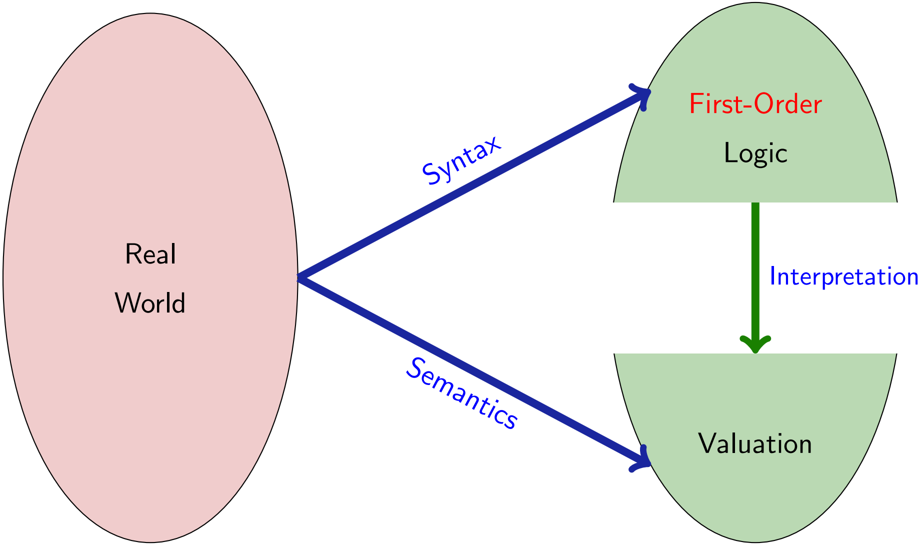
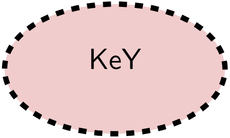
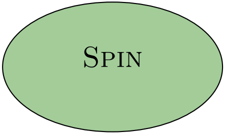


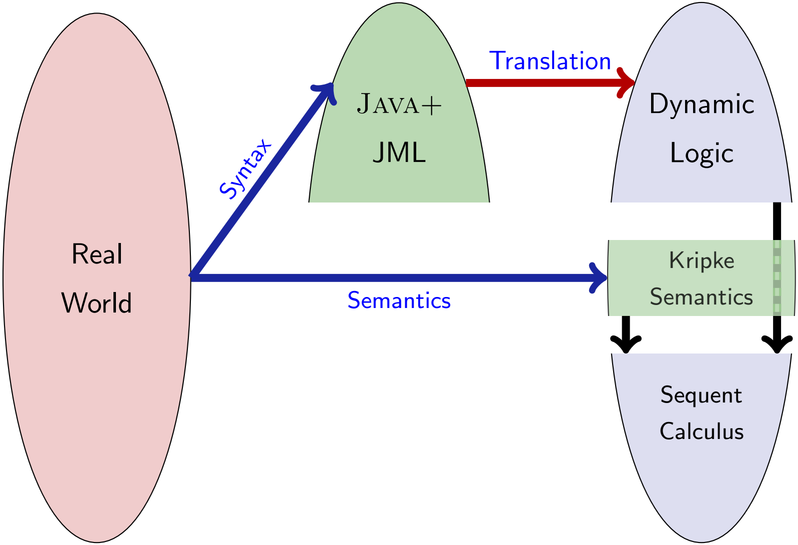
Approaches to Formal Software Verification



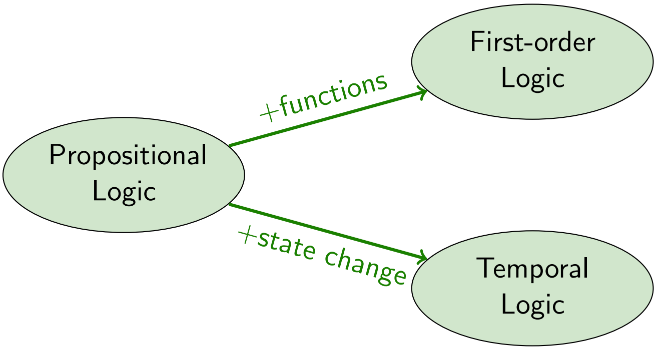
Concrete programs, Complex properties	Concrete programs, Simple properties
Abstract programs, Complex properties	Abstract programs, Simple properties



Formal Verification: Deduction



Beyond Propositional Logic



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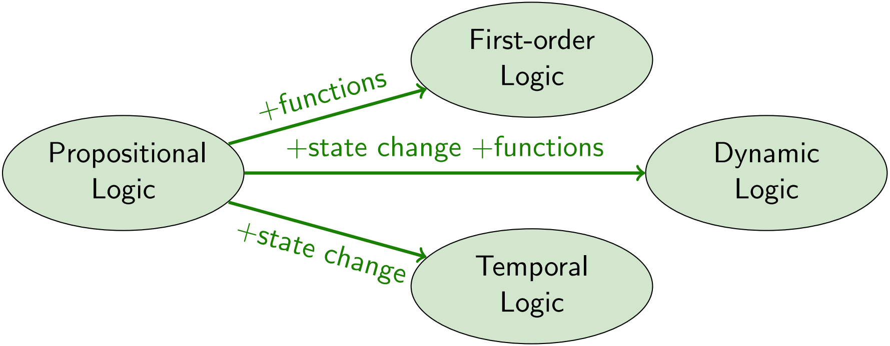

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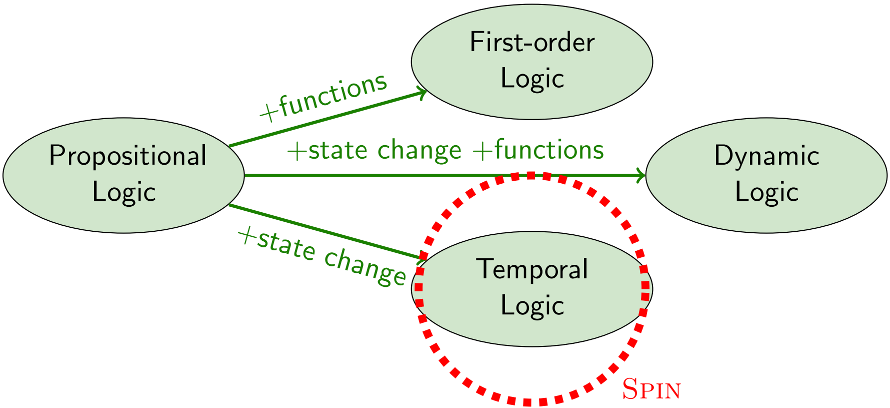

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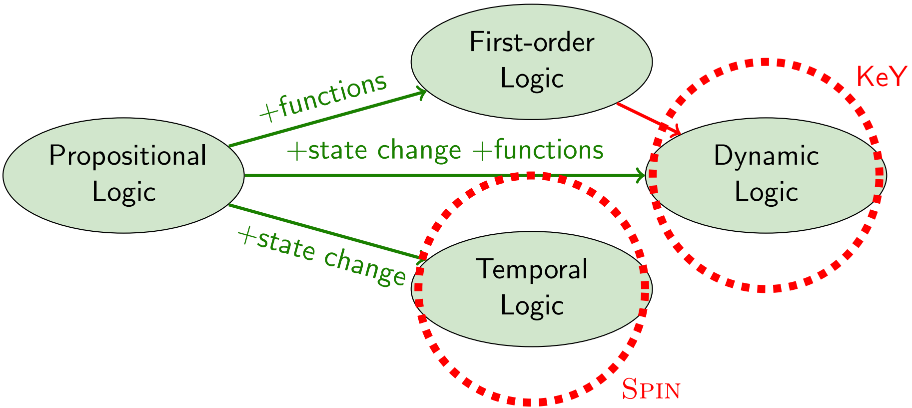

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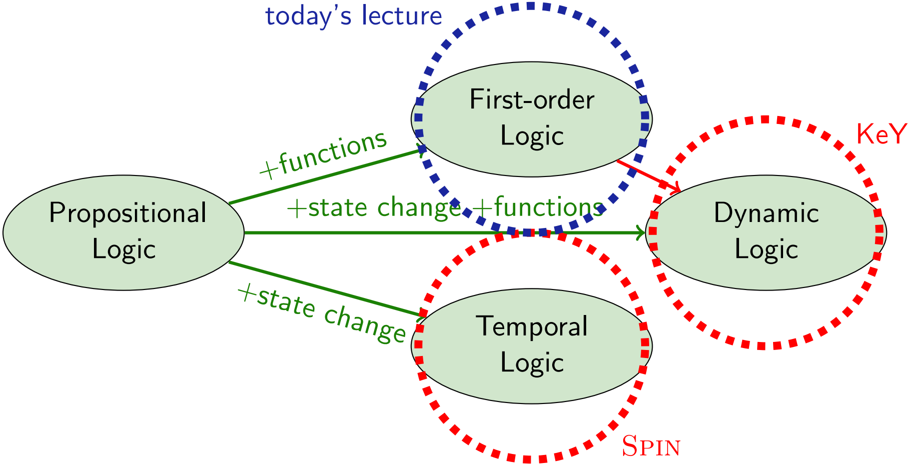
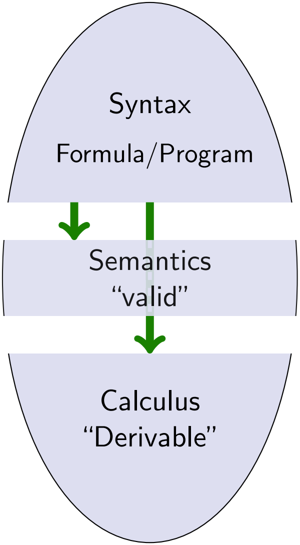


Syntax, Semantics, Calculus



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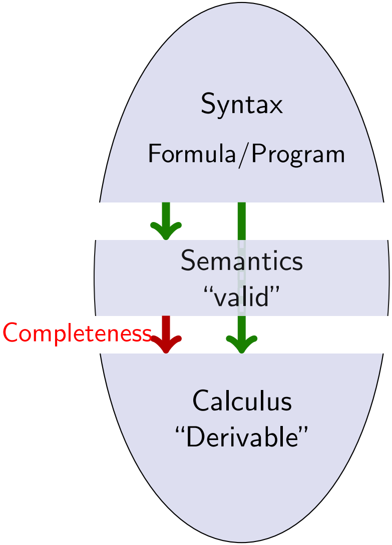

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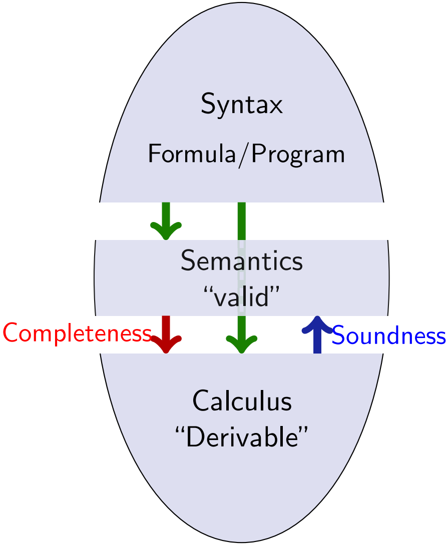


Limitations of Propositional Logic

Fixed, finite number of objects

Cannot express: let g be group with arbitrary number of elements

No functions or relations with arguments

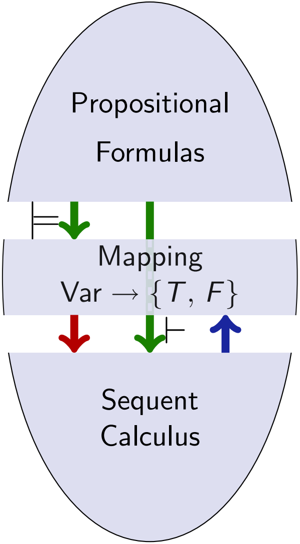
- ✔ Can express: finite function/relation table with indexed variables p_{ij}
- ✘ Cannot express:
  properties of function/relation on all arguments: "+" associative

Static interpretation

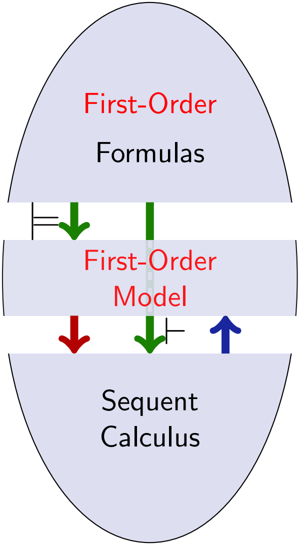
Programs change value of their variables, e.g., via assignment, call, etc.
Propositional formulas look at one single interpretation at a time

Outline

Propositional Logic



First-Order Logic



Syntax of First-Order Logic: Signature

Definition (First-Order Signature)

First-order signature $\Sigma = (\text{PSym}, \text{FSym}, \alpha)$

Predicate or Relation Symbols $\text{PSym} = \{p_i \mid i \in \mathbb{N}\}$
Function Symbols $\text{FSym} = \{f_i \mid i \in \mathbb{N}\}$

Typing function α, set of types $\mathcal{T}$

- $\alpha(p) \in \mathcal{T}^*$ for all $p \in \text{PSym}$
- $\alpha(f) \in \mathcal{T}^* \times \mathcal{T}$ for all $f \in \text{FSym}$

Definition (Variables)

$\text{VSym} = \{x_i \mid i \in \mathbb{N}\}$ set of typed variables

- In contrast to "standard" FOL, our symbols are typed
  Necessary to model a typed programming language such as JAVA!
- Allow any non-reserved name for symbols, not merely p_3, $f_{17}, \ldots$

Syntax of First-Order Logic: Signature

Declaration of signature symbols

- Write $T\ x$; to declare variable x of type T
- Write $p(T_1, \ldots, T_r)$; for $\alpha(p) = (T_1, \ldots, T_r)$
- Write $T\ f(T_1, \ldots, T_r)$; for $\alpha(f) = ((T_1, \ldots, T_r), T)$

Similar convention as in JAVA, no overloading of symbols
Case $r = 0$ is allowed, then write p instead of $p()$, etc.

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Example

Variables **boolean** `b;` **int** `i;`

Predicates `isEmpty(List); alertOn;`

Functions **int** `arrayLookup(`**int**`); java.lang.Object o;`

OO Type Hierarchy

We want to model the behavior of JAVA programs
Admissible types $\mathcal{T}$ form object-oriented type hierarchy

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We want to model the behavior of JAVA programs
Admissible types $\mathcal{T}$ form object-oriented type hierarchy

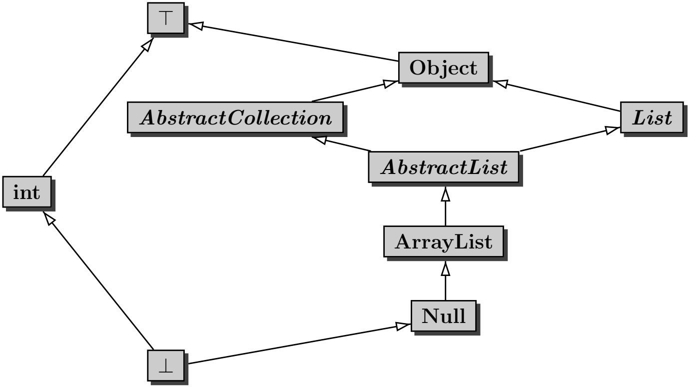
Definition (OO Type Hierarchy)

- $\mathcal{T}$ is finite set of types (not parameterized)
- Given subtype relation $\sqsubseteq$, assume $\mathcal{T}$ has all supertypes, i.e., $\mathcal{T}$ is $\sqcap$-closed
- Dynamic types $\mathcal{T}_d \subseteq \mathcal{T}$, where $\top \in \mathcal{T}_d$
- Abstract types $\mathcal{T}_a \subseteq \mathcal{T}$, where $\bot \in \mathcal{T}_a$
- $\mathcal{T}_d \cap \mathcal{T}_a = \emptyset$
- $\mathcal{T}_d \cup \mathcal{T}_a = \mathcal{T}$
- $\bot \sqsubseteq T \sqsubseteq \top$ for all $T \in \mathcal{T}$

OO Type Hierarchy

Example

Using UML notation



OO Type Hierarchy

- Dynamic types are those with direct elements
- Abstract types for abstract classes and interfaces
- JAVA 1.5+ is $\sqcap$-closed
- In JAVA primitive (value) and object types incomparable
- $\bot$ is abstract and hence no object ever can have this type
  $\bot$ cannot occur in declaration of signature symbols
- Each abstract type except $\bot$ has a non-empty dynamic subtype
- In JAVA $\top$ is chosen to have no direct elements
- JAVA has infinitely many types: **int** [], **int** [] [],...
  Restrict $\mathcal{T}$ to the finitely many types that occur in a given program

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  Restrict $\mathcal{T}$ to the finitely many types that occur in a given program

Example (The Minimal Type Hierarchy)

$\mathcal{T} = \{\bot, \top\}$

All signature symbols have same type $\top$: drop type, untyped logic

Reserved Signature Symbols

Reserved signature symbols

- Equality symbol $\doteq \in$ PSym declared as $\doteq(\top, \top)$

  Written infix: $x \doteq 0$

- Type predicate symbol $\sqsubseteq T \in$ PSym for each $T \in \mathcal{T}$

  Declared as $\sqsubseteq T(\top)$

  Written prefix: `i`$\sqsubseteq$**int** — read "instance of"

- Type cast symbol $(T) \in$ FSym for each $T \in \mathcal{T}$

  Declared as $T\ (T)(\top)$
  Written prefix: `(String)o` — read "cast o to String"

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So far, we have a type system and a signature — where is the logic?

Terms

First-order terms, informally

- Think of first-order terms as expressions in a programming language
  Built up from variables, constants, function symbols
- First-order terms have no side effects (like PROMELA, unlike JAVA)
- First-order terms have a type and must respect type hierarchy
  - type of $f(g(x))$ is result type in declaration of function f
  - in $f(g(x))$ the result type of g is subtype of argument type of f, etc.

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Definition (First-Order Terms $\{\mathrm{Term}_T\}_{T \in \mathcal{T}}$ with type $T \in \mathcal{T}$)

- x is term of type T for variable declared as $T\ x$;
- $f(t_1, \ldots, t_r)$ is term of type T for
  - function symbol declared as $T\ f(T_1, \ldots, T_r)$; and
  - terms t_i of type $T_i' \sqsubseteq T_i$ for $1 \leq i \leq r$
- There are no other terms (inductive definition)

Terms

Example

Signature: **int** `i`; **short** `j`; `List l`; **int** `f`(**int**);

- `f(i)` has result type **int** and is contained in $\mathrm{Term}_{\mathbf{int}}$
- `f(j)` has result type **int** (when $\mathbf{short} \sqsubseteq \mathbf{int}$)
- `f(l)` is ill-typed (when **int**, `List` incomparable)
- `f(i,i)` is not a term (doesn't match declaration)
- (**int**)`j` is term of type **int**
- even (**int**)`l` is term of type **int** (type cast always well-formed)

Terms

Example

Signature: **int** `i`; **short** `j`; `List l`; **int** `f(`**int**`)`;

- `f(i)` has result type **int** and is contained in $\text{Term}_{\textbf{int}}$
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- `f(l)` is ill-typed (when **int**, `List` incomparable)
- `f(i,i)` is not a term (doesn't match declaration)
- `(`**int**`)j` is term of type **int**
- even `(`**int**`)l` is term of type **int** (type cast always well-formed)

- If f is constant ($r = 0$) write f instead of $f()$
- Use infix notation liberally, where appropriate: declare **int** `+(`**int**`,` **int**`)`; then write `i+j`, etc.
- Use brackets to disambiguiate parsing: `(i+j)*i`

First-Order Atomic Formulas

Definition (Atomic First-Order Formulas)

$p(t_1, \ldots, t_r)$ is atomic first-order formula for

- predicate symbol declared as $p(T_1, \ldots, T_r)$; and
- terms t_i of type $T_i' \sqsubseteq T_i$ for $1 \leq i \leq r$

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Example

Signature: **int** `i`; **short** `j`; `List l`; `<(`**int**, **int**`)`;

- `i < i` is an atomic first-order formula
- `i < j` is an atomic first-order formula (when **short**$\sqsubseteq$**int**)
- `i < l` is ill-typed (when **int**, `List` incomparable)
- `i`$\doteq$`j` and even `i`$\doteq$`l` are atomic first-order formulas
- `i`$\sqsubseteq$**short** is an atomic first-order formula

First-Order Formulas

Definition (Set of First-Order Formulas *For*)

- Truth constants true, false and all first-order atomic formulas are first-order formulas
- If ϕ and ψ are first-order formulas then

$$!\,\phi, \quad (\phi \;\&\; \psi), \quad (\phi \mid \psi), \quad (\phi \;\texttt{->}\; \psi), \quad (\phi \;\texttt{<->}\; \psi)$$

  are also first-order formulas
- If $T\ x$ is a variable declaration, ϕ a first-order formula, then $\forall\, T\ x;\ \phi$ and $\exists\, T\ x;\ \phi$ are first-order formulas
  Any occurrence of x in ϕ must be well-typed

- $\forall\, T\ x;\ \phi$ called universally quantified formula
- $\exists\, T\ x;\ \phi$ called existentially quantified formula

First-Order Formulas

- In $\forall\ T\ x;\ \phi$ and $\exists\ T\ x;\ \phi$ call ϕ the scope of x bound by $\forall/\exists$
- Variables bound in quantified formulas similar to program locations declared as local variables/formal parameters

Example

- $\forall\,\mathbf{int}\ i;\ \exists\,\mathbf{int}\ j;\ i < j$ is a first-order formula
- $\forall\,\mathbf{int}\ i;\ \exists\,\texttt{List}\ l;\ i < l$ is ill-typed
- $\forall\,\mathbf{int}\ i;\ i < j$ is a first-order formula if j is a constant compatible with $\mathbf{int}$
- $(\forall\,\mathbf{int}\ i;\ \forall\,\mathbf{int}\ j;\ i < j)\ \mid\ (\forall\,\mathbf{int}\ i;\ \forall\,\mathbf{int}\ j;\ i > j)$ is a first-order formula

Remark on Concrete Syntax

	Text book	SPIN	KeY	JAVA
Negation	$\neg$	`!`	`!`	`!`
Conjunction	$\wedge$	`&&`	`&`	`&&`
Disjunction	$\vee$	$\vert\vert$	$\vert$	$\vert\vert$
Implication	$\rightarrow$, $\supset$	`->`	`->`	n/a
Equivalence	$\leftrightarrow$	`<->`	`<->`	n/a
Universal Quantifier	$\forall x; \phi$	n/a	`\forall` $T\ x; \phi$	n/a
Existential Quantifier	$\exists x; \phi$	n/a	`\exists` $T\ x; \phi$	n/a
Value equality	$\doteq$	`==`	`=`	`==`

Remark on Concrete Syntax

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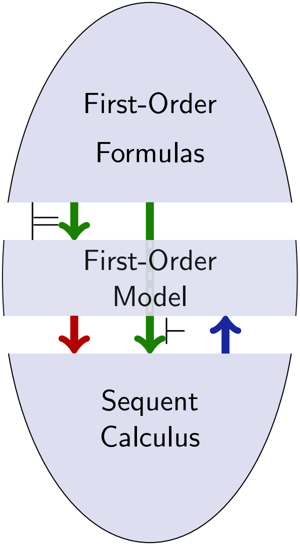
Implication	$\rightarrow$, $\supset$	->	->	n/a
Equivalence	$\leftrightarrow$	<->	<->	n/a
Universal Quantifier	$\forall x; \phi$	n/a	`\forall` $T\ x; \phi$	n/a
Existential Quantifier	$\exists x; \phi$	n/a	`\exists` $T\ x; \phi$	n/a
Value equality	$\doteq$	==	=	==

For quantifiers we normally use textbook syntax and suppress type information to ease readability

For propositional connectives we use KeY syntax

Outline

First-Order Semantics



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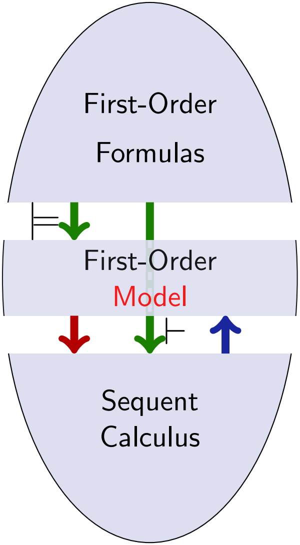


First-Order Semantics

From propositional to first-order semantics

- In prop. logic, interpretation of variables with $\{T, F\}$
- In first-order logic we must assign meaning to:
  - variables bound in quantifiers
  - constant and function symbols
  - predicate symbols
- Each variable or function value may denote a different object
- Respect typing: **int** `i`, `List l` must denote different objects

What we need (to interpret a first-order formula)

1. A collection of typed universes of objects (akin to heap objects)
2. A mapping from variables to objects
3. A mapping from function arguments to function values
4. The set of argument tuples where a predicate is true

First-Order Domains/Universes

1. A collection of typed universes of objects

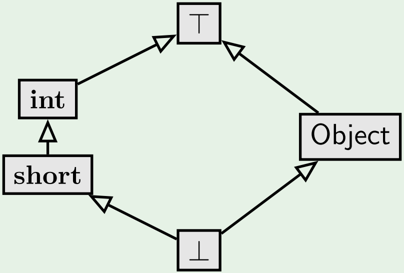
Definition (Universe/Domain)

A non-empty set $\mathcal{D}$ of objects is a universe or domain
Each element of $\mathcal{D}$ has a fixed type given by $\delta : \mathcal{D} \to \mathcal{T}_d$

- Like heap objects and values in JAVA
- Notation for the domain elements type-compatible with $T \in \mathcal{T}$:
  $\mathcal{D}^T = \{d \in \mathcal{D} \mid \delta(d) \sqsubseteq T\}$
- For each dynamic type $T \in \mathcal{T}_d$ there must be at least one domain element type-compatible with it: $\mathcal{D}^T \neq \emptyset$

First-Order Universes

Example



- $\mathcal{D} = \{17, o\}$
- $\delta(17) = \textbf{short}$, $\delta(o) = \text{Object}$
- Then $\mathcal{D}^{\textbf{short}} = \mathcal{D}^{\textbf{int}} = \{17\}$, $\mathcal{D}^{\text{Object}} = \{o\}$, $\mathcal{D}^{\top} = \mathcal{D} = \{17, o\}$, and $\mathcal{D}^{\bot} = \{\}$

First-Order Models

3. A mapping from function arguments to function values
4. The set of argument tuples where a predicate is true

Definition (First-Order Model)

Let $\mathcal{D}$ be a domain with typing function δ

Let f be declared as $T\ f(T_1, \ldots, T_r)$;

Let p be declared as $p(T_1, \ldots, T_r)$;

Let $\mathcal{I}(f) : \mathcal{D}^{T_1} \times \cdots \times \mathcal{D}^{T_r} \rightarrow \mathcal{D}^{T}$

Let $\mathcal{I}(p) \subseteq \mathcal{D}^{T_1} \times \cdots \times \mathcal{D}^{T_r}$

Then $\mathcal{M} = (\mathcal{D}, \delta, \mathcal{I})$ is a first-order model

First-Order Models

Example

Signature: `int i; short j; int f(int); Object obj; <(int,int);`

$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

$$\mathcal{I}(i) = 17$$
$$\mathcal{I}(j) = 17$$
$$\mathcal{I}(\texttt{obj}) = o$$

$\mathcal{D}^{\textbf{int}}$	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\textbf{int}} \times \mathcal{D}^{\textbf{int}}$	in $\mathcal{I}(<)$?
$(2,2)$	F
$(2,17)$	T
$(17,2)$	F
$(17,17)$	F

One of uncountably many possible first-order models!

Semantics of Reserved Signature Symbols

Definition

- Equality symbol $\doteq$ declared as $\doteq(\top, \top)$

  Model is fixed as $\mathcal{I}(\doteq) = \{(d, d) \mid d \in \mathcal{D}\}$
  "Referential Equality" (holds if arguments refer to identical object)

  Exercise: write down the predicate table for example domain

- Type predicate symbol $\sqsubseteq T$ for any T, declared as $\sqsubseteq T(\top)$

  $\mathcal{I}(\sqsubseteq T) = \mathcal{D}^T$

  Exercise: what is $\mathcal{I}(\sqsubseteq\text{Object})$?

- Type cast symbol (T) for each T, declared as $T\ (T)(\top)$

$$\mathcal{I}((T))(x) = \begin{cases} x & \text{if cast succeeds } (\delta(x) \sqsubseteq T) \\ d & \text{otherwise, for an arbitrary fixed } d \in \mathcal{D}^T \end{cases}$$

  Exercise: what is $\mathcal{I}((\mathbf{int}))(17)$?

Signature Symbols vs. Domain Elements

- Domain elements are not just the terms representing them
- First-order formulas and terms have no access to domain
- As in JAVA: identity and memory layout of values/objects hidden
- Think of a first-order model as a “heap” of first-order logic

Example

Signature: `Object obj1, obj2;`
Domain: $\mathcal{D} = \{o\}$

In this model, necessarily $\mathcal{I}(\texttt{obj1}) = \mathcal{I}(\texttt{obj2}) = o$

Effect similar to aliasing in JAVA with reference types

Variable Assignments

2. A mapping from variables to objects

Think of variable assignment as environment for storage of local variables

Definition (Variable Assignment)

A variable assignment β maps variables to domain elements
It respects the variable type, i.e., if x has type T then $\beta(x) \in \mathcal{D}^T$

Definition (Modified Variable Assignment)

Let y be variable of type T, β variable assignment, $d \in \mathcal{D}^T$:

$$\beta_y^d(x) := \begin{cases} \beta(x) & \text{if } x \neq y \\ d & \text{if } x = y \end{cases}$$

Semantic Evaluation of Terms

Given a first-order model $\mathcal{M}$ and a variable assignment β
it is possible to evaluate first-order terms under $\mathcal{M}$ and β

Analogy

Evaluating an expression in a programming language
with respect to a given heap ($\mathcal{M}$) and binding of local variables (β)

Definition (Valuation of Terms)

$val_{\mathcal{M},\beta} : \text{Term} \rightarrow \mathcal{D}$ such that $val_{\mathcal{M},\beta}(t) \in \mathcal{D}^T$ for $t \in \text{Term}_T$:

- $val_{\mathcal{M},\beta}(x) = \beta(x)$ (recall that β respects typing)
- $val_{\mathcal{M},\beta}(f(t_1, \ldots, t_r)) = \mathcal{I}(f)(val_{\mathcal{M},\beta}(t_1), \ldots, val_{\mathcal{M},\beta}(t_r))$

Semantic Evaluation of Terms

Example

Signature: **int** `i`; **short** `j`; **int** `f`(**int**);

$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

Variables: `Object obj`; **int** `x`;

$\mathcal{I}(\texttt{i}) = 17$
$\mathcal{I}(\texttt{j}) = 17$

$\mathcal{D}^{\textbf{int}}$	$\mathcal{I}(\texttt{f})$
2	17
17	2

Var	β
obj	o
x	17

- $val_{\mathcal{M},\beta}(\texttt{f}(\texttt{f}(\texttt{i})))$?
- $val_{\mathcal{M},\beta}(x)$?
- $val_{\mathcal{M},\beta}((\textbf{int})\texttt{obj})$?

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17	2

Var	β
`obj`	o
`x`	17

- $val_{\mathcal{M},\beta}(\texttt{f(f(i))})$? $= 17$
- $val_{\mathcal{M},\beta}(x)$?
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Semantic Evaluation of Terms

Example

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17	2

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17	2

Var	β
obj	o
x	17

- $val_{\mathcal{M},\beta}(\mathtt{f}(\mathtt{f}(\mathtt{i})))\ ?\ = 17$
- $val_{\mathcal{M},\beta}(x)\ ?\ = 17$
- $val_{\mathcal{M},\beta}((\textbf{int})\mathtt{obj})\ ?\ = 2$, say

Semantic Evaluation of Formulas

Formulas are true or false

A validity relation is more convenient than a function

Definition (Validity Relation for Formulas)

$\mathcal{M}, \beta \models \phi$ for $\phi \in For$ "$\mathcal{M}, \beta$ models ϕ"

- $\mathcal{M}, \beta \models p(t_1, \ldots, t_r)$ iff $(val_{\mathcal{M},\beta}(t_1), \ldots, val_{\mathcal{M},\beta}(t_r)) \in \mathcal{I}(p)$
- $\mathcal{M}, \beta \models \phi \ \& \ \psi$ iff $\mathcal{M}, \beta \models \phi$ and $\mathcal{M}, \beta \models \psi$
- ... as in propositional logic
- $\mathcal{M}, \beta \models \forall\, T\ x; \phi$ iff $\mathcal{M}, \beta_x^d \models \phi$ for all $d \in \mathcal{D}^T$
- $\mathcal{M}, \beta \models \exists\, T\ x; \phi$ iff $\mathcal{M}, \beta_x^d \models \phi$ for at least one $d \in \mathcal{D}^T$

Semantic Evaluation of Formulas

Example

Signature: `short j; int f(int); Object obj; <(int,int);`

$\mathcal{D} = \{17,\ 2,\ o\}$ where all numbers are short

$\mathcal{I}(j) = 17$
$\mathcal{I}(\texttt{obj}) = o$

$\mathcal{D}^{\mathbf{int}}$	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\mathbf{int}} \times \mathcal{D}^{\mathbf{int}}$	in $\mathcal{I}(<)$?
$(2, 2)$	F
$(2, 17)$	T
$(17, 2)$	F
$(17, 17)$	F

- $\mathcal{M}, \beta \models f(j) < j$?
- $\mathcal{M}, \beta \models \exists\, \mathbf{int}\ x;\ f(x) \doteq x$?
- $\mathcal{M}, \beta \models \forall\, \texttt{Object}\ o1;\ \forall\, \texttt{Object}\ o2;\ o1 \doteq o2$?

Semantic Notions

Definition (Satisfiability, Truth, Validity)

$$
\begin{array}{llll}
\mathcal{M}, \beta \models \phi & & & (\phi \text{ is } \textbf{satisfiable}) \\
\mathcal{M} \models \phi & \text{iff} & \text{for all } \beta : \quad \mathcal{M}, \beta \models \phi & (\phi \text{ is } \textbf{true} \text{ in } \mathcal{M}) \\
\models \phi & \text{iff} & \text{for all } \mathcal{M} : \quad \mathcal{M} \models \phi & (\phi \text{ is } \textbf{valid})
\end{array}
$$

Closed formulas that are satisfiable are also true: one top-level notion

Semantic Notions

Definition (Satisfiability, Truth, Validity)

$\mathcal{M}, \beta \models \phi$				(ϕ is **satisfiable**)
$\mathcal{M} \models \phi$	iff	for all β :	$\mathcal{M}, \beta \models \phi$	(ϕ is **true** in $\mathcal{M}$)
$\models \phi$	iff	for all $\mathcal{M}$:	$\mathcal{M} \models \phi$	(ϕ is **valid**)

Closed formulas that are satisfiable are also true: one top-level notion

Example

- $f(j) < j$ is true in $\mathcal{M}$
- $\exists\, \mathbf{int}\ x;\ i \doteq x$ is valid
- $\exists\, \mathbf{int}\ x;\ !(x \doteq x)$ is not satisfiable

Outline

Untyped First-Order Logic

Most logic textbooks introduce untyped logic

How to obtain untyped logic as a special case

- Minimal Type Hierarchy: $\mathcal{T} = \{\bot, \top\}$
- $\mathcal{D} = \mathcal{D}^\top \neq \emptyset$: only one populated type $\top$, drop all typing info
- Signature merely specifies arity of functions and predicates: Write $f/1$, $</2$, $i/0$, etc.
- Untyped logic is suitable whenever we model a uniform domain
- Typical applications: pure mathematics such as algebra

Untyped First-Order Logic

Example (Axiomatization of a group in first-order logic)

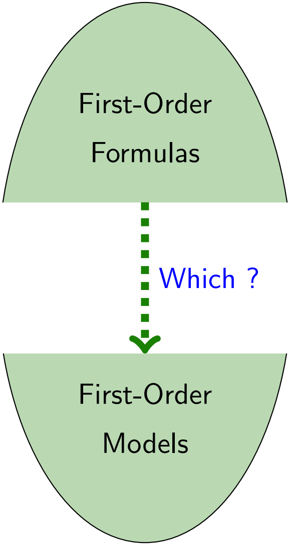
Signature Σ_G: FSym $= \{\circ/2, \mathbf{e}/0\}$, PSym $= \{\doteq/2\}$
Let G be the following formulas:

Left identity	$\forall x;\ \mathbf{e} \circ x \doteq x$
Left inverse	$\forall x;\ \exists y;\ y \circ x \doteq \mathbf{e}$
Associativity	$\forall x;\ \forall y;\ \forall z;\ (x \circ y) \circ z \doteq x \circ (y \circ z)$

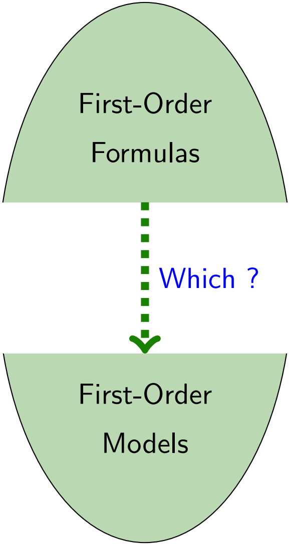
Let ϕ be Σ_G-formula.
Whenever $\models G \rightarrow \phi$, then ϕ is a theorem of group theory

Outline

Modeling with First-Order Logic

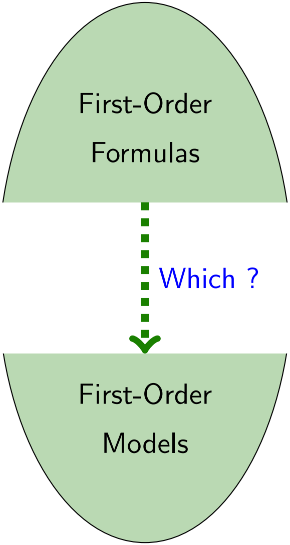


Modeling with First-Order Logic



Example (At least two elements)

Modeling with First-Order Logic

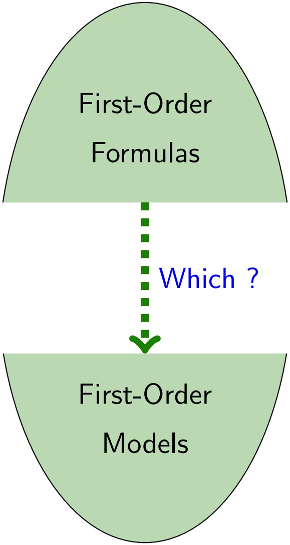


Example (At least two elements)

$\exists x;\ \exists y;\ !(x \doteq y)$

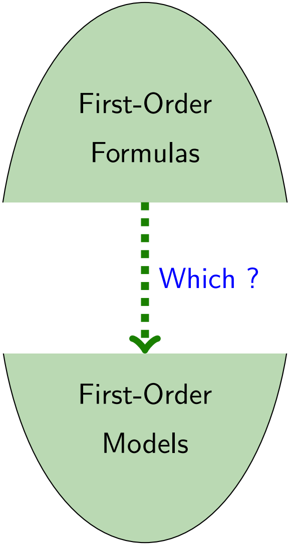
How to do this without built-in equality?

Modeling with First-Order Logic



Example (Strict partial order)

Modeling with First-Order Logic



Example (Strict partial order)

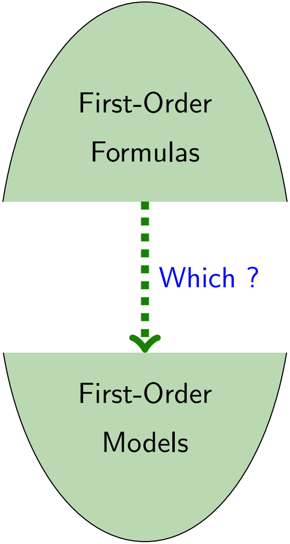
$PSym = \{< /2\}$

Irreflexivity $\forall x; !(x < x)$

Asymmetry $\forall x; \forall y; (x < y \text{ -> } !(y < x))$

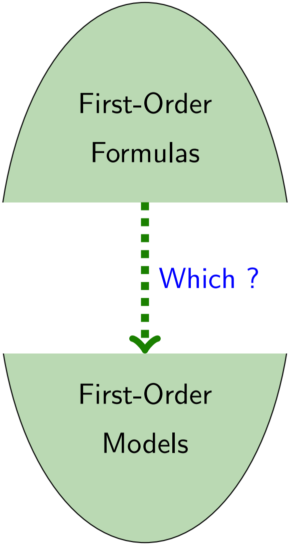
Transitivity $\forall x; \forall y; \forall z;$
$(x < y \,\&\, y < z \text{ -> } x < z)$

Modeling with First-Order Logic



Example (All models have infinite domain)

Modeling with First-Order Logic

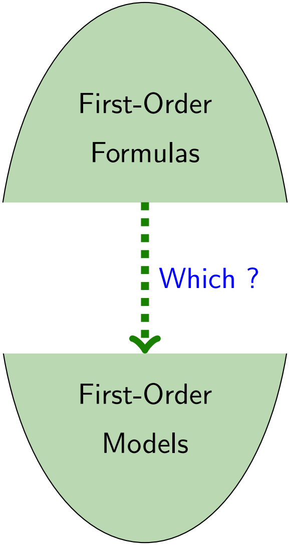


Example (All models have infinite domain)

Signature and axioms of irreflexive order plus

Existence Successor $\forall x;\, \exists y;\, x < y$

Modeling with First-Order Logic



Example (Abstract data types)

$$\text{FSym} = \{\ \text{Stack push}(\mathbf{int}, \text{Stack});$$
$$\mathbf{int}\ \text{pop}(\text{Stack});$$
$$\text{Stack nil};\ \}$$

$$\forall\, \mathbf{int}\ i;\ \forall\, \text{Stack}\ s;\ \text{pop}(\text{push}(i, s)) \doteq s$$

...

Why such a Complicated First-Order Semantics?

Why not take terms and properties at their "face value"?

Definition (Herbrand Model (untyped logic))

A first-order model where

- Domain $\mathcal{D}$ are all variable-free (i.e., ground) terms
- Each domain element is represented as a term
- Interpretation of function symbols is identity:
  $\mathcal{I}(f)(d_1, \dots, d_r) = f(d_1, \dots, d_r)$

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Major limitations of Herbrand models

- Too many different domain elements: $1 + 2$, $2 + 1$
- Natural to represent program locations as terms and domain elements as their values whose exact representation we don't know
- There are theoretical limitations as well

Outline

1. Formal Modeling
2. First-Order Logic
   - Signature
   - Terms
   - Formulas
3. Semantics
   - Domain
   - Model
   - Variable Assignment
   - Term Valuation
   - Formula Valuation
   - Semantic Notions
4. Untyped Logic
5. Modeling with FOL
6. **Summary**

Summary and Outlook

Summary

- First-order formulas defined over a signature of typed symbols
- Hierarchical OO type system with abstract and dynamic types
- Quantification over variables, no "free" variables in formulas
- Semantic domain like objects in a JAVA heap
- First-order model assigns semantic value to terms and formulas
- Semantic notions satisfiability and validity

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- Semantic notions satisfiability and validity

Semantic evaluation is not feasible in practice

- Infinite (uncountable) number of first-order models
- Evaluation of quantified formula may involve infinitely many cases
- Next goal: a syntactic calculus allowing mechanical validity checking

Background Literature

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