Logical Foundations & Cyber-Physical Systems

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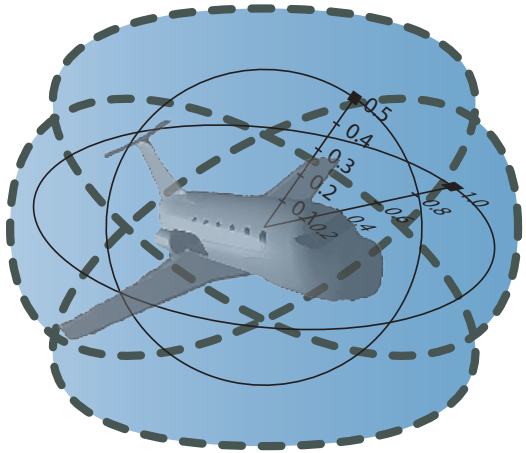
http://symbolaris.com/



Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
2. Dynamic Logic for Multi-Dynamical Systems
   - Syntax
   - Semantics
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Radical Invariants
5. Applications
6. Summary

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Can you trust a computer to control physics?

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Rationale

1. Safety guarantees require analytic foundations
2. Foundations revolutionized digital computer science & society
3. Need even stronger foundations when software reaches out into our physical world

Cyber-physical Systems

CPS combine cyber capabilities with physical capabilities to solve problems that neither part could solve alone.

> How can we provide people with cyber-physical systems they can bet their lives on?
> — Jeannette Wing

Benefits of Logical Foundations for V & V

Proving

Safety	Formalize system properties: What is "Safe"? "Reach goal"?
Models	Formalize system models, clarify behavior
Assumptions	Make assumptions explicit rather than silent
Constraints	Reveal invariants, switching conditions, operating conditions
Design	Invariants guide safe controller design
Constructive	Construct system models along with their proofs

Byproducts

Analysis	Determine design trade-offs & feasibility early
Synthesis	Turn high-level models into code & correctness monitors
Certificate	Proofs as artifacts for certification

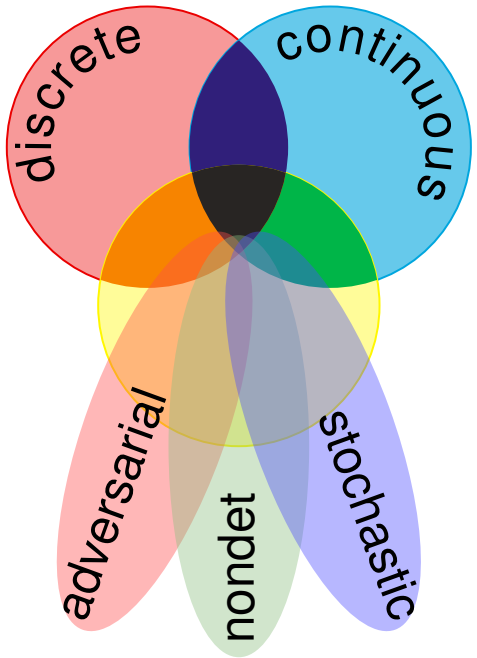
Tools

KeYmaera	Theorem prover for CPS

CPS are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combine multiple simple dynamical effects.

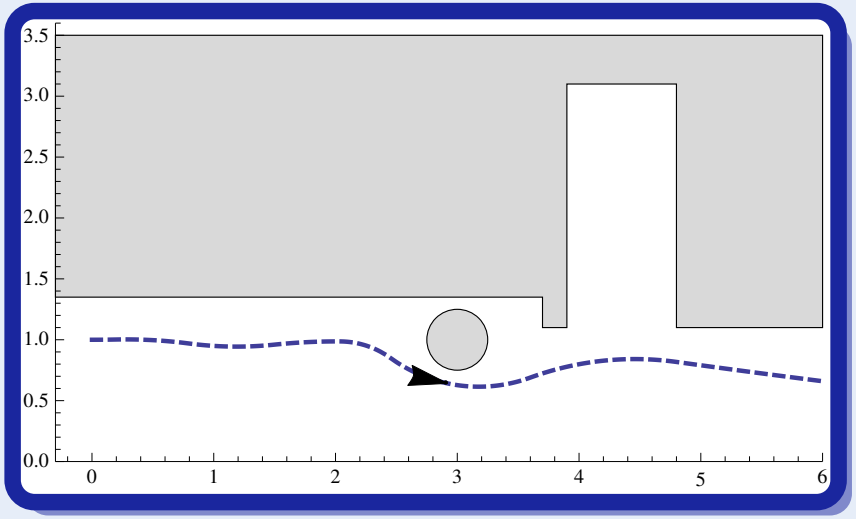
Tame Parts

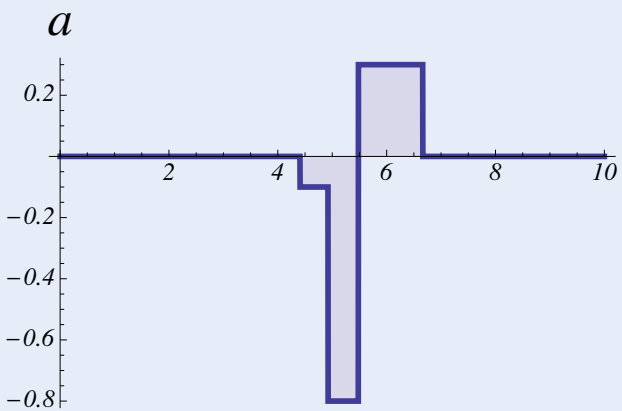
Exploiting compositionality tames CPS complexity.

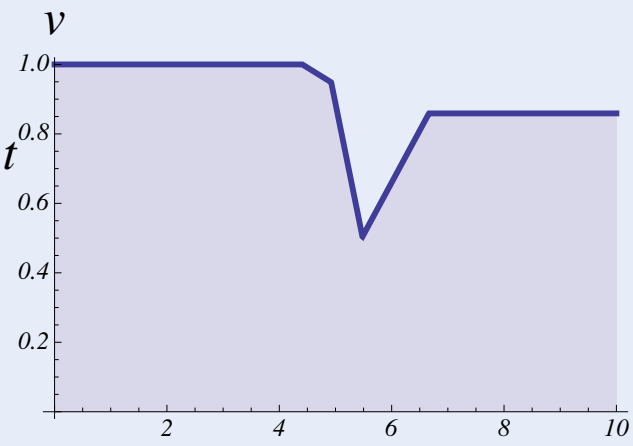
Challenge (Hybrid Systems)

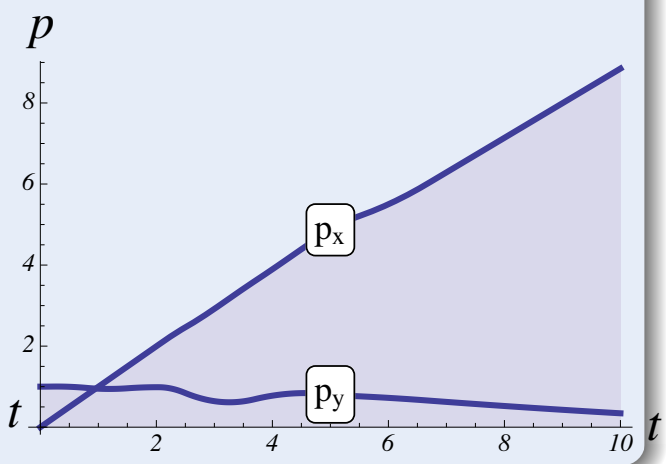
Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)









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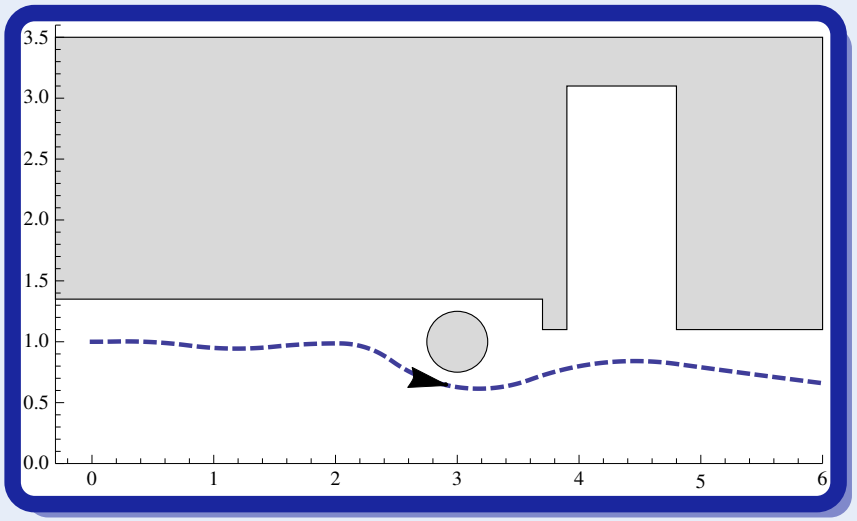

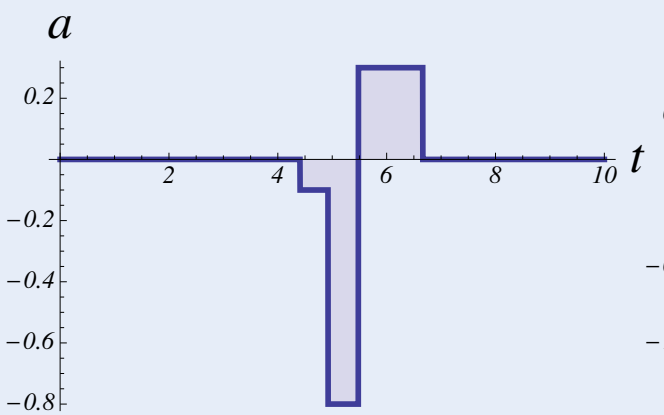

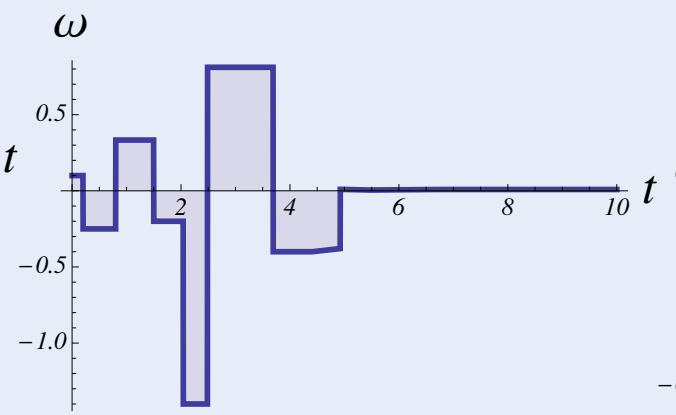

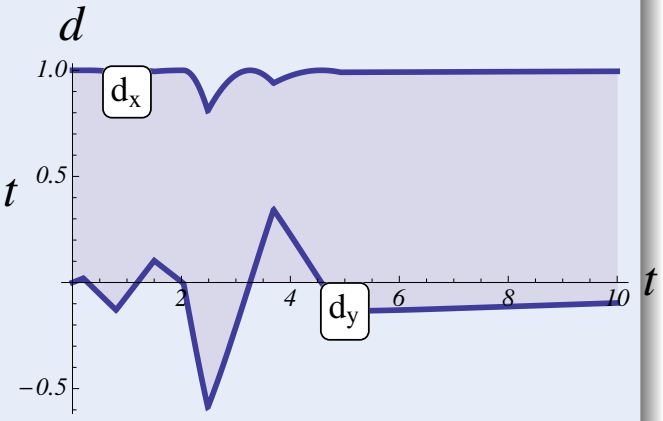

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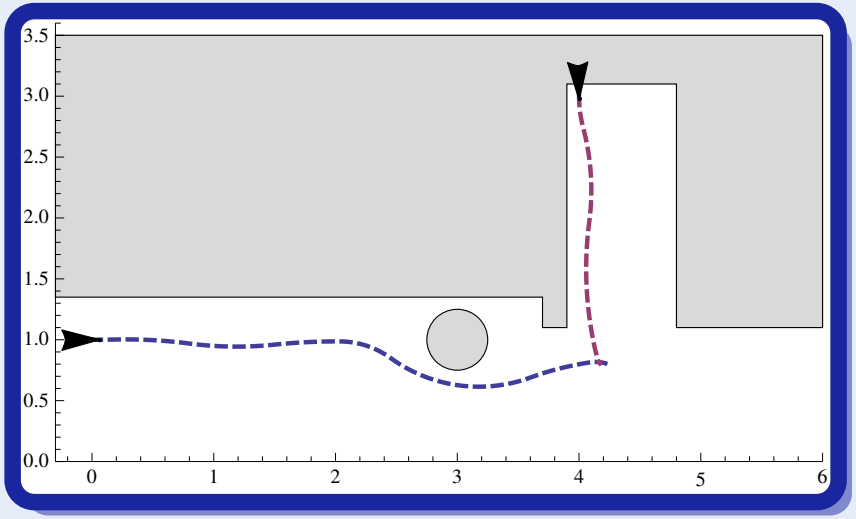

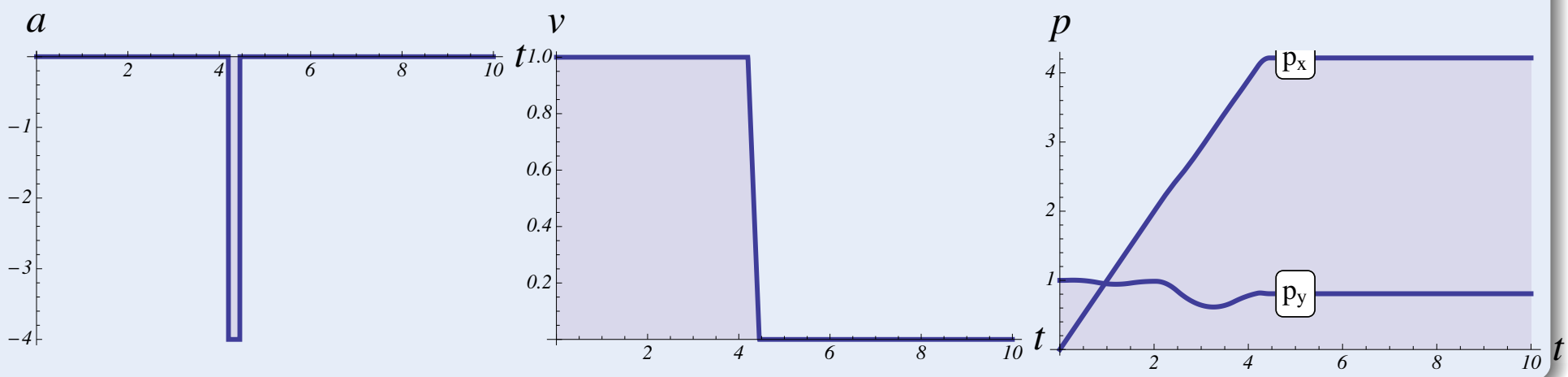

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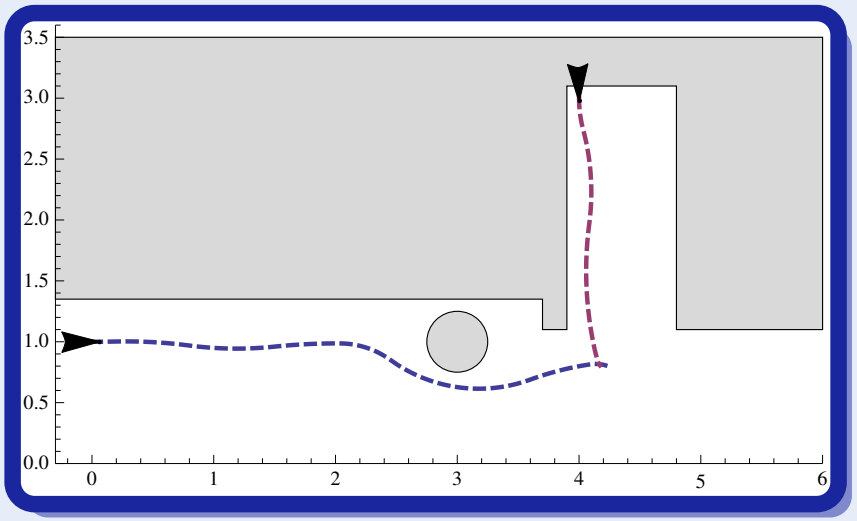
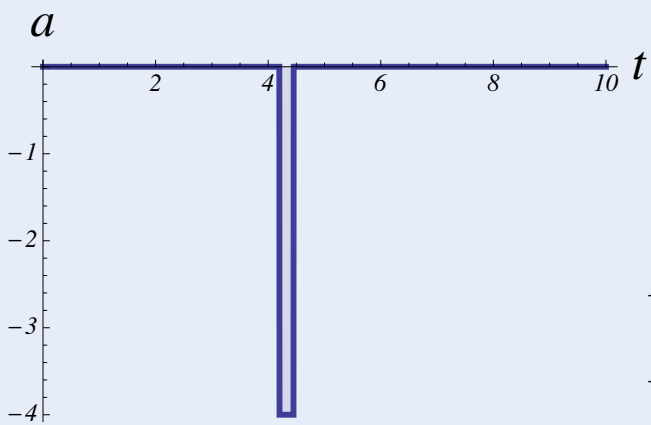
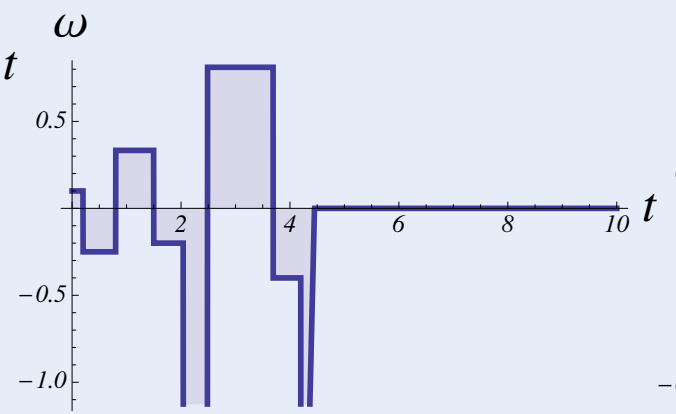
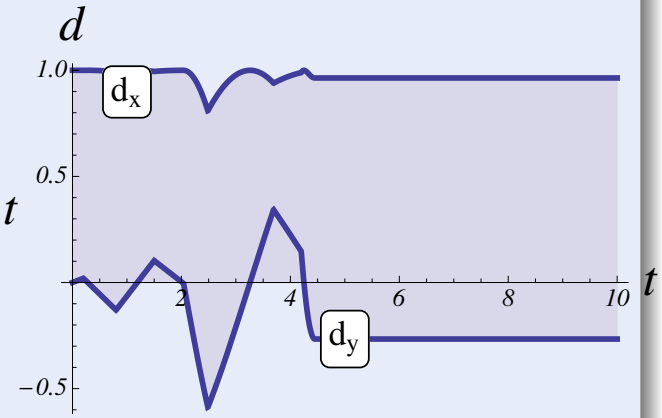
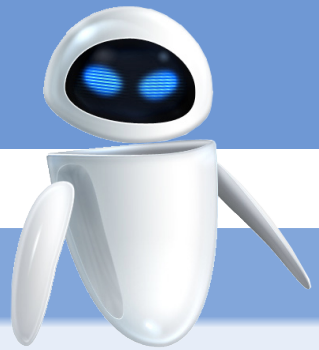






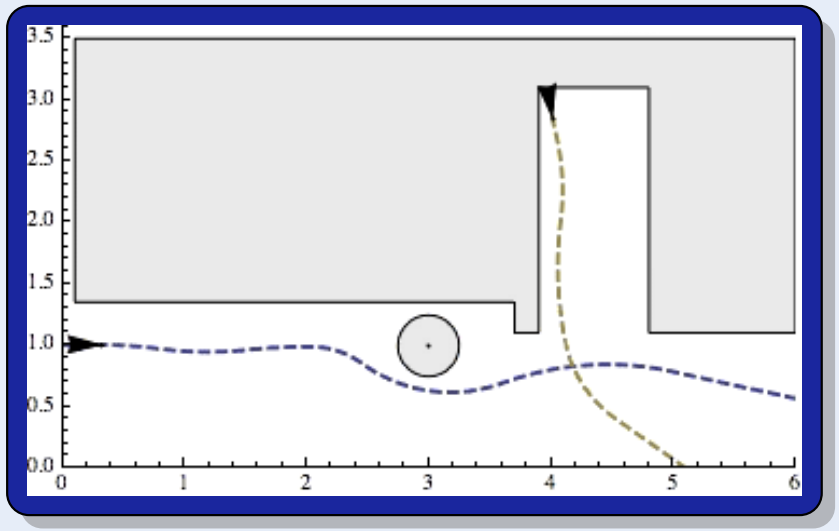


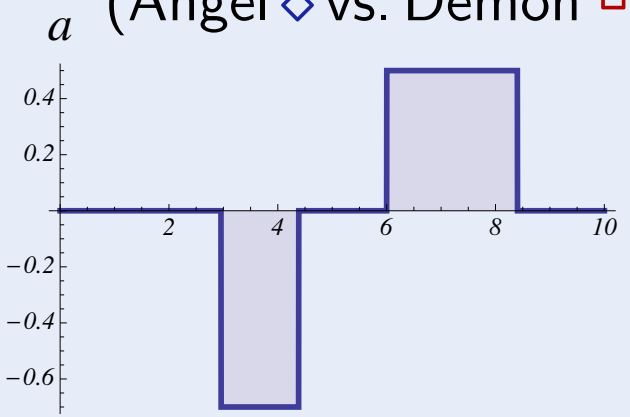


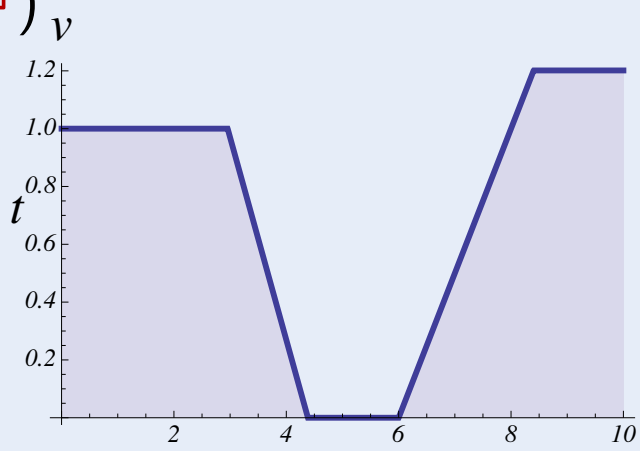
Challenge (Hybrid Games)

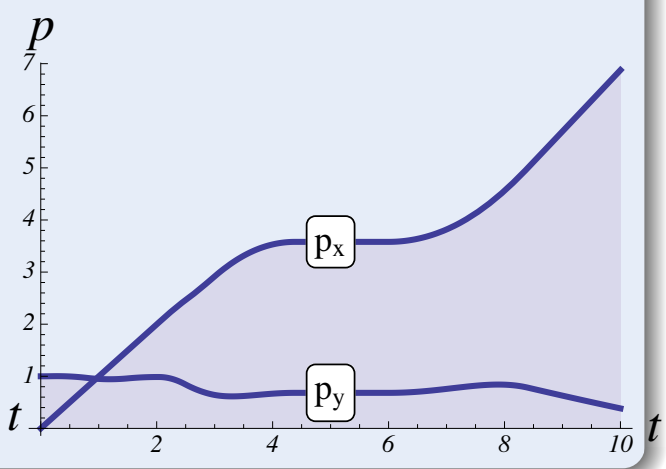
Game rules describing play evolution with

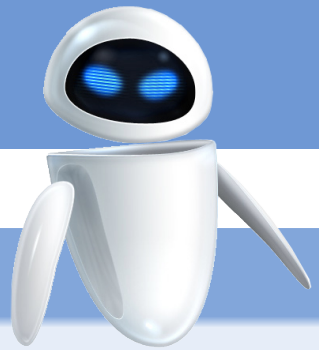
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)











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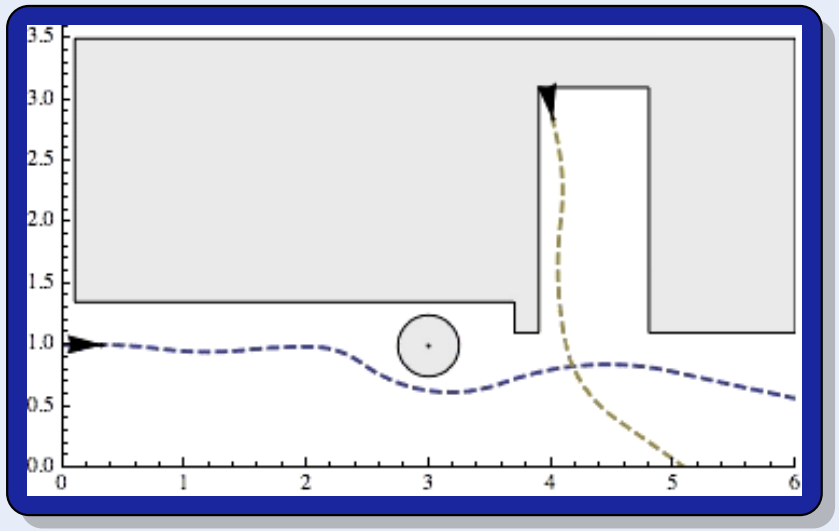

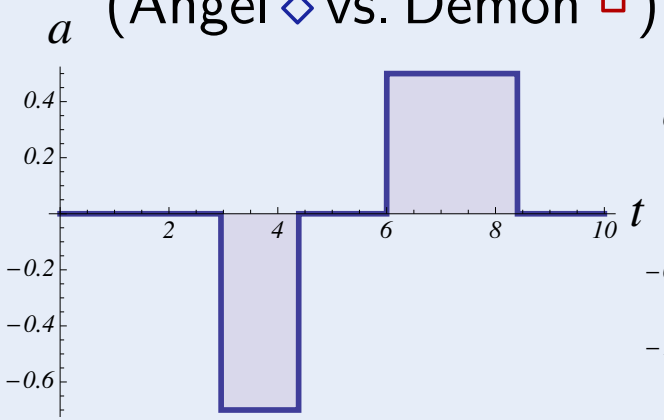

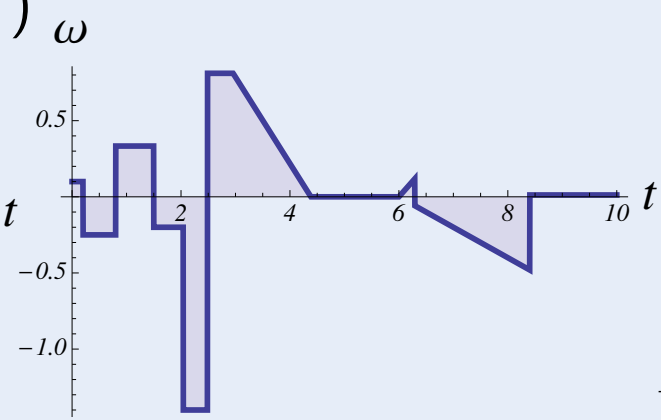

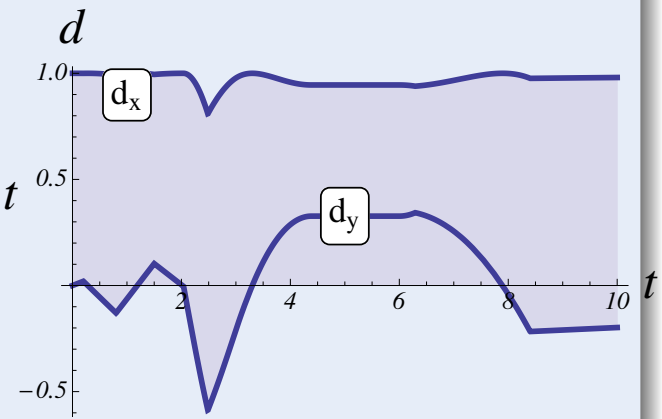

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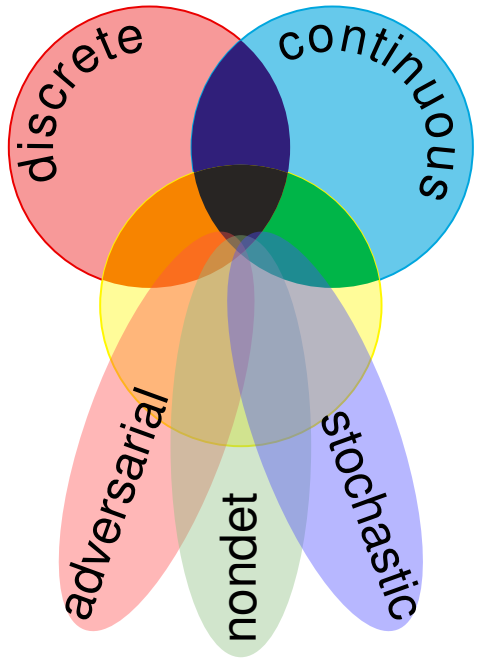
CPS are Multi-Dynamical Systems

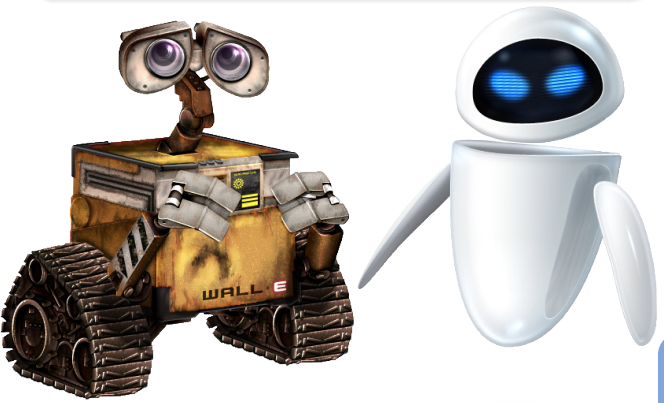
hybrid systems

HS = discrete + ODE



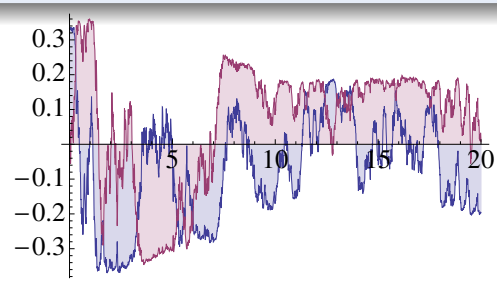
hybrid games

HG = HS + adversary



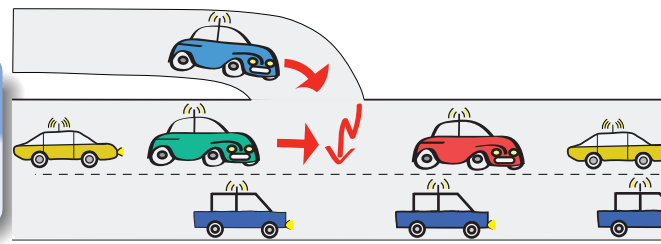
stochastic hybrid sys.

SHS = HS + stochastics

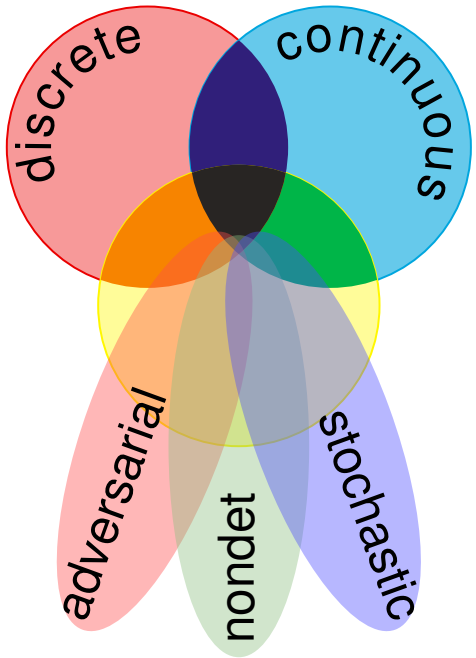


distributed hybrid sys.

DHS = HS + distributed



Family of Differential Dynamic Logics

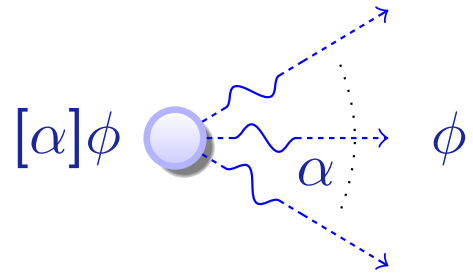


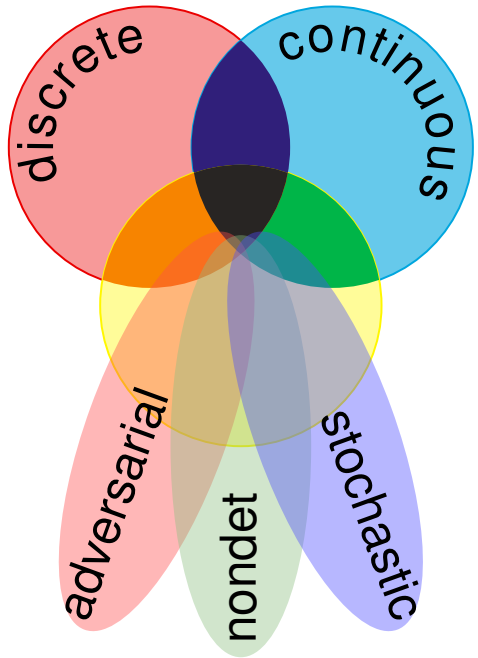
JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Family of Differential Dynamic Logics

differential dynamic logic

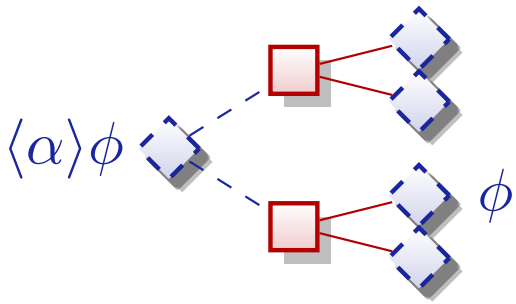
$$\mathsf{d}\mathcal{L} = \mathsf{DL} + \mathsf{HP}$$





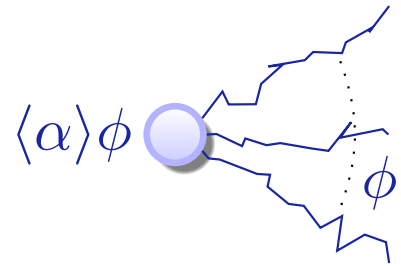
differential game logic

$$\mathsf{dG}\mathcal{L} = \mathsf{GL} + \mathsf{HG}$$



stochastic differential DL

$$\mathsf{Sd}\mathcal{L} = \mathsf{DL} + \mathsf{SHP}$$



quantified differential DL

$$\mathsf{Qd}\mathcal{L} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$$

JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Benefits

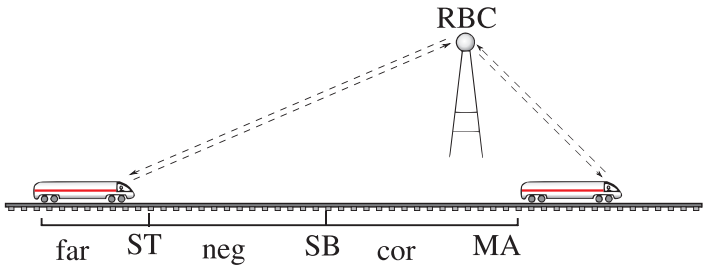
Logical foundations of cyber-physical systems

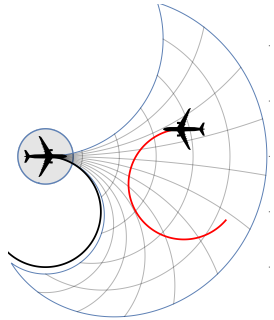
1. Multi-dynamical systems
2. Tame complexity by combinations of simple dynamics
3. Compositional programming language for CPS
4. Compositional logics and proof calculi
5. Proofs for differential equations
6. Elegant theory
7. Many useful applications
8. Education: undergrad course Foundations of CPS

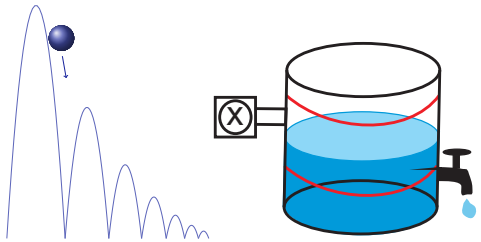
Basis for other technology

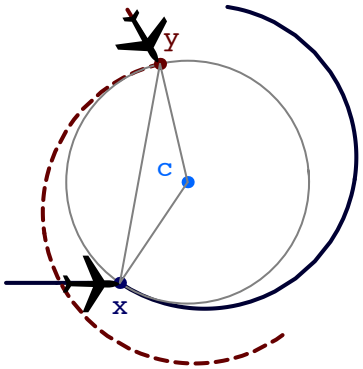
1. ModelPlex transfers CPS model $\rightsquigarrow$ CPS implementation safety RV'14
2. Proof-aware refactoring to co-evolve model + proof FM'14
3. Control envelope design ACC'12

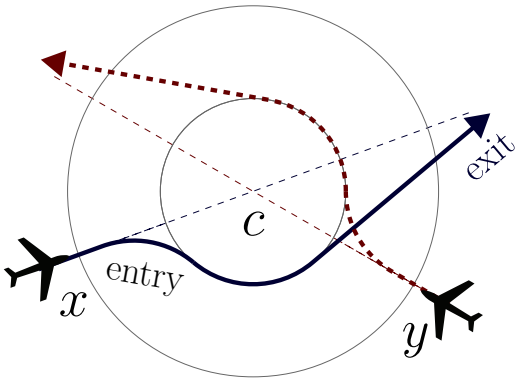
Successful CPS Proofs

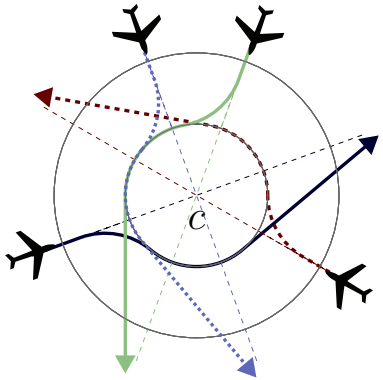


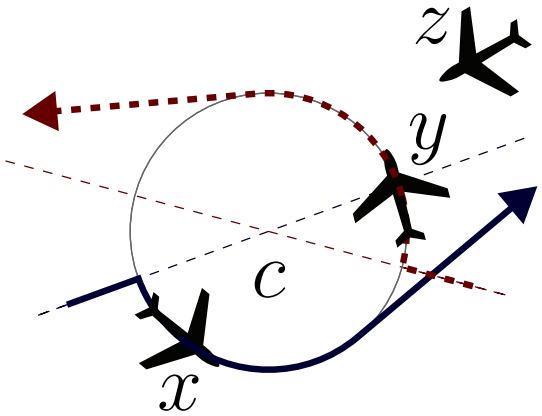


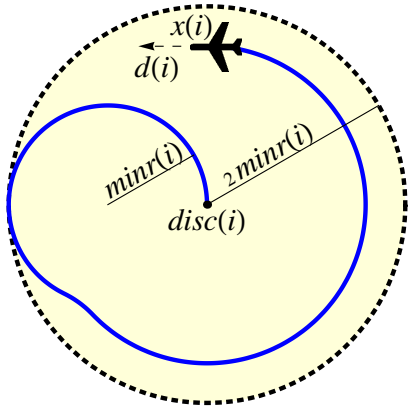


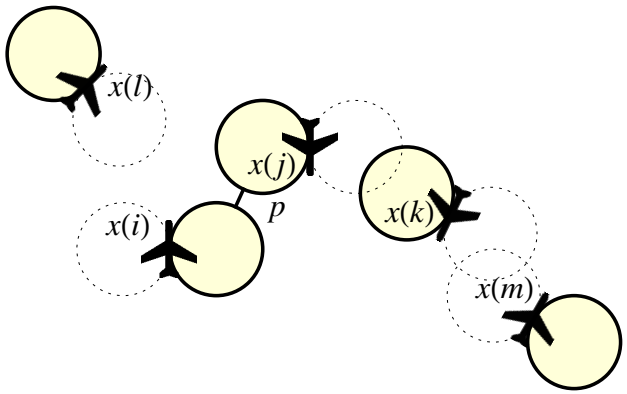






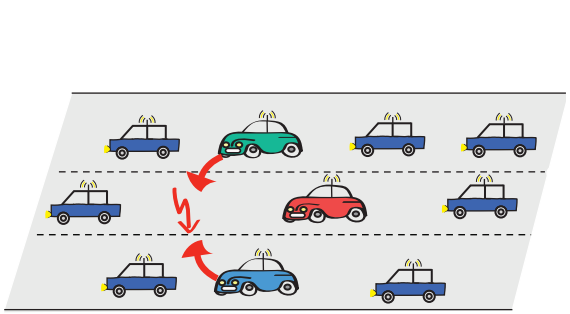


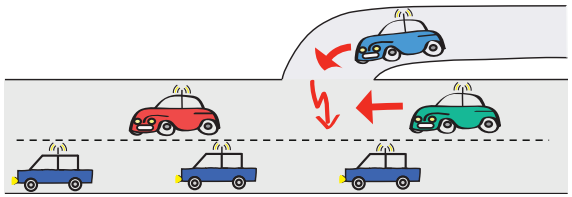


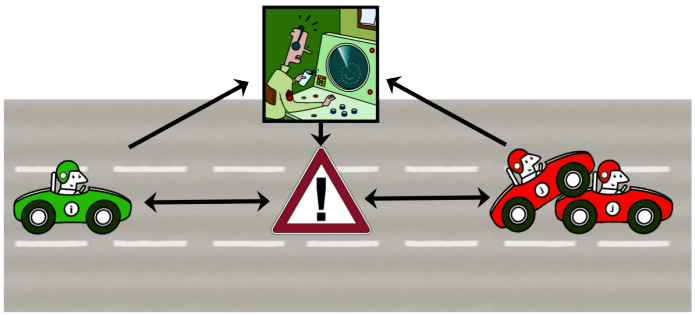


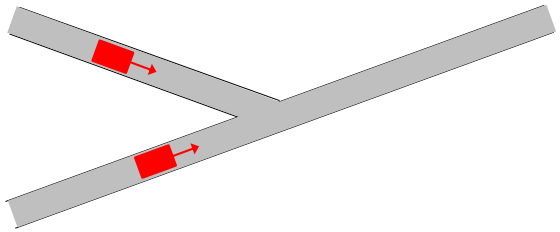
ICFEM'09,JAIS'14,CAV'08,FM'09,HSCC'11,HSCC'13

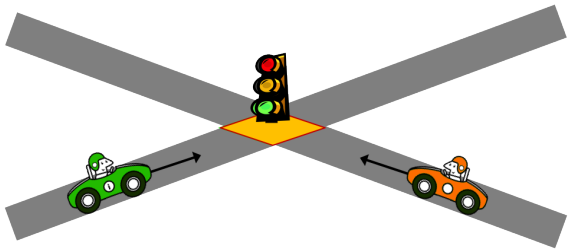
Successful CPS Proofs

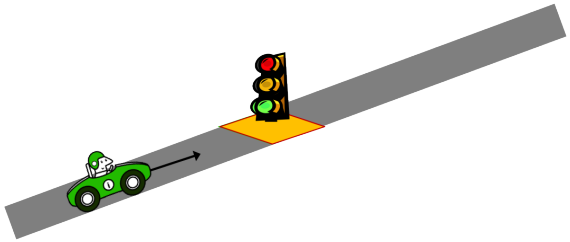


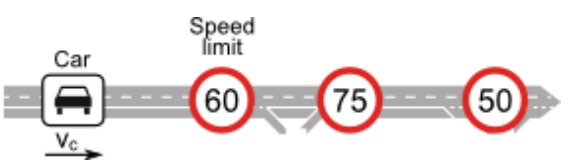


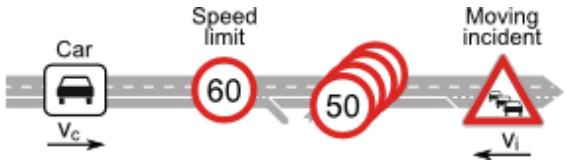


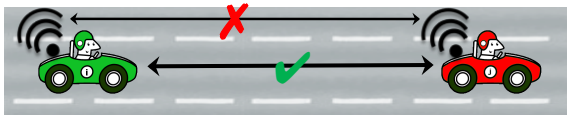


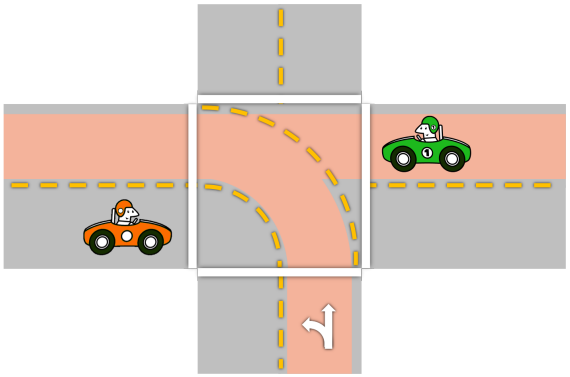


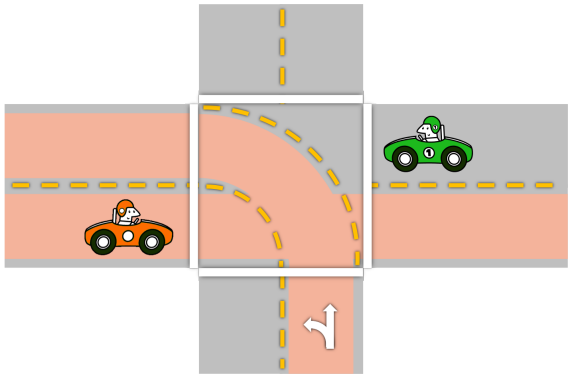


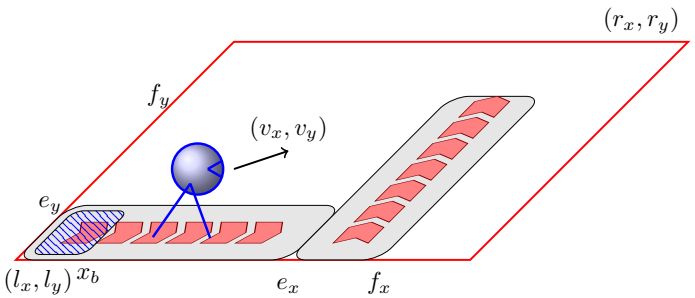






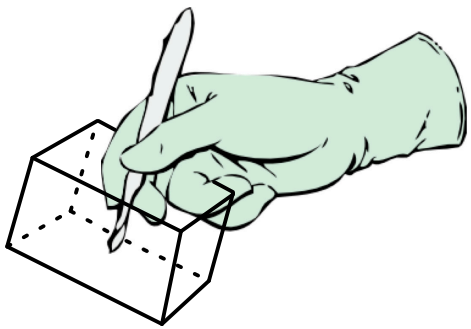


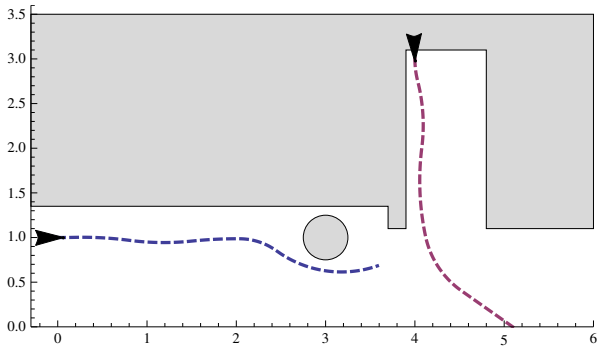


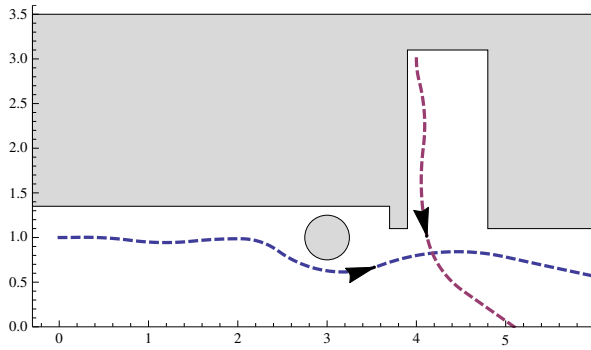


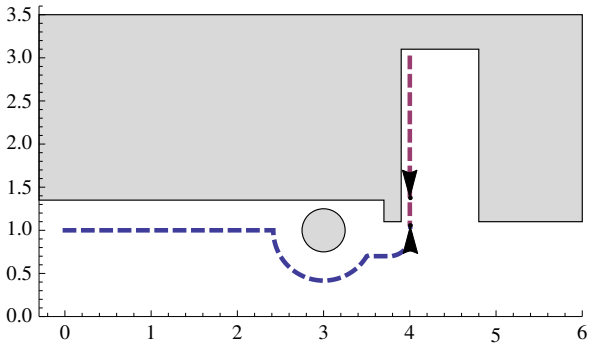
FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

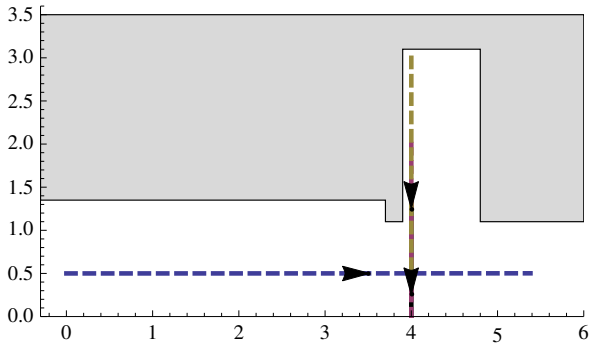
Successful CPS Proofs

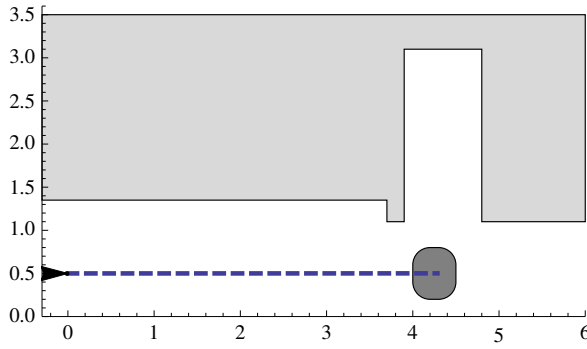


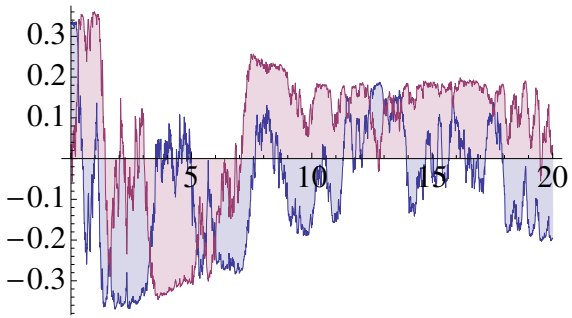


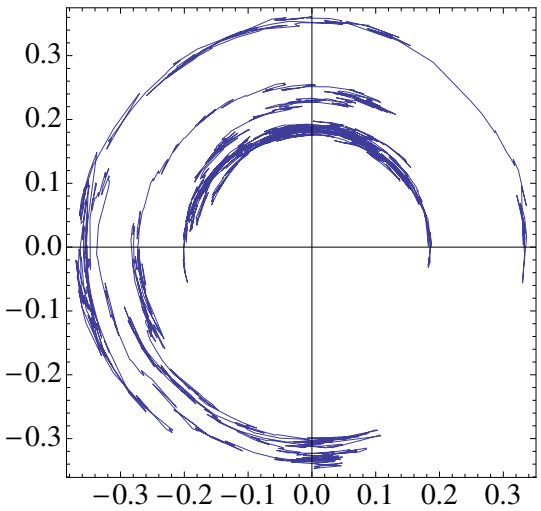


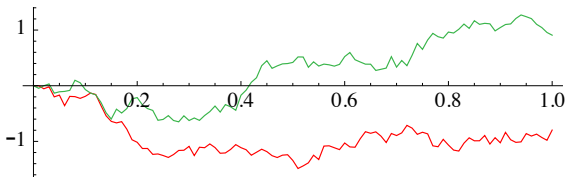






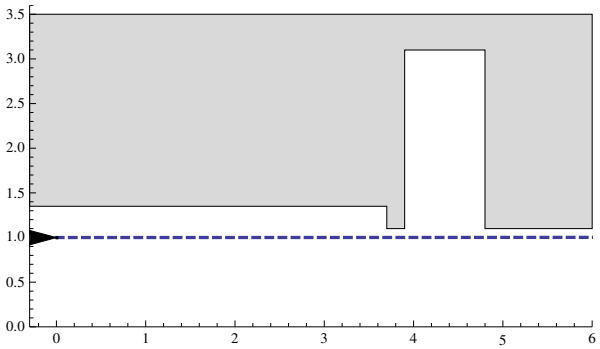
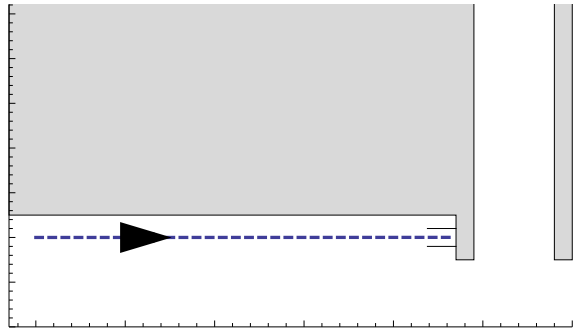
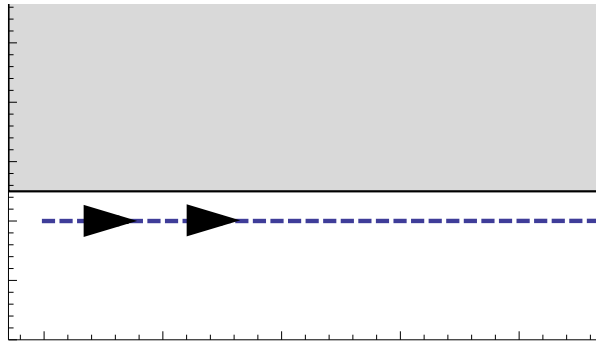
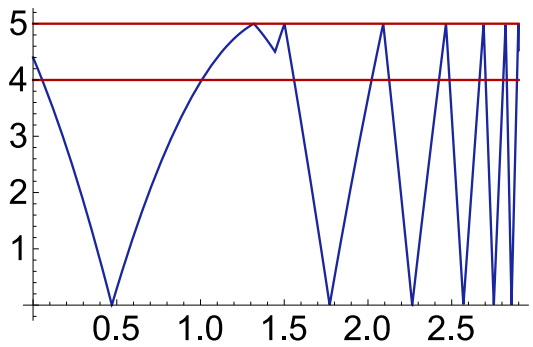
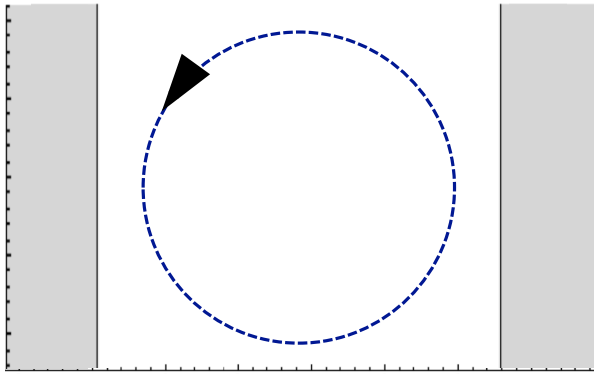
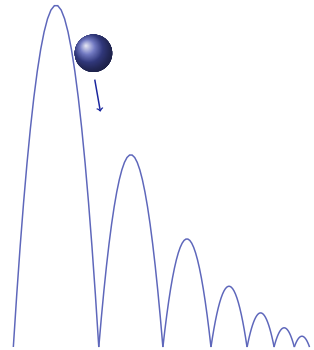
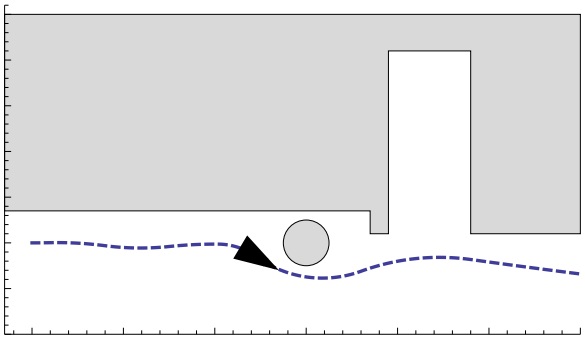
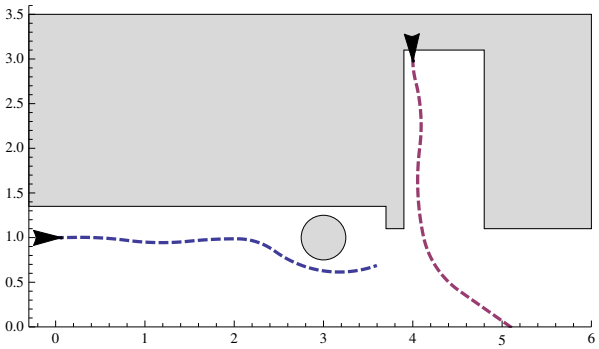
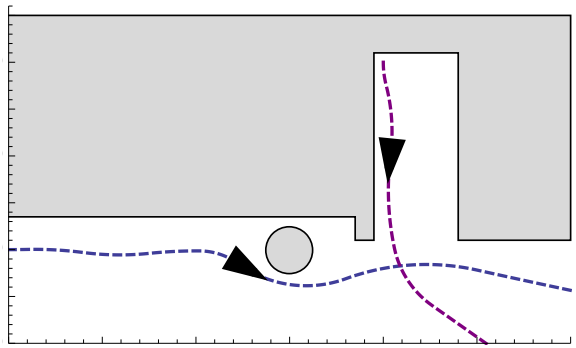






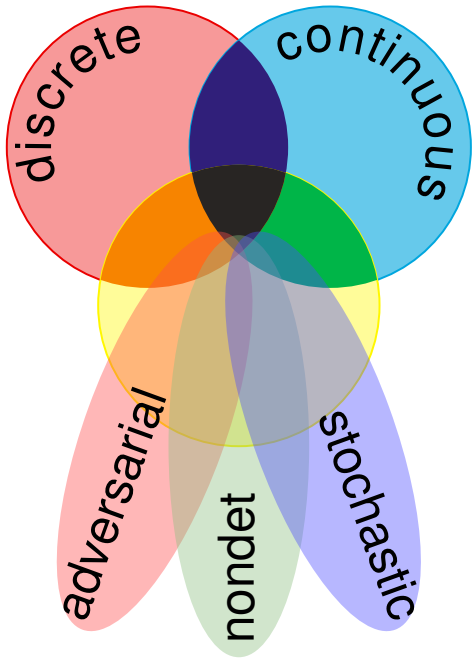
HSCC'13,RSS'13,CADE'12

Successful CPS Proofs By Undergrads

15-424/624 *Foundations of Cyber-Physical Systems* students

Family of Differential Dynamic Logics

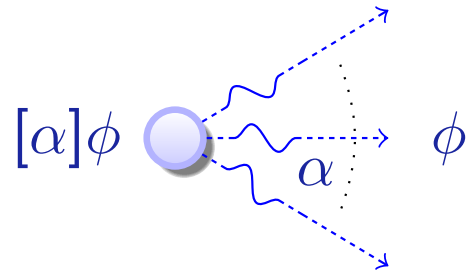


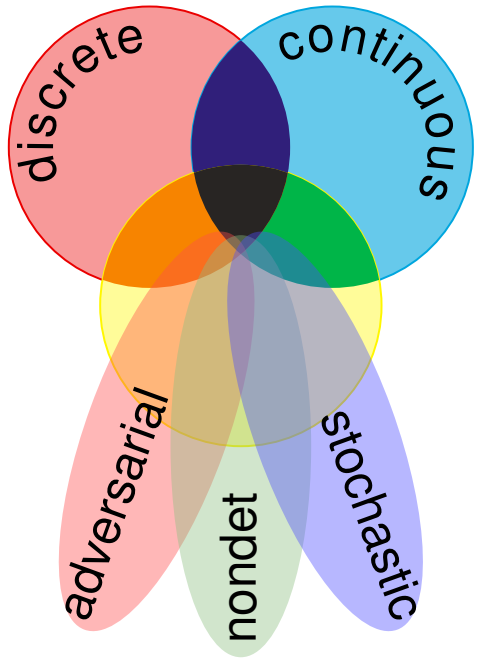
JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Family of Differential Dynamic Logics

differential dynamic logic

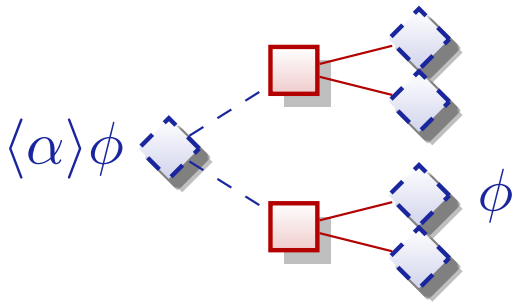
$$\mathsf{d}\mathcal{L} = \mathsf{DL} + \mathsf{HP}$$





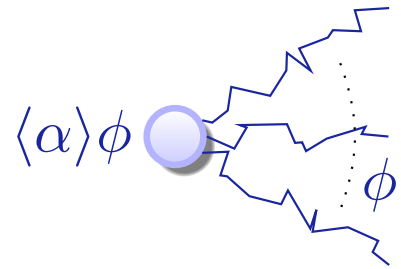
differential game logic

$$\mathsf{dG}\mathcal{L} = \mathsf{GL} + \mathsf{HG}$$



stochastic differential DL

$$\mathsf{Sd}\mathcal{L} = \mathsf{DL} + \mathsf{SHP}$$



quantified differential DL

$$\mathsf{Qd}\mathcal{L} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$$

JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Differential Dynamic Logic d$\mathcal{L}$: Syntax

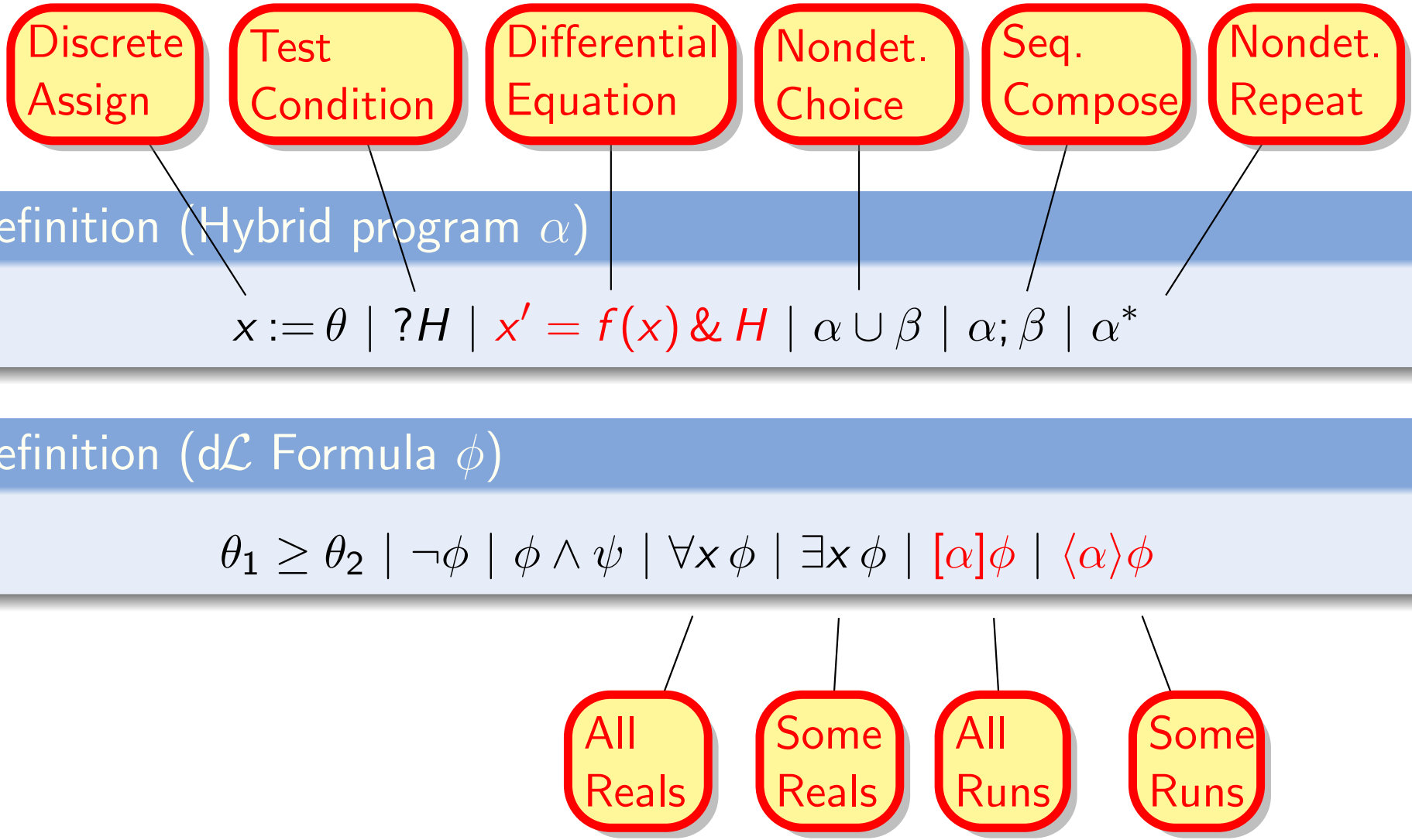
Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x)\,\&\,H \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Differential Dynamic Logic dℒ: Syntax



Differential Dynamic Logic dℒ: Semantics

Definition (Hybrid program α) (Reachability Semantics)

$$\begin{aligned}
\rho(x := \theta) &= \{(v,w) : w = v \text{ except } [\![x]\!]_w = [\![\theta]\!]_v\} \\
\rho(?H) &= \{(v,v) : v \models H\} \\
\rho(x' = f(x)) &= \{(\varphi(0),\varphi(r)) : \varphi \models x' = f(x) \text{ for some duration } r\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha;\beta) &= \rho(\beta) \circ \rho(\alpha) \\
\rho(\alpha^*) &= \bigcup_{n\in\mathbb{N}} \rho(\alpha^n)
\end{aligned}$$

Definition (dℒ Formula ϕ) (Modal Semantics)

$v \models \phi \wedge \psi$ iff $v \models \phi$ and $v \models \psi$

$v \models [\alpha]\phi$ iff $w \models \phi$ for all w with $(v,w) \in \rho(\alpha)$

$v \models \langle\alpha\rangle\phi$ iff $w \models \phi$ for some w with $(v,w) \in \rho(\alpha)$

$v \models \forall x\, \phi$ iff $w \models \phi$ for all w that agree with v except for x

$v \models \exists x\, \phi$ iff $w \models \phi$ for some w that agrees with v except for x

Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
2. Dynamic Logic for Multi-Dynamical Systems
   - Syntax
   - Semantics
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   - Differential Radical Invariants
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6. Summary

Differential Dynamic Logic: Axiomatization

equations of truth

$[:=]$ $[x := \theta]\phi(x) \leftrightarrow \phi(\theta)$

$[?]$ $[?H]\phi \leftrightarrow (H \rightarrow \phi)$

$[']$ $[x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\, [x := y(t)]\phi$ $\qquad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$

$[;]$ $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$

$[^*]$ $[\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi$

K $[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$

I $[\alpha^*](\phi \rightarrow [\alpha]\phi) \rightarrow (\phi \rightarrow [\alpha^*]\phi)$

C $[\alpha^*]\forall v{>}0\, (\varphi(v) \rightarrow \langle\alpha\rangle\varphi(v-1)) \rightarrow \forall v\, (\varphi(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\, \varphi(v))$

LICS'12

Differential Dynamic Logic: Axiomatization

equations of truth

$$\mathsf{G}\ \frac{\phi}{[\alpha]\phi}$$

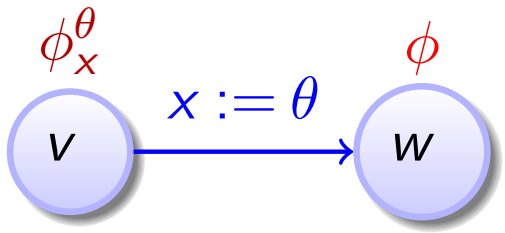
$$\mathsf{MP}\ \frac{\phi \to \psi \quad \phi}{\psi}$$

$$\forall\ \frac{\phi}{\forall x\, \phi}$$

LICS'12

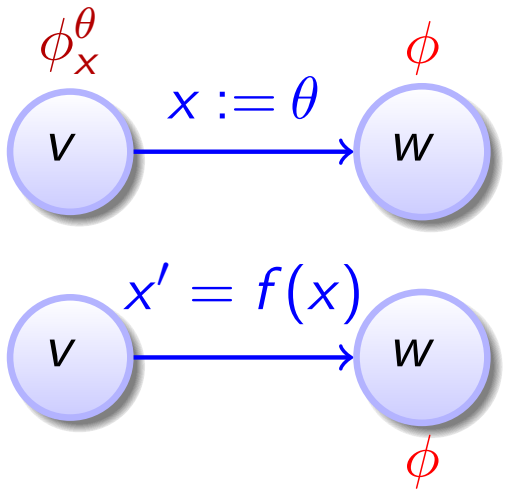
Proofs for Hybrid Systems

$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$



$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$

$$\frac{\forall t{\geq}0\,[x := y_x(t)]\phi}{[x' = f(x)]\phi}$$



Proofs for Hybrid Systems

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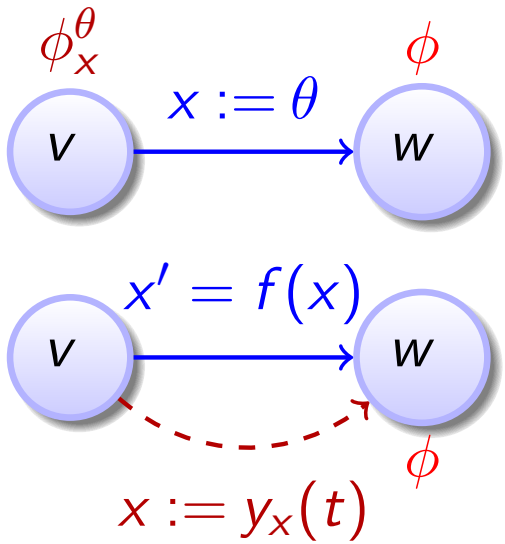
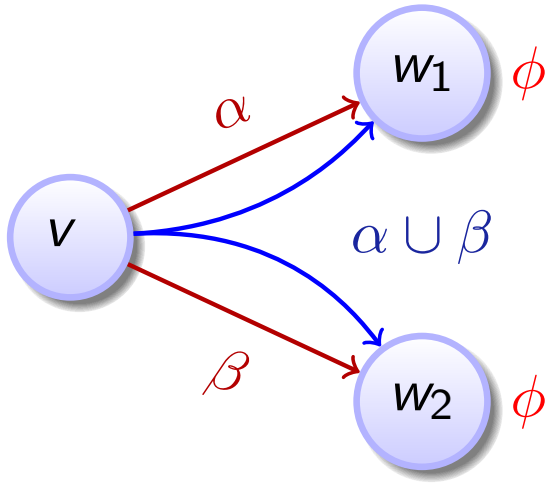


Proofs for Hybrid Systems

compositional semantics $\Rightarrow$ compositional rules!

$$\frac{[\alpha]\phi \wedge [\beta]\phi}{[\alpha \cup \beta]\phi}$$



Proofs for Hybrid Systems

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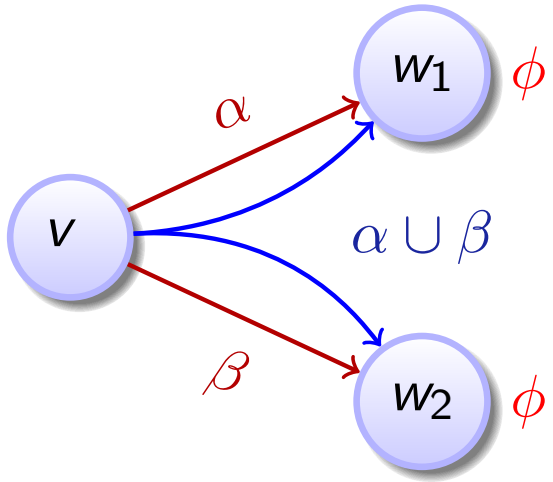
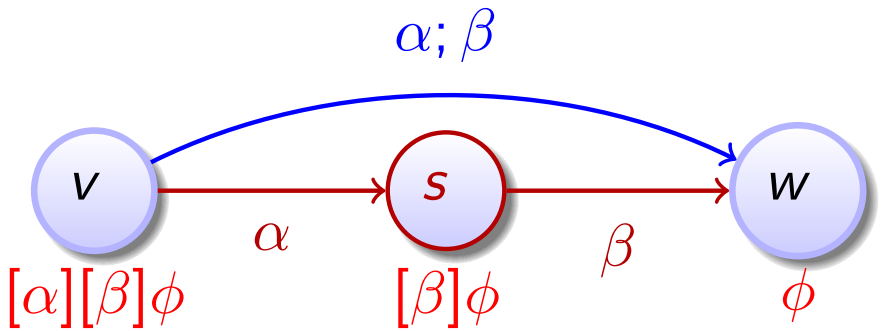


$$\frac{[\alpha][\beta]\phi}{[\alpha;\beta]\phi}$$



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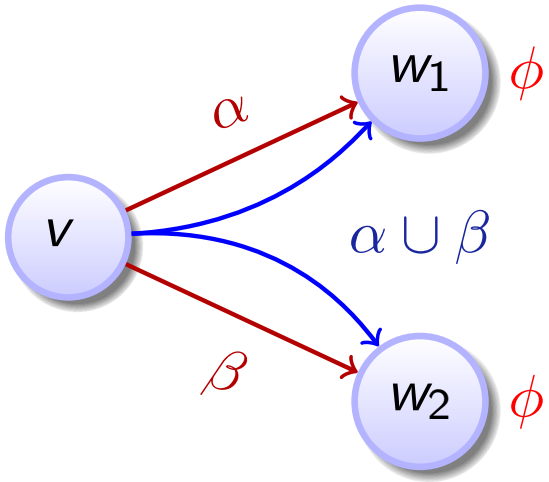
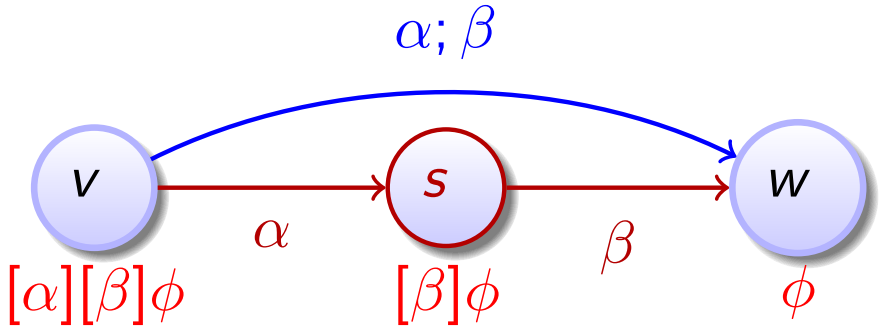
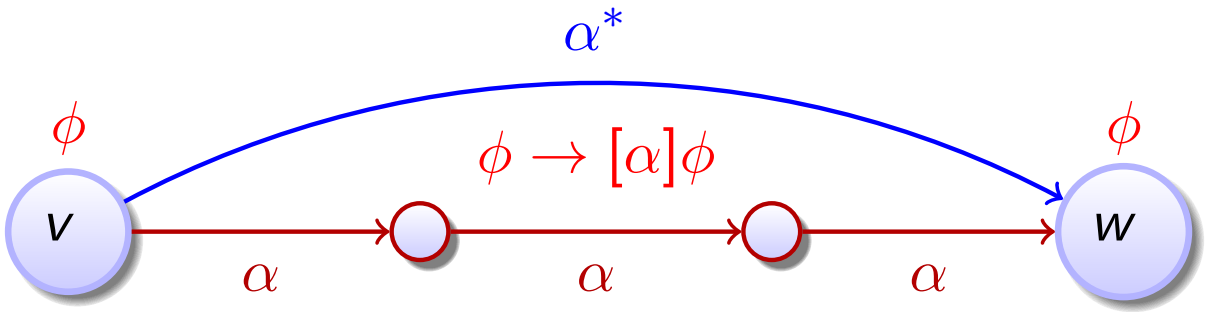


$$\frac{[\alpha][\beta]\phi}{[\alpha;\beta]\phi}$$



$$\frac{\phi \quad (\phi \rightarrow [\alpha]\phi)}{[\alpha^*]\phi}$$



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Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (J.Autom.Reas. 2008, LICS'12)

$d\mathcal{L}$ calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations **or** *discrete dynamics.* ▸ Proof 25pp

Corollary (Complete Proof-theoretical Alignment & Bridging)

proving continuous = proving hybrid = proving discrete

JAutomReas'08,LICS'12

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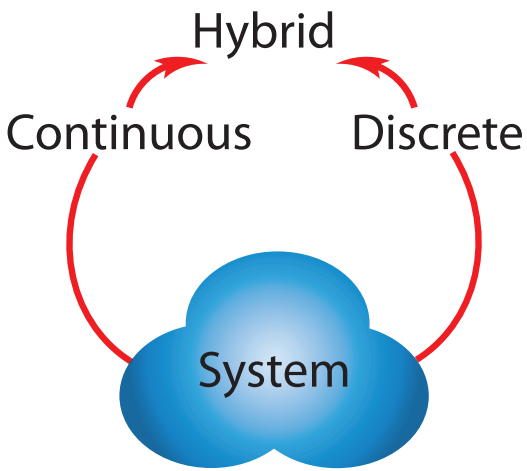

JAutomReas'08,LICS'12

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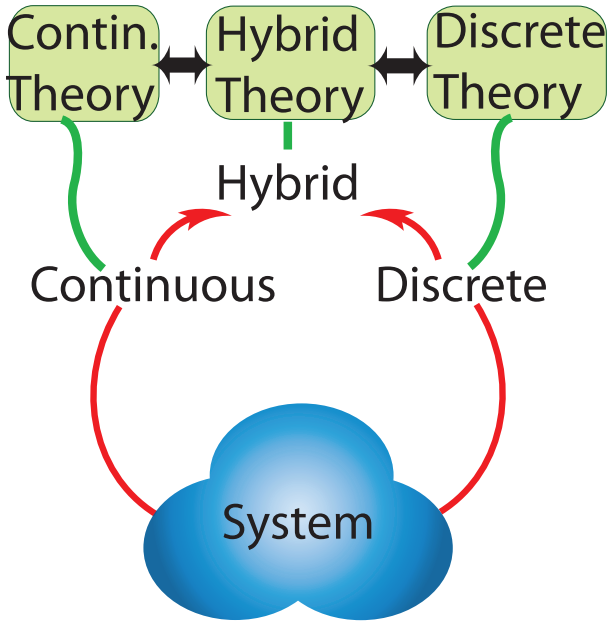
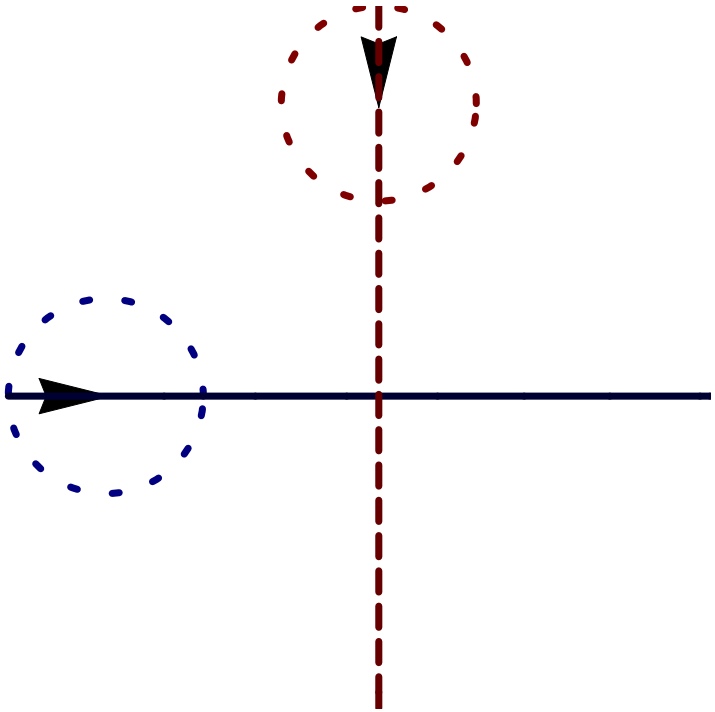
Corollary (Complete Proof-theoretical Alignment & Bridging)

proving continuous = proving hybrid = proving discrete

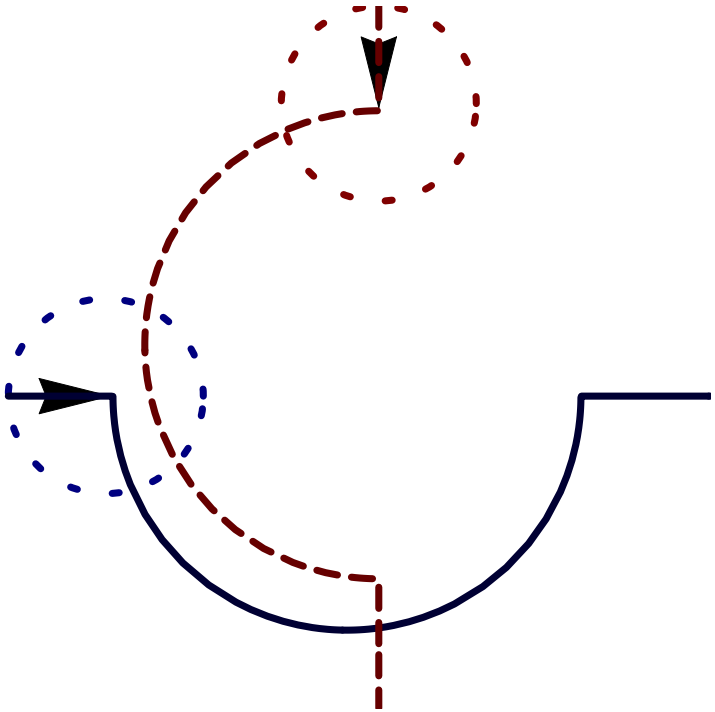


JAutomReas'08,LICS'12

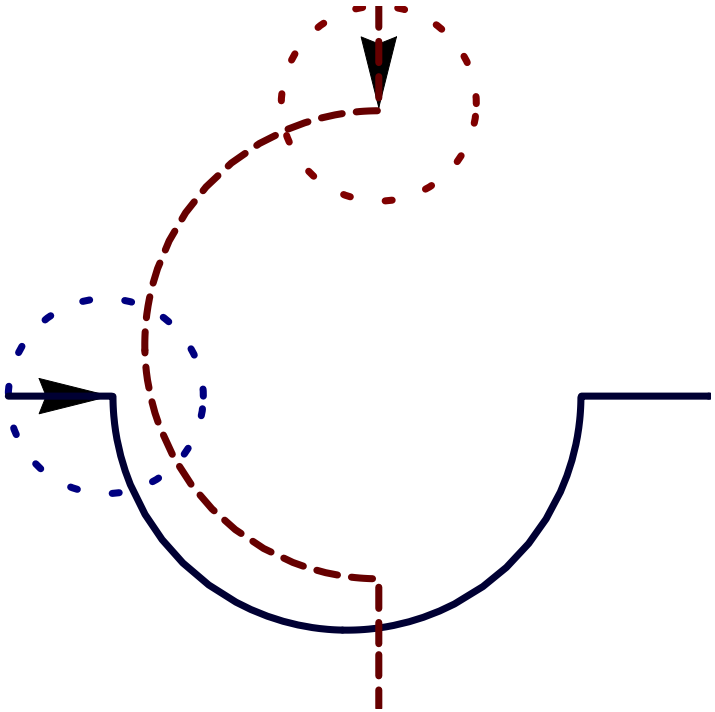
Air Traffic Control



Air Traffic Control



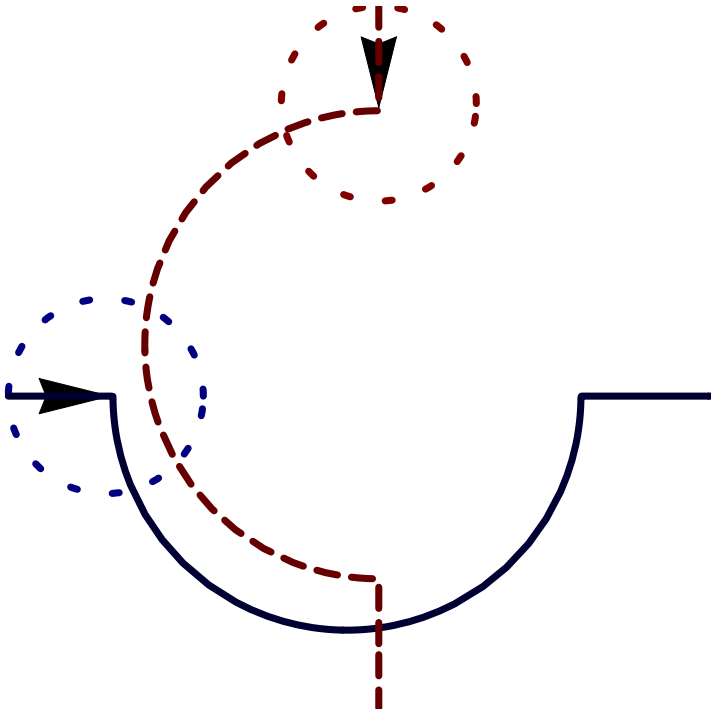
Air Traffic Control

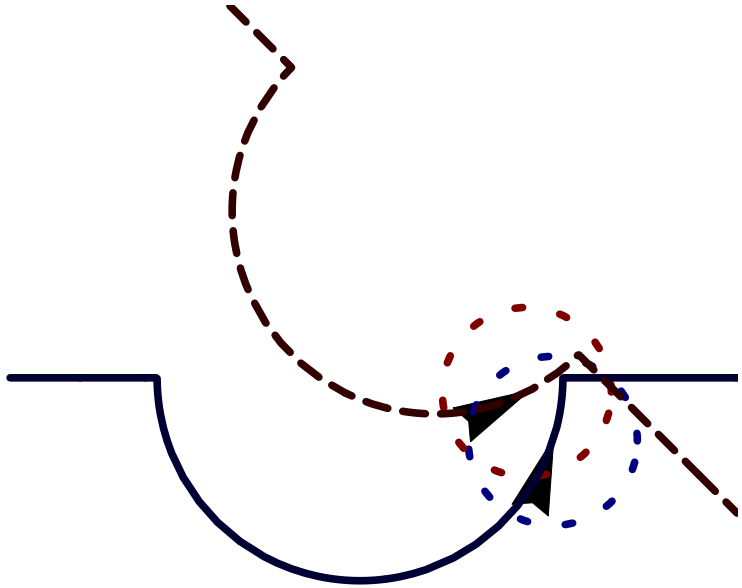


Verification?

looks correct

Air Traffic Control

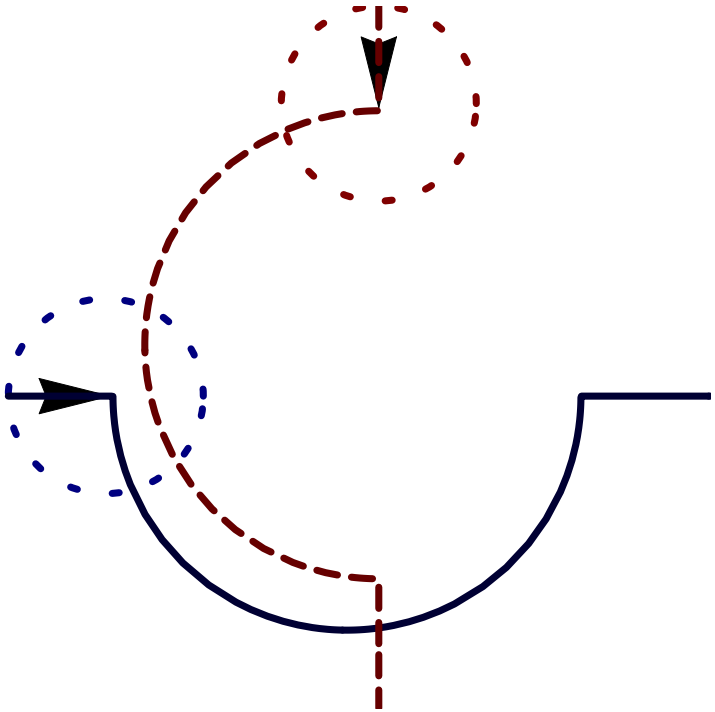


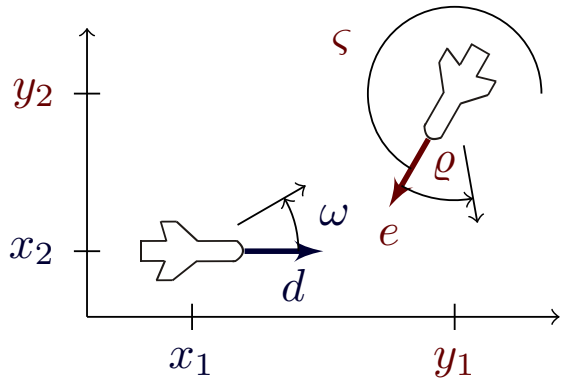


Verification?

looks correct NO!

Air Traffic Control



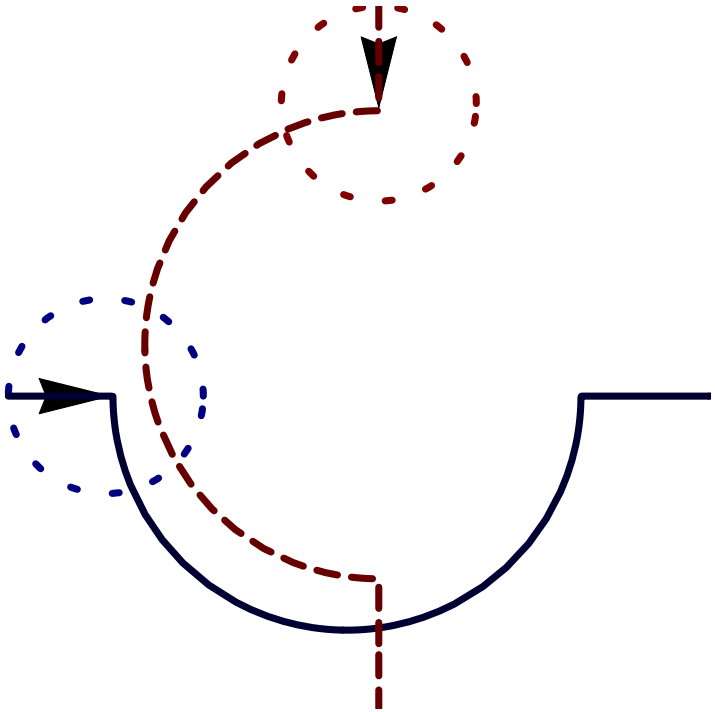


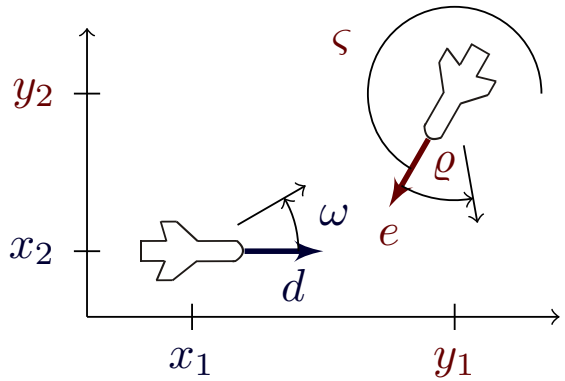
$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Verification?

looks correct NO!

Air Traffic Control



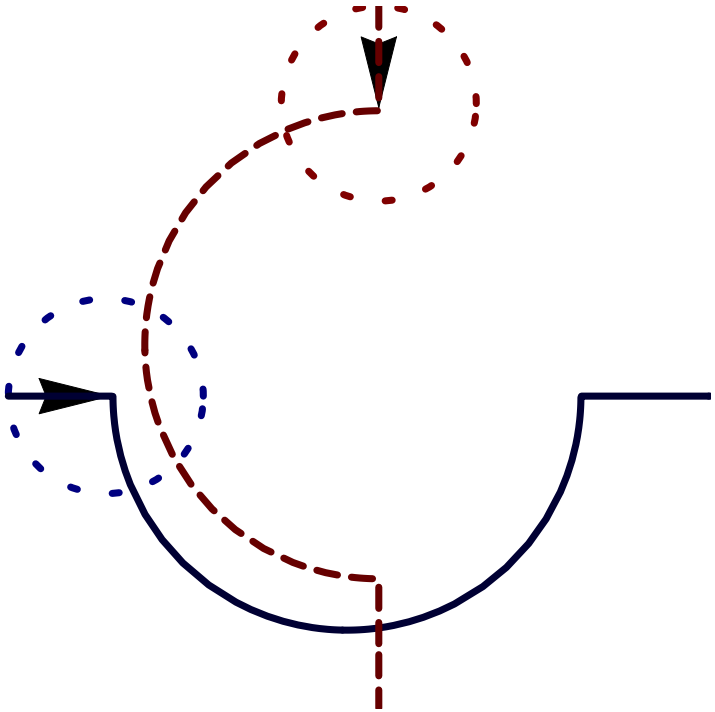


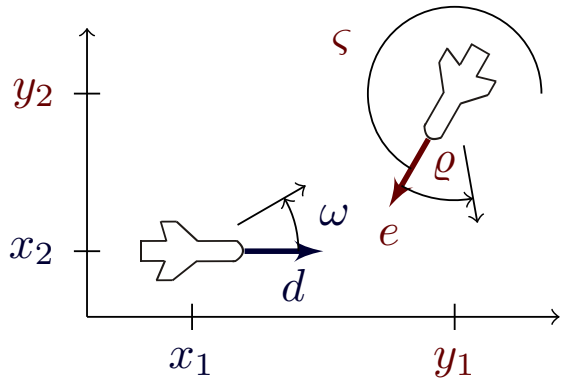
$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$

Air Traffic Control





$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$\forall t \geq 0 \quad \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$

```
\forall R ts2.
  (   0 <= ts2 & ts2 <= t2_0
   ->      (  (om_1)^-1
              * (omb_1)^-1
              * (  om_1 * omb_1 * x1 * Cos(om_1 * ts2)
                 + om_1 * v2 * Cos(om_1 * ts2) * (1 + -1 * (Cos(u))^2)^(1 / 2)
                 + -1 * omb_1 * v1 * Sin(om_1 * ts2)
                 + om_1 * omb_1 * x2 * Sin(om_1 * ts2)
                 + om_1 * v2 * Cos(u) * Sin(om_1 * ts2)
                 + -1 * om_1 * v2 * Cos(omb_1 * ts2) * Cos(u) * Sin(om_1 * ts2)
                 + om_1 * v2 * Cos(om_1 * ts2) * Cos(u) * Sin(omb_1 * ts2)
                 + om_1 * v2 * Cos(om_1 * ts2) * Cos(omb_1 * ts2) * Sin(u)
                 + om_1 * v2 * Sin(om_1 * ts2) * Sin(omb_1 * ts2) * Sin(u)))
           ^2
         +   (  (om_1)^-1
              * (omb_1)^-1
              * (  -1 * omb_1 * v1 * Cos(om_1 * ts2)
                 + om_1 * omb_1 * x2 * Cos(om_1 * ts2)
                 + omb_1 * v1 * (Cos(om_1 * ts2))^2
                 + om_1 * v2 * Cos(om_1 * ts2) * Cos(u)
                 + -1 * om_1 * v2 * Cos(om_1 * ts2) * Cos(omb_1 * ts2) * Cos(u)
                 + -1 * om_1 * omb_1 * x1 * Sin(om_1 * ts2)
                 +   -1
                   * om_1
                   * v2
                   * (1 + -1 * (Cos(u))^2)^(1 / 2)
                   * Sin(om_1 * ts2)
                 + omb_1 * v1 * (Sin(om_1 * ts2))^2
                 + -1 * om_1 * v2 * Cos(u) * Sin(om_1 * ts2) * Sin(omb_1 * ts2)
                 + -1 * om_1 * v2 * Cos(omb_1 * ts2) * Sin(om_1 * ts2) * Sin(u)
                 + om_1 * v2 * Cos(om_1 * ts2) * Sin(omb_1 * ts2) * Sin(u)))
           ^2
         >= (p)^2),
  t2_0 >= 0,
  x1^2 + x2^2 >= (p)^2
==>
```

```
\forall R t7.
  (   t7 >= 0
   ->      (  (om_3)^-1
            * (    om_3
                 * (  (om_1)^-1
                    * (omb_1)^-1
                    * (  om_1 * omb_1 * x1 * Cos(om_1 * t2_0)
                       +   om_1
                         * v2
                         * Cos(om_1 * t2_0)
                         * (1 + -1 * (Cos(u))^2)^(1 / 2)
                       + -1 * omb_1 * v1 * Sin(om_1 * t2_0)
                       + om_1 * omb_1 * x2 * Sin(om_1 * t2_0)
                       + om_1 * v2 * Cos(u) * Sin(om_1 * t2_0)
                       +   -1
                         * om_1
                         * v2
                         * Cos(omb_1 * t2_0)
                         * Cos(u)
                         * Sin(om_1 * t2_0)
                       +   om_1
                         * v2
                         * Cos(om_1 * t2_0)
                         * Cos(u)
                         * Sin(omb_1 * t2_0)
                       +   om_1
                         * v2
                         * Cos(om_1 * t2_0)
                         * Cos(omb_1 * t2_0)
                         * Sin(u)
                       +   om_1
                         * v2
                         * Sin(om_1 * t2_0)
                         * Sin(omb_1 * t2_0)
                         * Sin(u)))
```

```
  * Cos(om_3 * t5)
+   v2
  * Cos(om_3 * t5)
  *   (  1
       +   -1
         * (Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
   ^(1 / 2)
+ -1 * v1 * Sin(om_3 * t5)
+   om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  -1 * omb_1 * v1 * Cos(om_1 * t2_0)
        + om_1 * omb_1 * x2 * Cos(om_1 * t2_0)
        + omb_1 * v1 * (Cos(om_1 * t2_0))^2
        + om_1 * v2 * Cos(om_1 * t2_0) * Cos(u)
        +   -1
          * om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(omb_1 * t2_0)
          * Cos(u)
        + -1 * om_1 * omb_1 * x1 * Sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * (1 + -1 * (Cos(u))^2)^(1 / 2)
          * Sin(om_1 * t2_0)
        + omb_1 * v1 * (Sin(om_1 * t2_0))^2
        +   -1
          * om_1
          * v2
          * Cos(u)
          * Sin(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
```

```
                  +   -1
                    * om_1
                    * v2
                    * Cos(omb_1 * t2_0)
                    * Sin(om_1 * t2_0)
                    * Sin(u)
                  +   om_1
                    * v2
                    * Cos(om_1 * t2_0)
                    * Sin(omb_1 * t2_0)
                    * Sin(u)))
            * Sin(om_3 * t5)
          +   v2
            * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
            * Sin(om_3 * t5)
          +   v2
            * (Cos(om_3 * t5))^2
            * Sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          +   v2
            * (Sin(om_3 * t5))^2
            * Sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)))
 ^2
+   (  (om_3)^-1
     * (  -1 * v1 * Cos(om_3 * t5)
        +   om_3
          * (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * Cos(om_1 * t2_0)
                + om_1 * omb_1 * x2 * Cos(om_1 * t2_0)
                + omb_1 * v1 * (Cos(om_1 * t2_0))^2
                + om_1 * v2 * Cos(om_1 * t2_0) * Cos(u)
                +   -1
                  * om_1
                  * v2
                  * Cos(om_1 * t2_0)
                  * Cos(omb_1 * t2_0)
```

```
          + -1 * om_1 * omb_1 * x1 * Sin(om_1 * t2_0)
          +   -1
            * om_1
            * v2
            * (1 + -1 * (Cos(u))^2)^(1 / 2)
            * Sin(om_1 * t2_0)
          + omb_1 * v1 * (Sin(om_1 * t2_0))^2
          +   -1
            * om_1
            * v2
            * Cos(u)
            * Sin(om_1 * t2_0)
            * Sin(omb_1 * t2_0)
          +   -1
            * om_1
            * v2
            * Cos(omb_1 * t2_0)
            * Sin(om_1 * t2_0)
            * Sin(u)
          +   om_1
            * v2
            * Cos(om_1 * t2_0)
            * Sin(omb_1 * t2_0)
            * Sin(u)))
  * Cos(om_3 * t5)
+ v1 * (Cos(om_3 * t5))^2
+   v2
  * Cos(om_3 * t5)
  * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
+   -1
  * v2
  * (Cos(om_3 * t5))^2
  * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
```

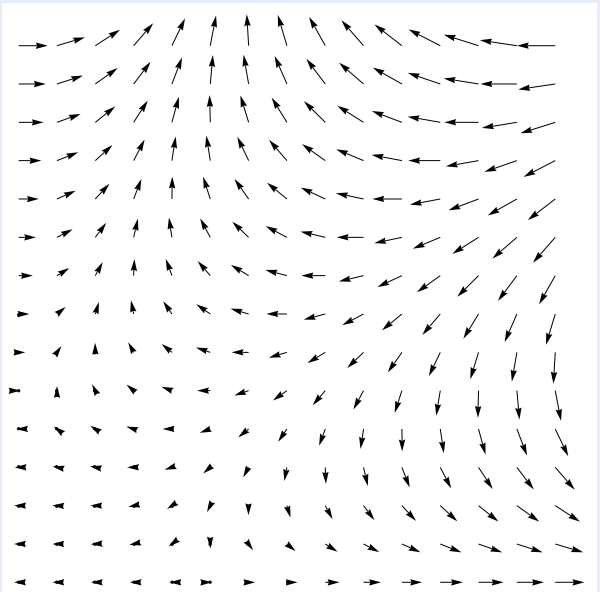
```
+   -1
  * om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  om_1 * omb_1 * x1 * Cos(om_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * (1 + -1 * (Cos(u))^2)^(1 / 2)
        + -1 * omb_1 * v1 * Sin(om_1 * t2_0)
        + om_1 * omb_1 * x2 * Sin(om_1 * t2_0)
        + om_1 * v2 * Cos(u) * Sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * Cos(omb_1 * t2_0)
          * Cos(u)
          * Sin(om_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(u)
          * Sin(omb_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(omb_1 * t2_0)
          * Sin(u)
        +   om_1
          * v2
          * Sin(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
          * Sin(u)))
  * Sin(om_3 * t5)
```

```
          +   -1
            * v2
            *   (  1
                 +   -1
                   * (Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
              ^(1 / 2)
            * Sin(om_3 * t5)
          + v1 * (Sin(om_3 * t5))^2
          +   -1
            * v2
            * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
            * (Sin(om_3 * t5))^2))
  ^2
>= (p)^2)
```

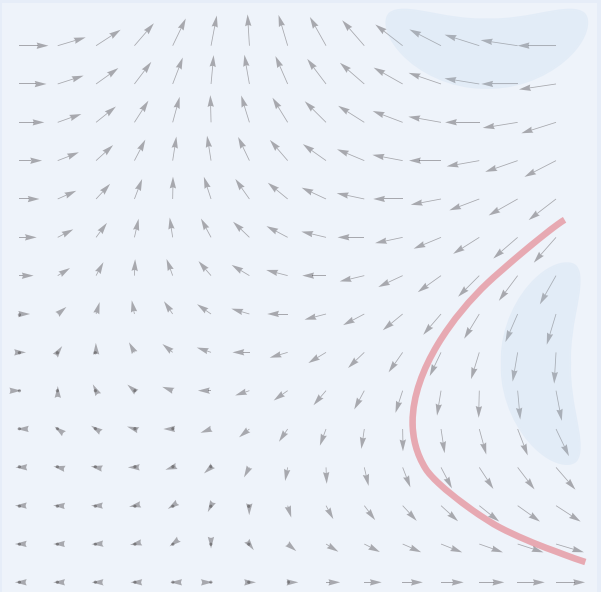
This is just one branch to prove for aircraft . . .

Differential Invariants for Differential Equations

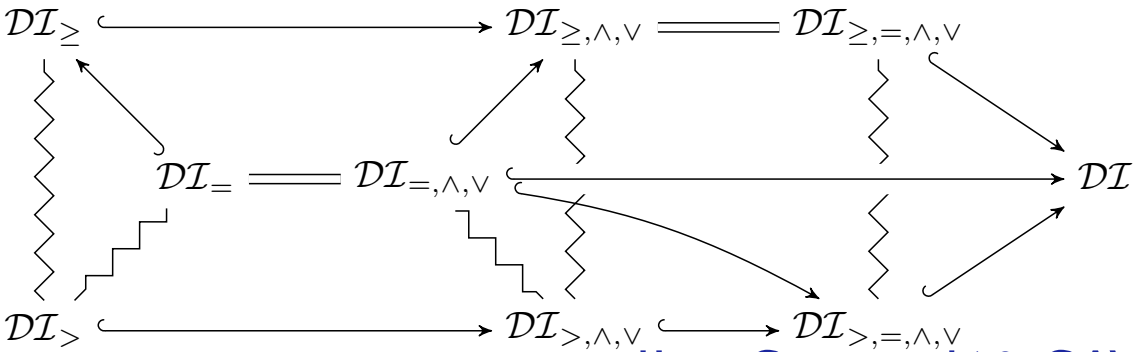
Differential Invariant



Differential Cut



Differential Ghost



Logic

Provability theory

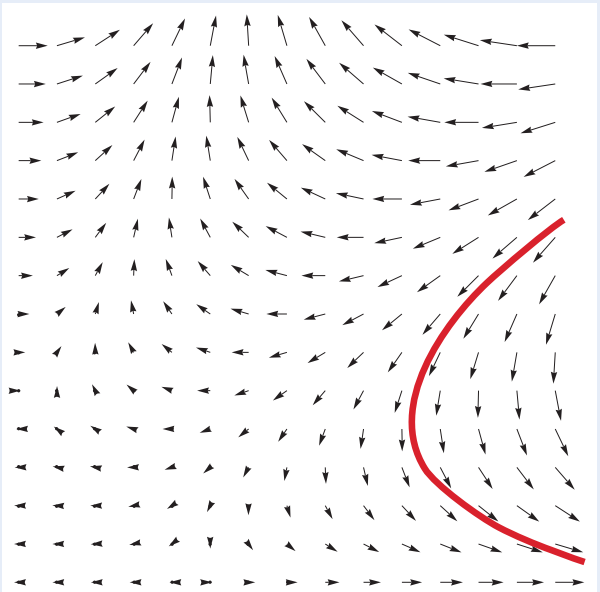
Math

Characteristic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



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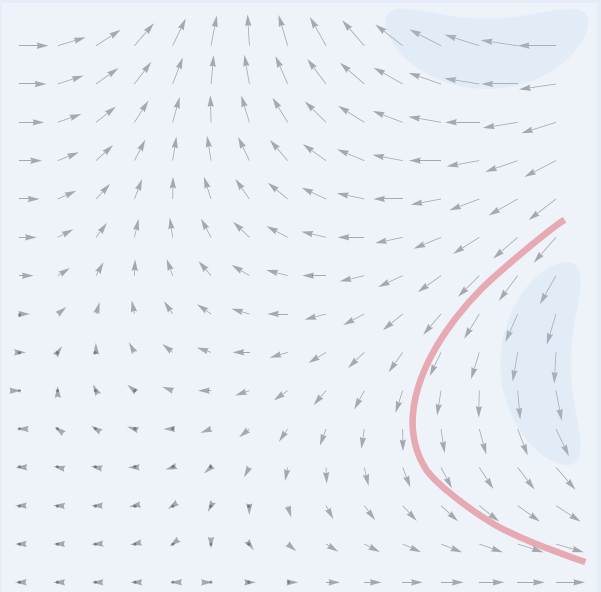

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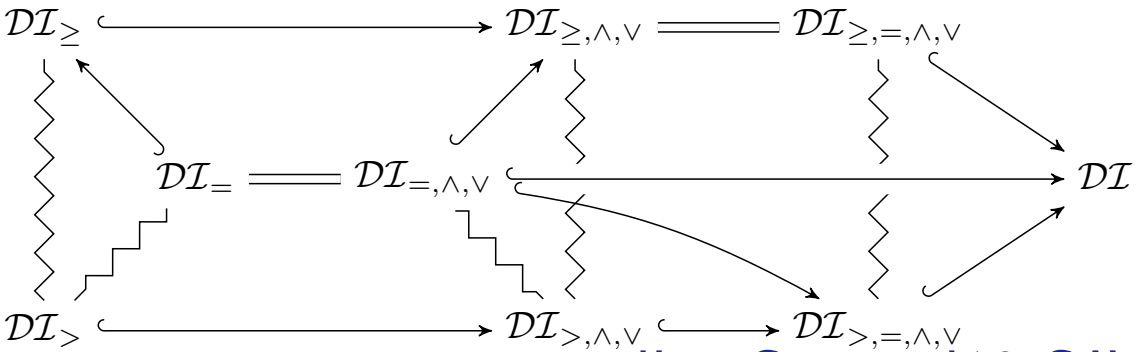

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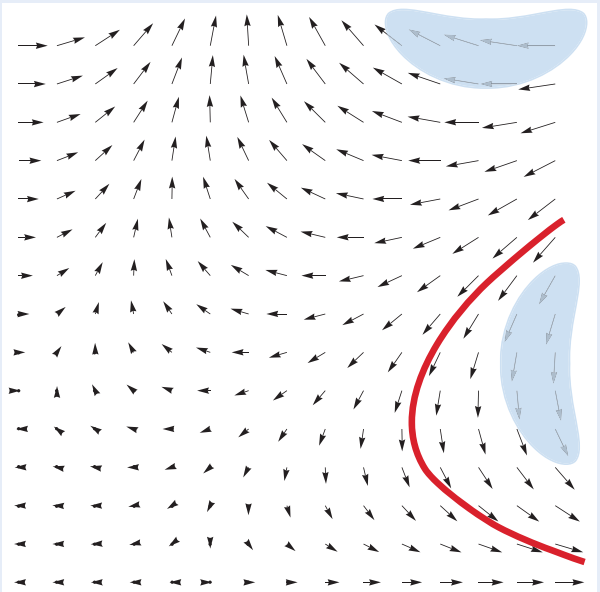
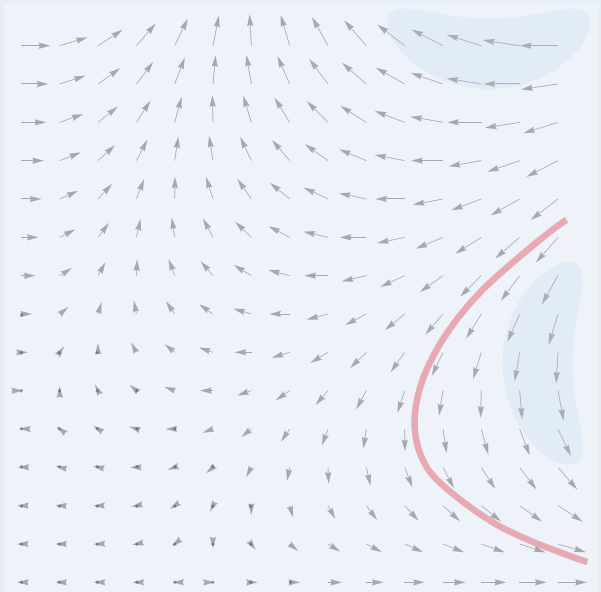
Differential Invariants for Differential Equations

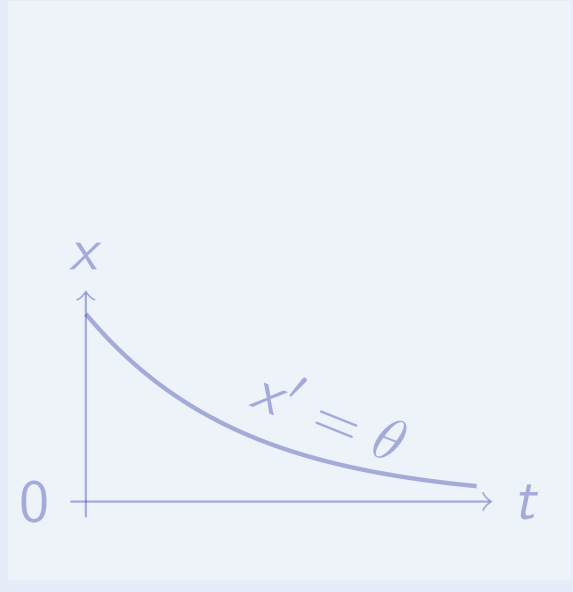
Differential Invariant

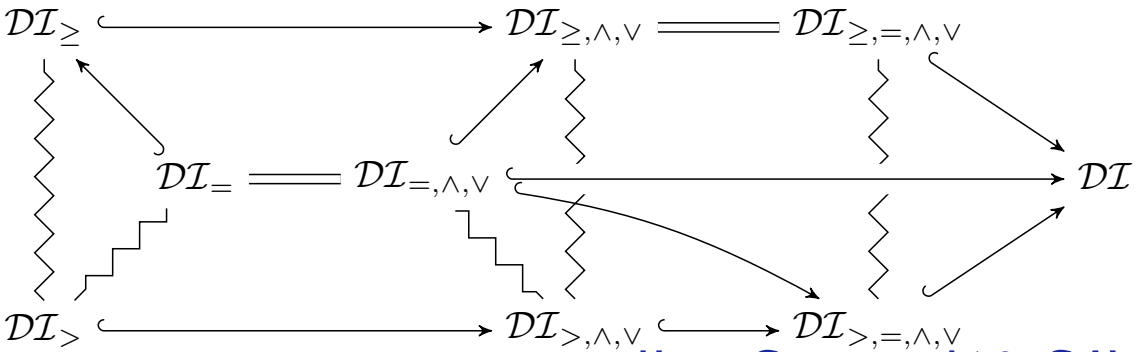


Differential Cut



Differential Ghost





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Characteristic PDE

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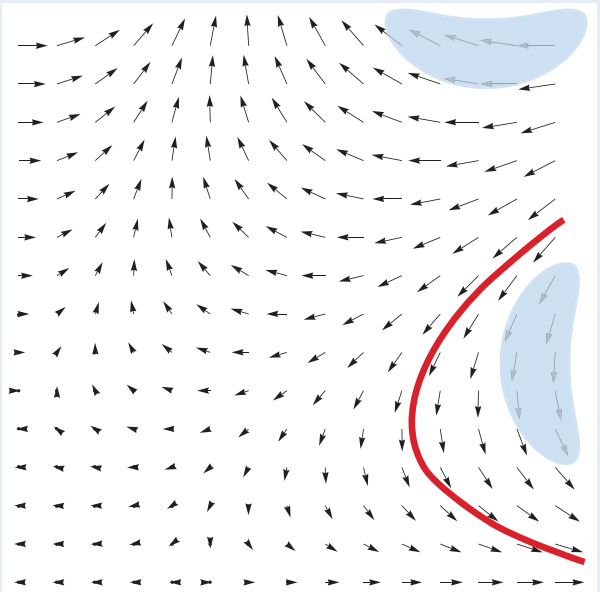
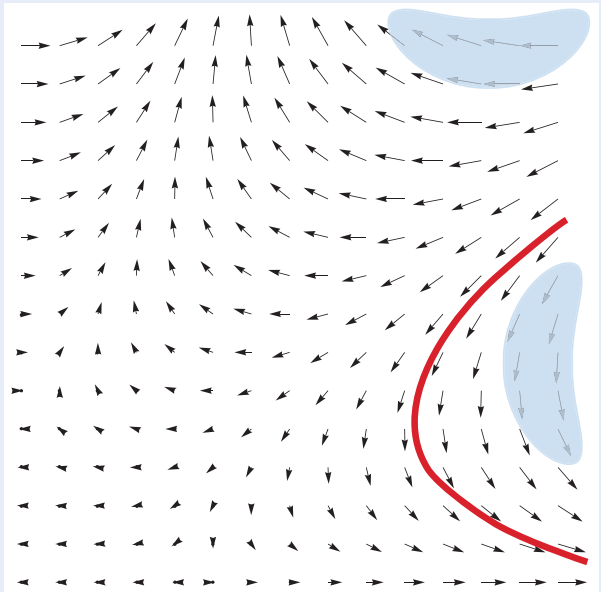
Differential Invariants for Differential Equations

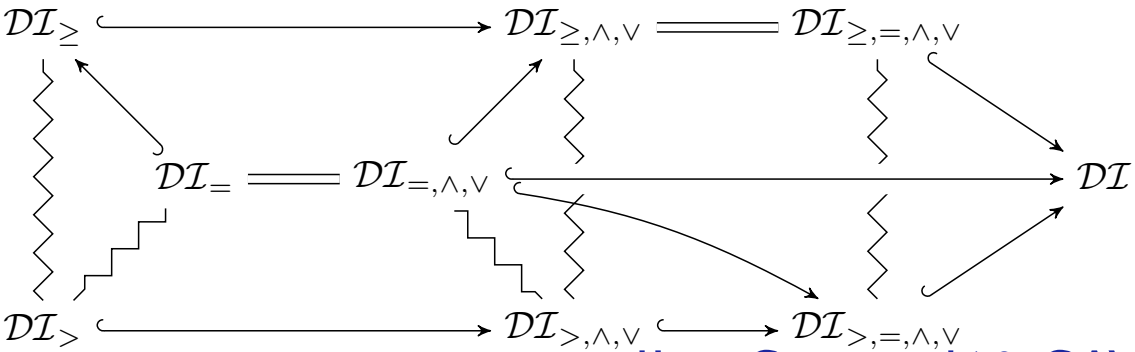
Differential Invariant



Differential Cut



Differential Ghost



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Differential Invariants for Differential Equations

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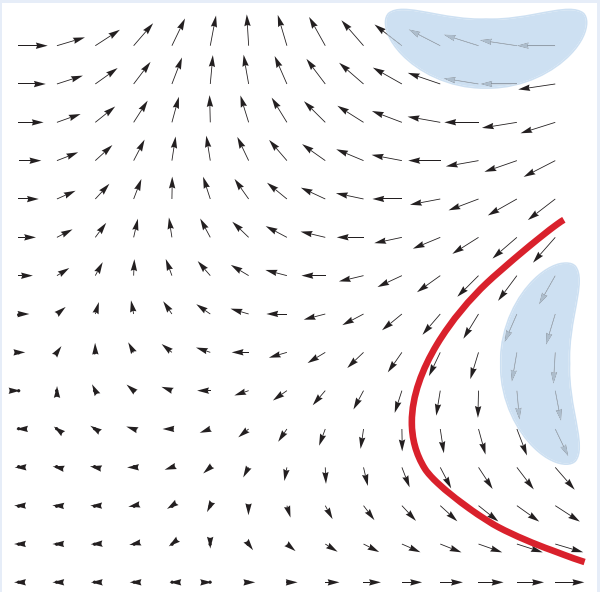
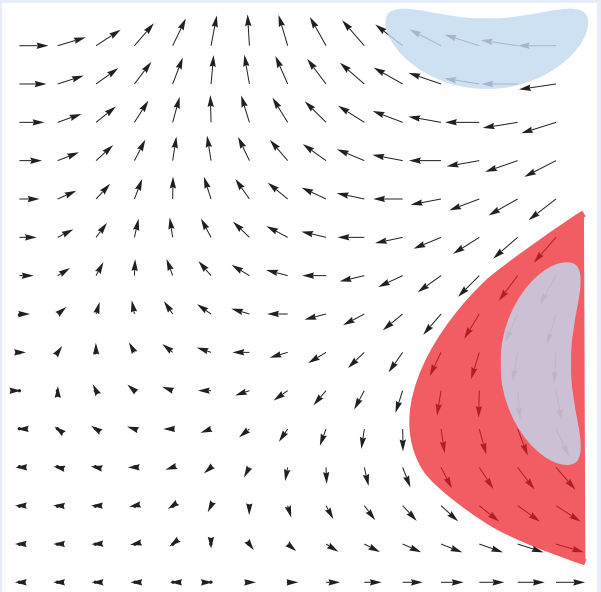


Differential Cut



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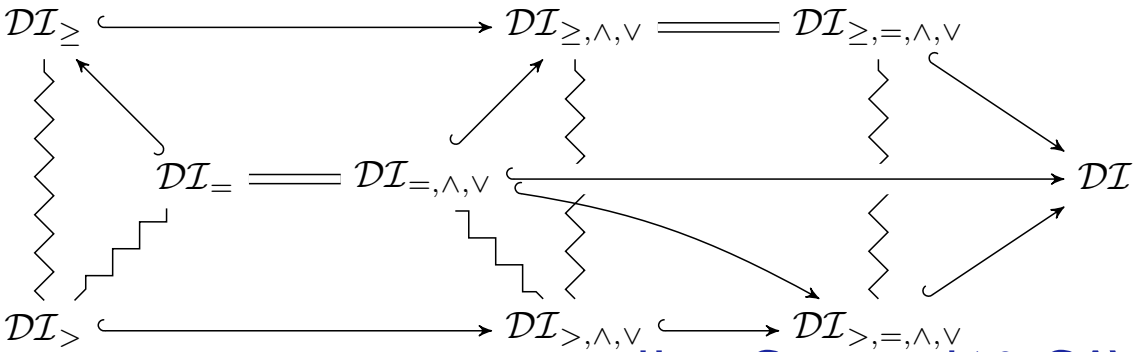

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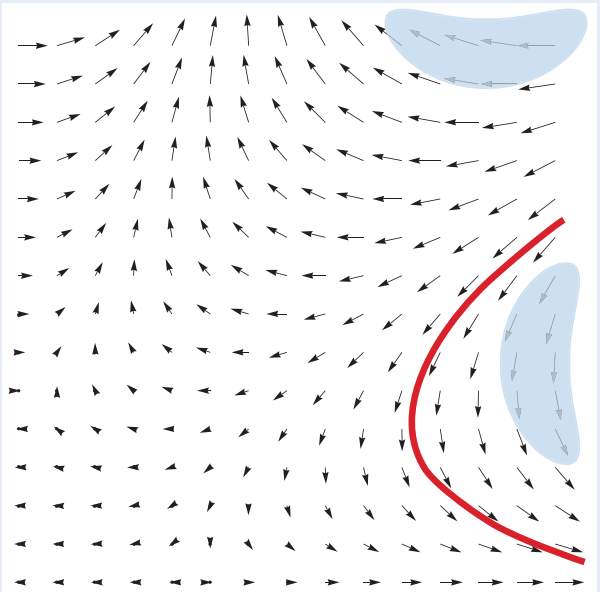
Characteristic PDE

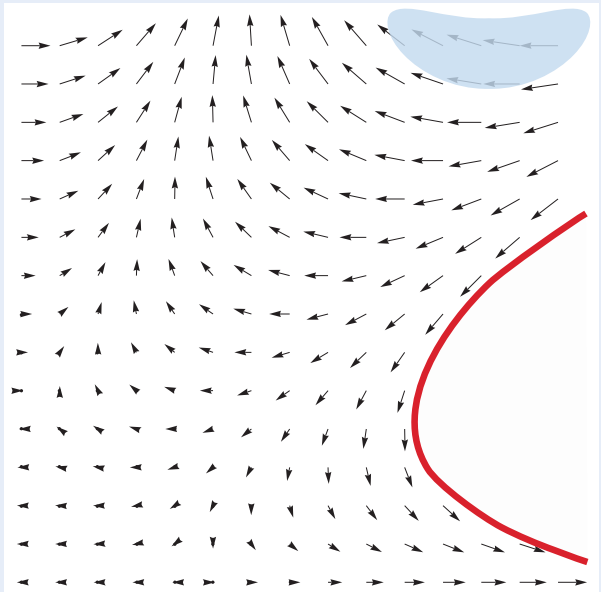
JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

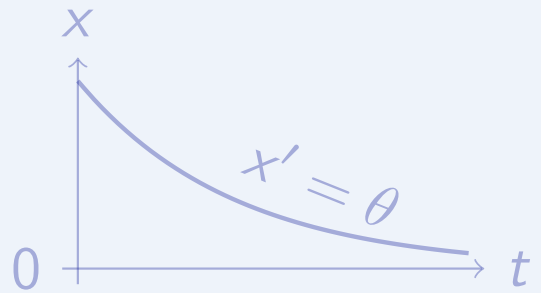
Differential Invariant

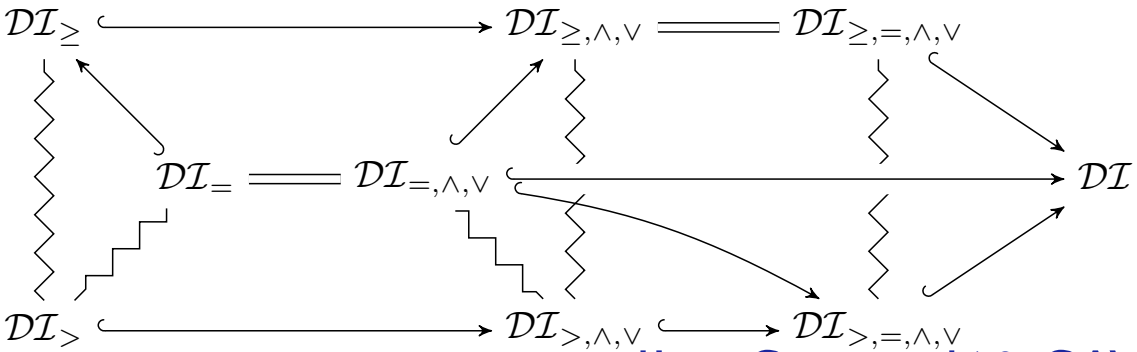


Differential Cut



Differential Ghost





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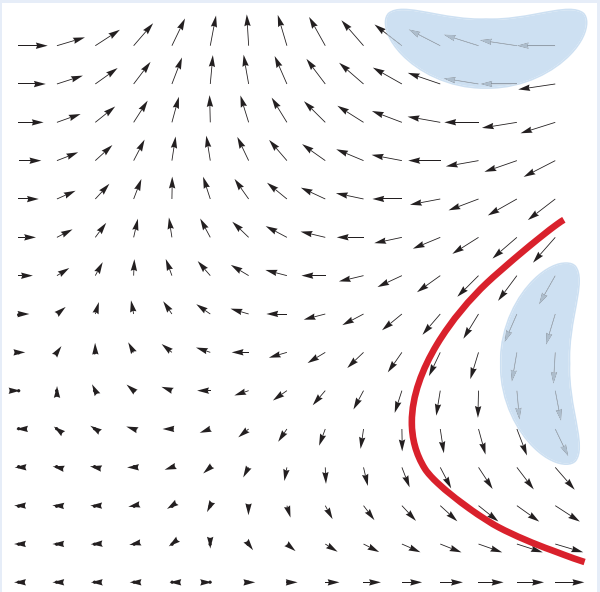

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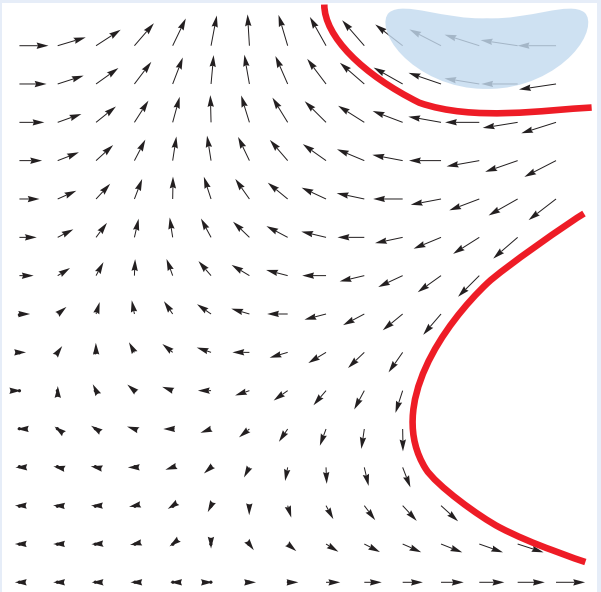

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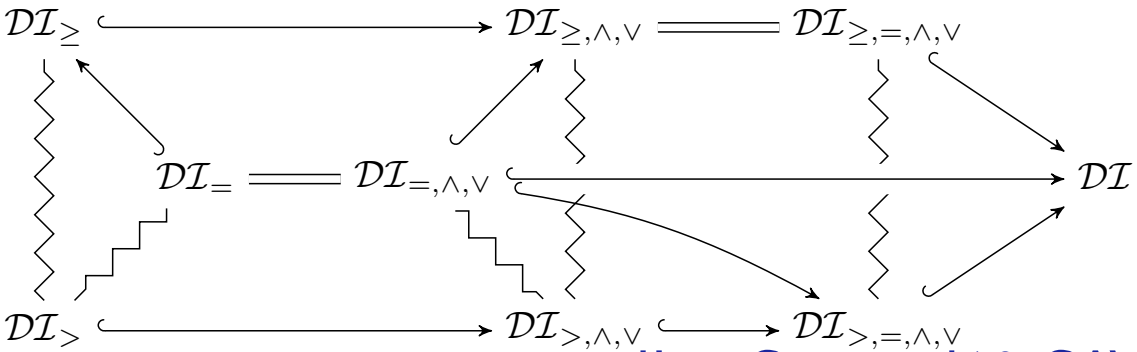

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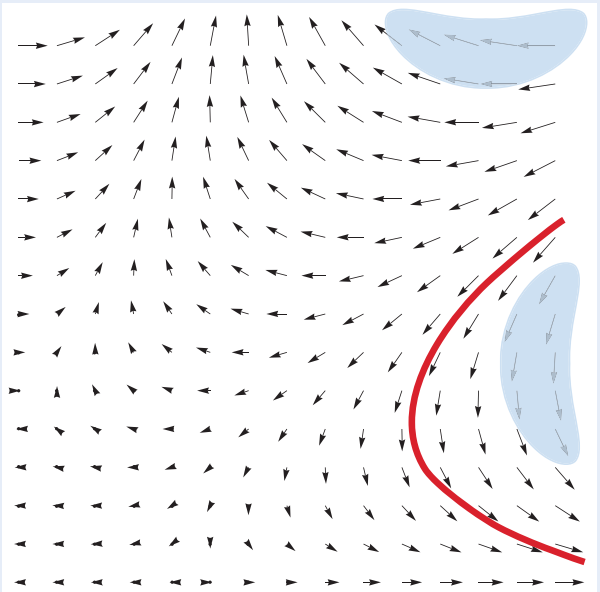

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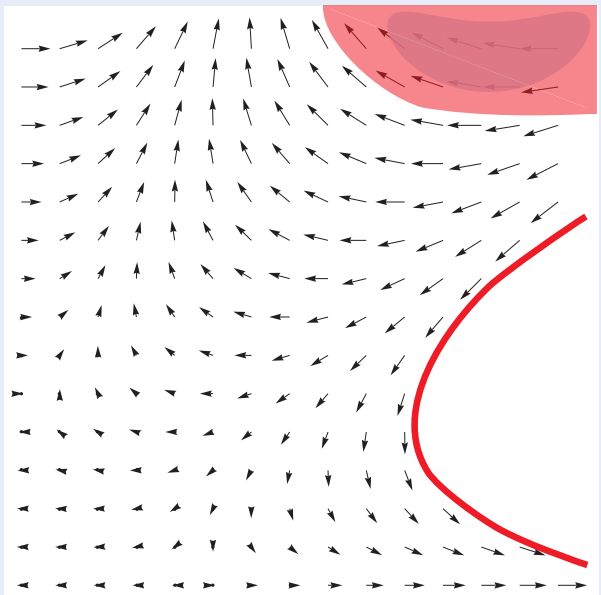

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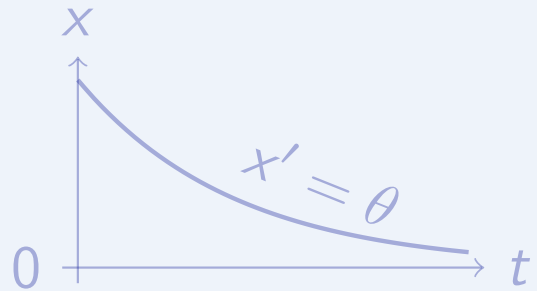

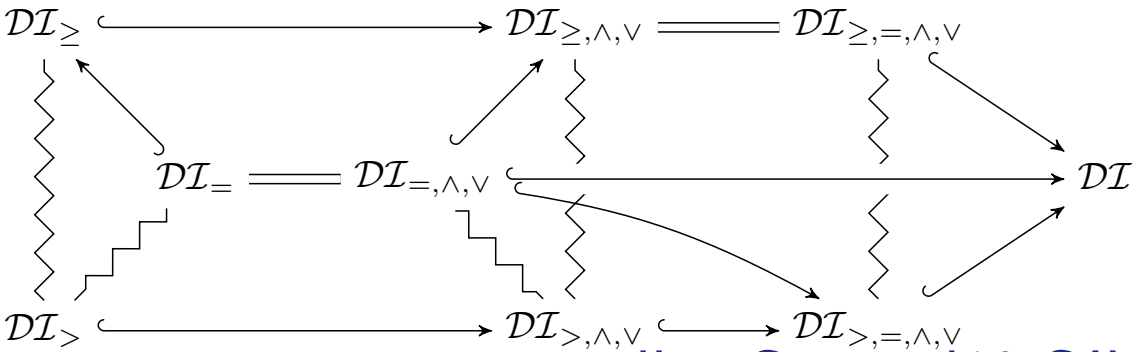

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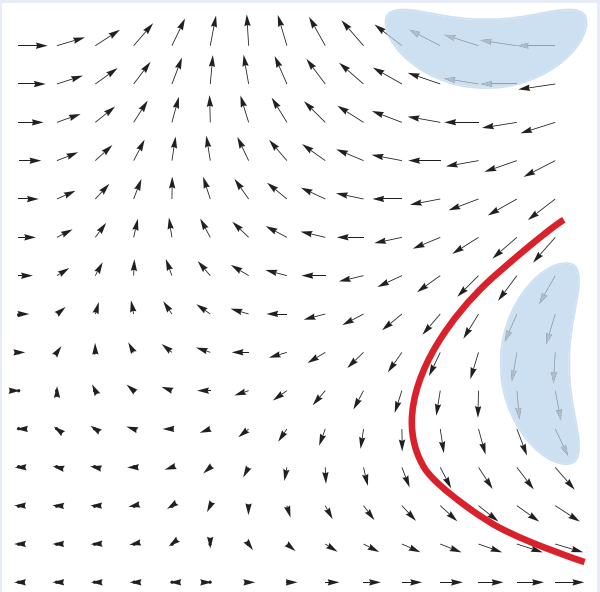

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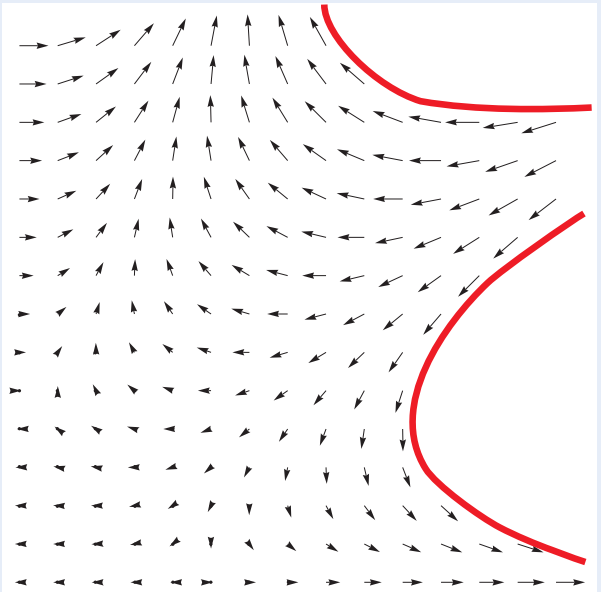

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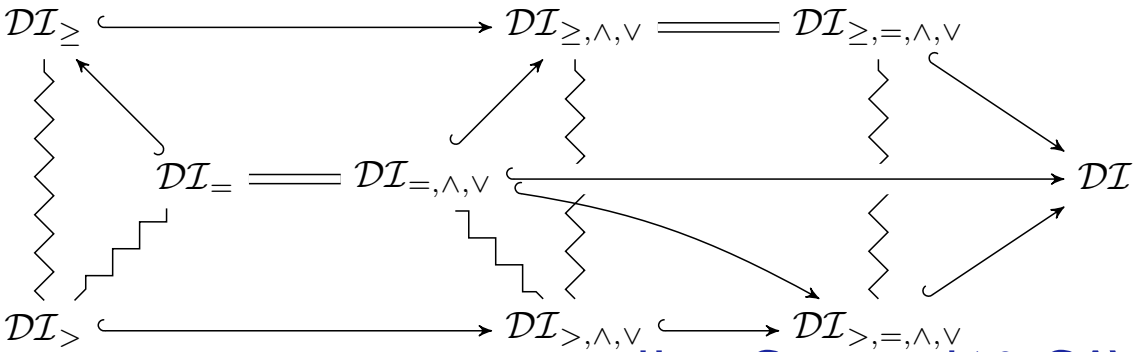

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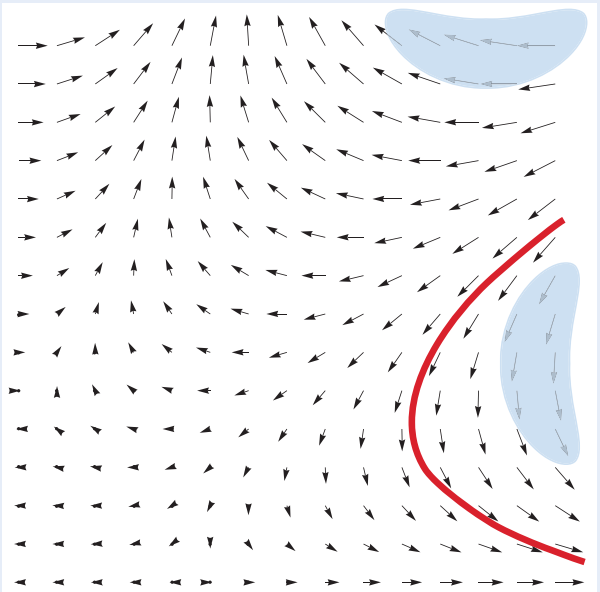

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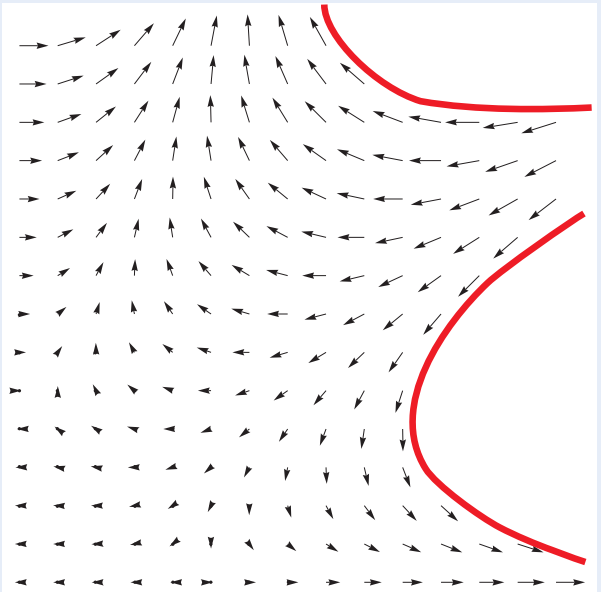

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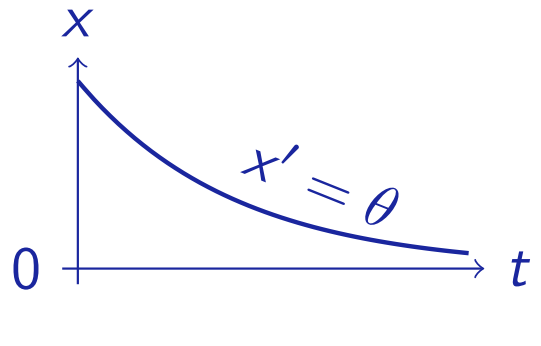

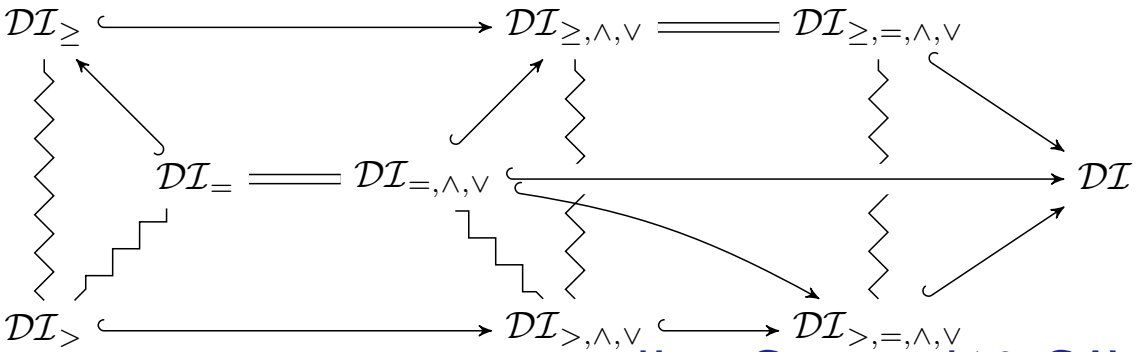

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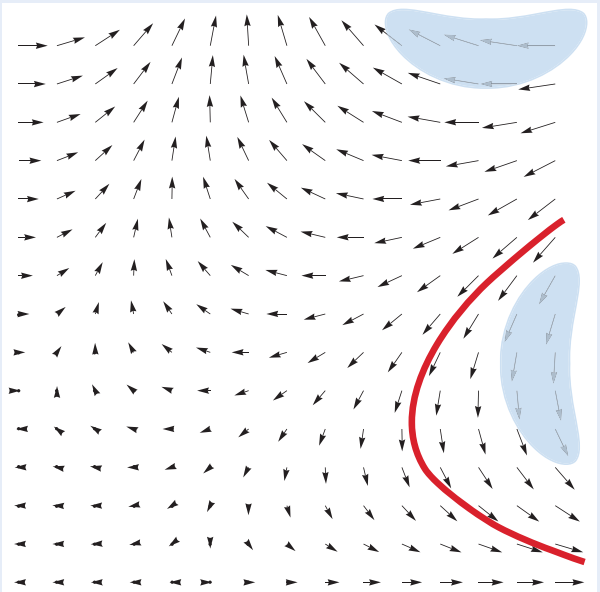

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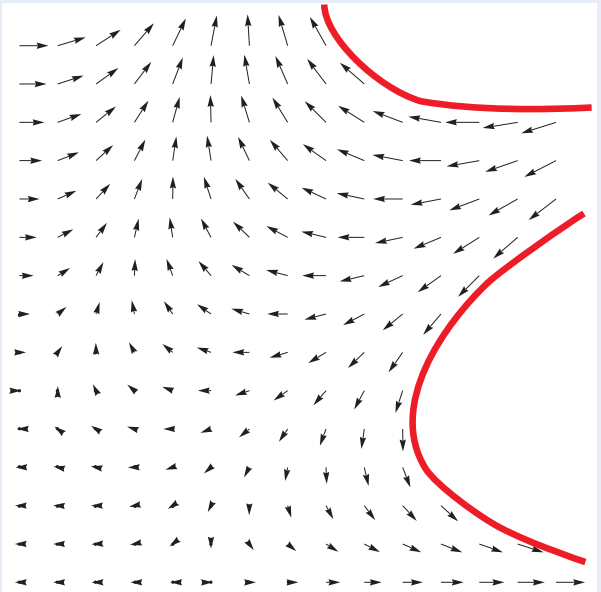

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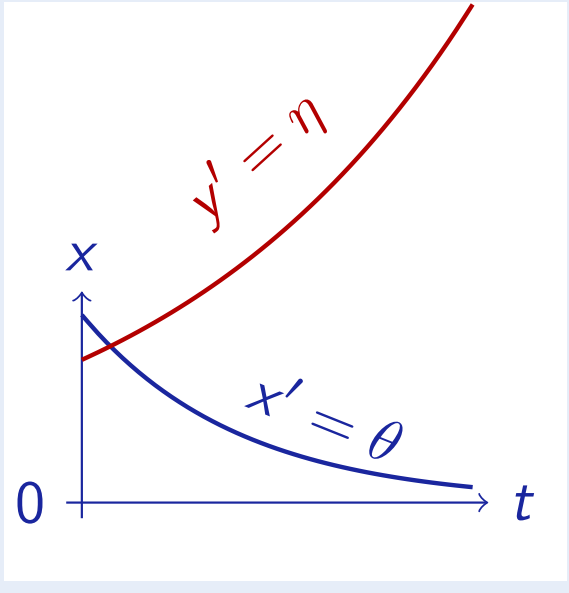

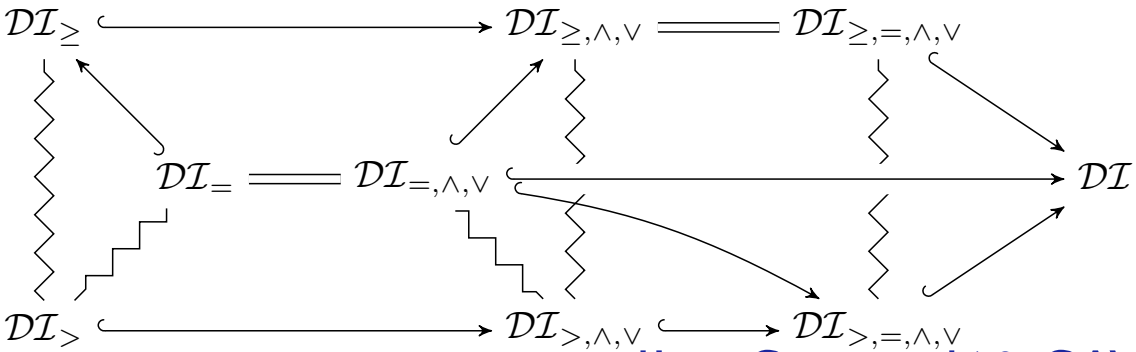

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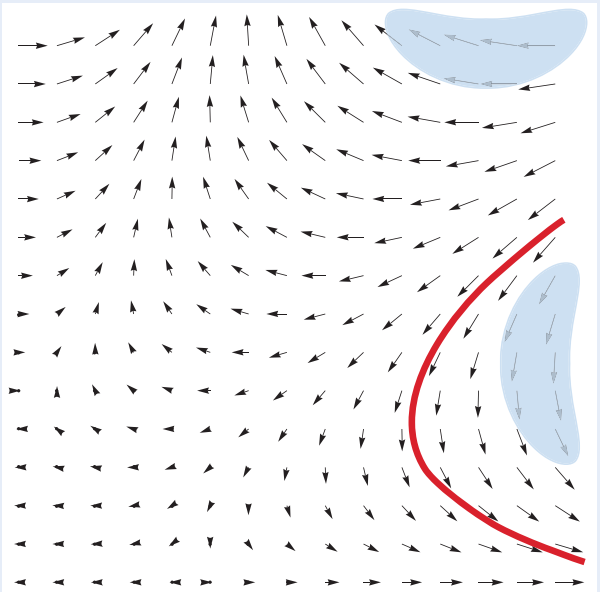

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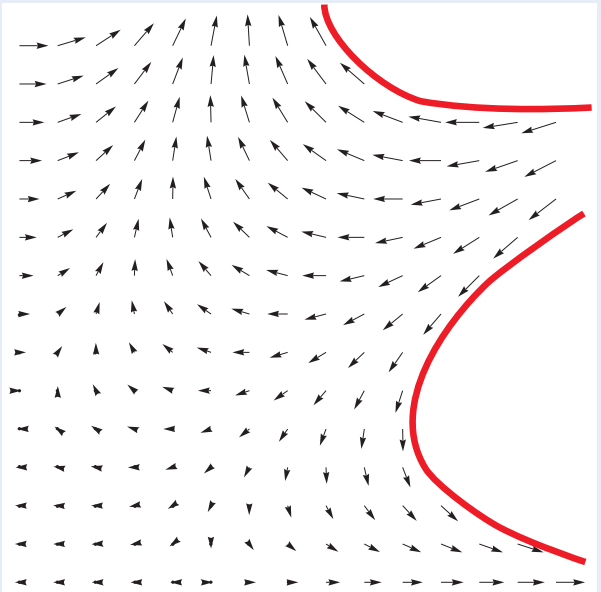

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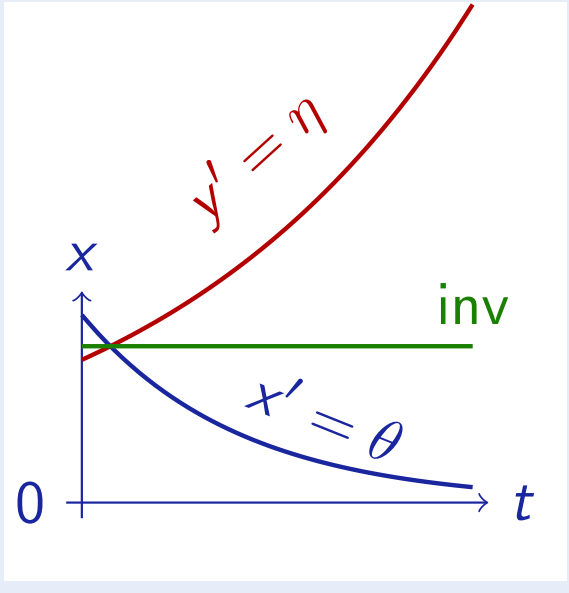

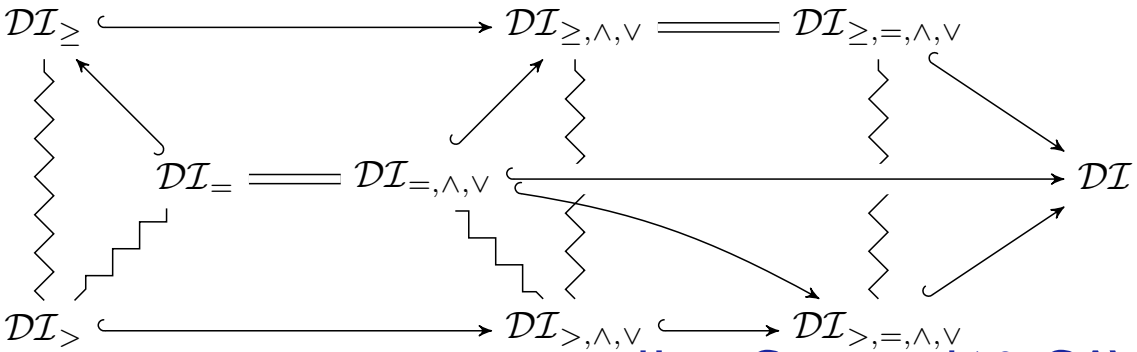

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Differential Invariant

$$\frac{H \rightarrow F'}{F \rightarrow [x' = f(x) \,\&\, H]F}$$

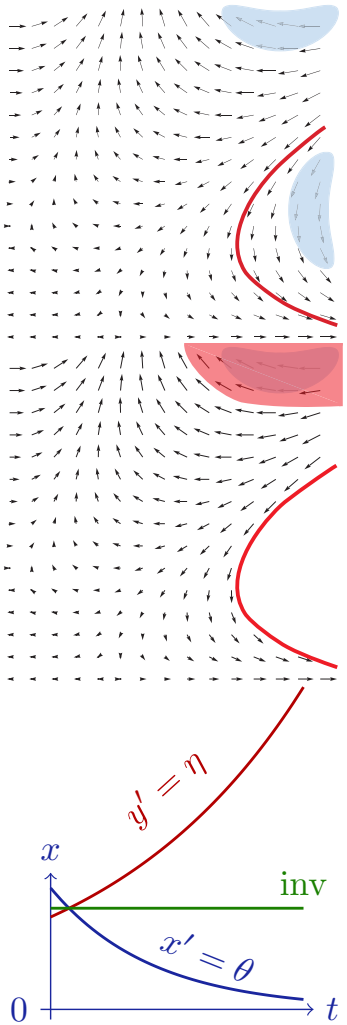
Differential Cut

$$\frac{F \rightarrow [x' = \theta \,\&\, H]C \quad F \rightarrow [x' = \theta \,\&\, (H \wedge C)]F}{F \rightarrow [x' = \theta \,\&\, H]F}$$

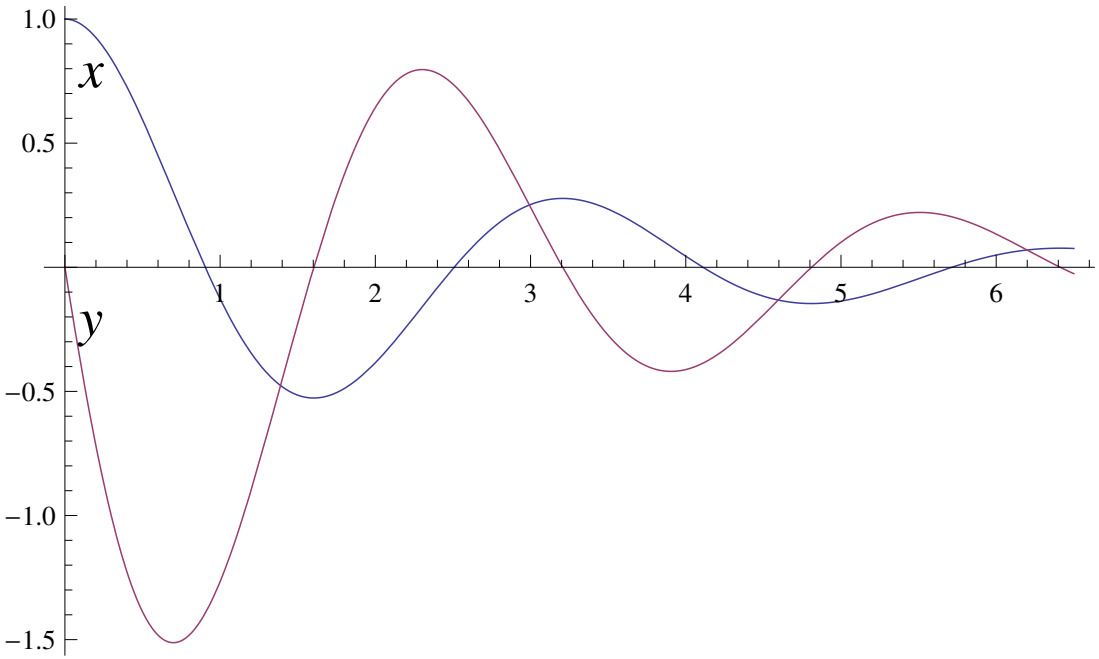
Differential Ghost

$$\frac{F \leftrightarrow \exists y\, G \quad G \rightarrow [x' = \theta, y' = \eta \,\&\, H]G}{F \rightarrow [x' = \theta \,\&\, H]F}$$

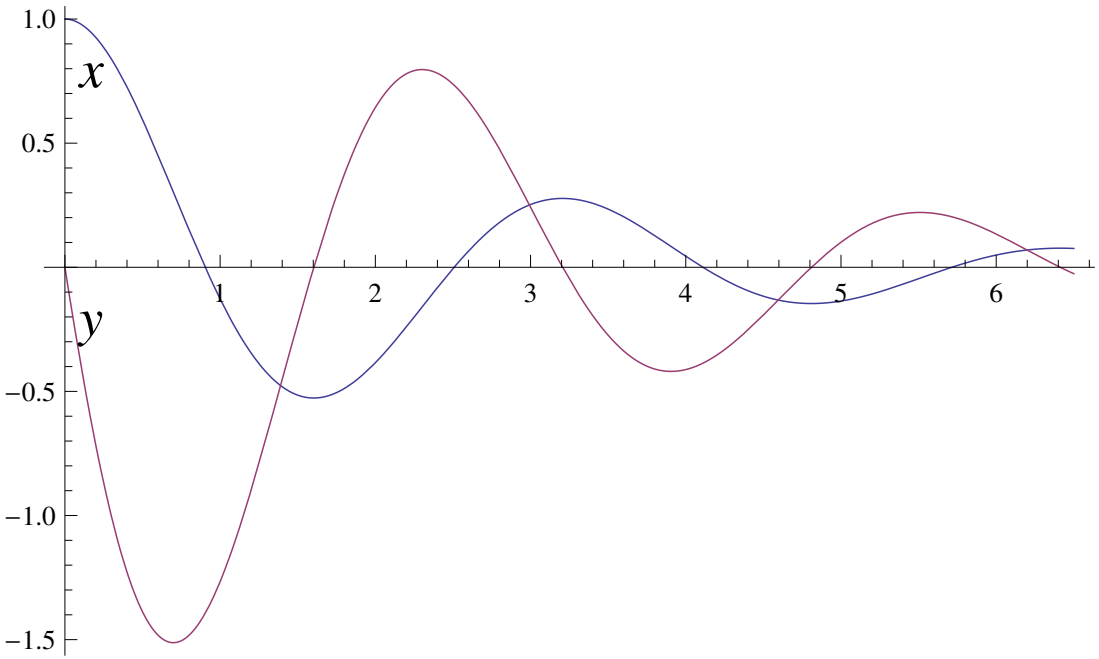
if new $y' = \eta$ has a global solution



$$\omega^2x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \wedge d\geq 0)]\, \omega^2x^2+y^2\leq c^2$$



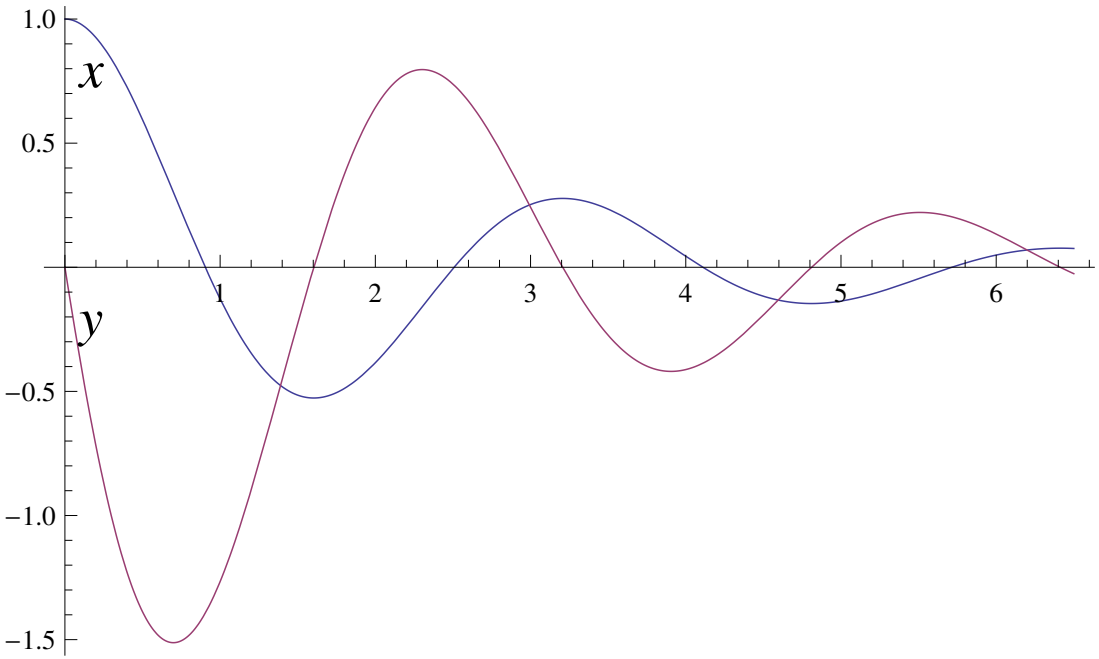
$$\frac{\omega\geq 0 \wedge d\geq 0 \rightarrow (2\omega^2 x x' + 2y y' \leq 0)^{y,\ -\omega^2 x - 2d\omega y}_{x',\ y'}}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \wedge d\geq 0)]\, \omega^2 x^2 + y^2 \leq c^2}$$



$$\omega \geq 0 \wedge d \geq 0 \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega \geq 0 \wedge d \geq 0 \rightarrow (2\omega^2 xx' + 2yy' \leq 0)^{y,\ -\omega^2 x - 2d\omega y}_{x',\ y'}$$

$$\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega \geq 0 \wedge d \geq 0)]\, \omega^2 x^2 + y^2 \leq c^2$$



$$*$$

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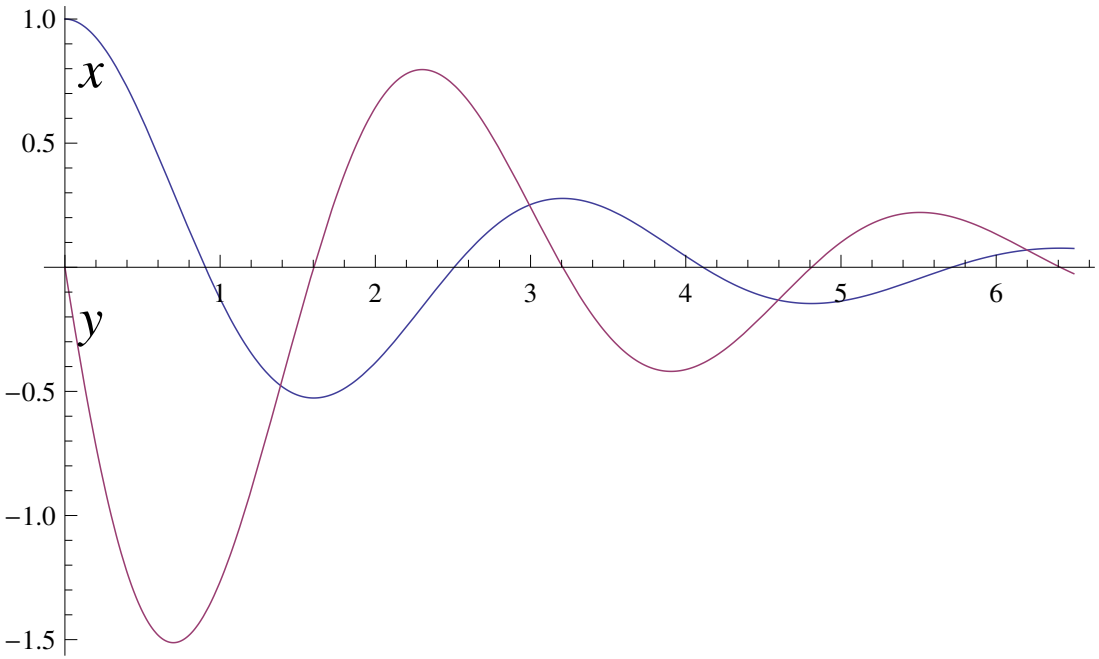

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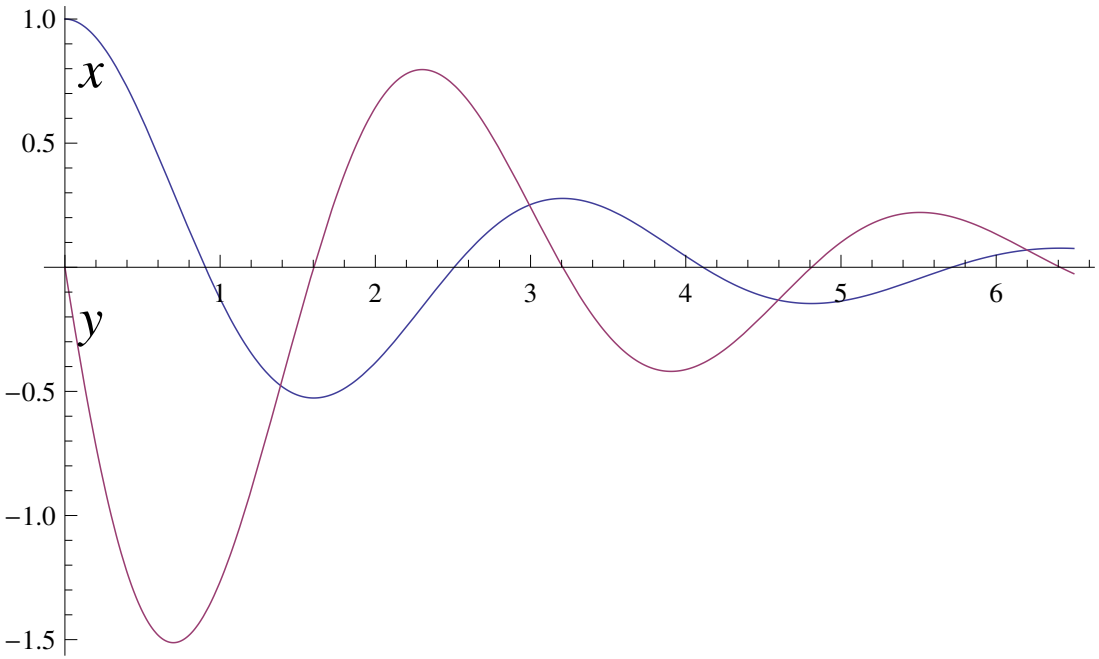
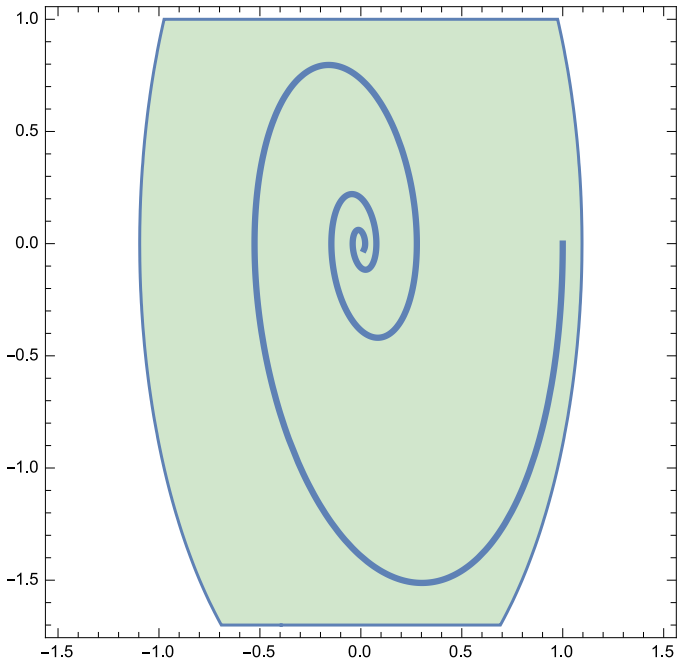
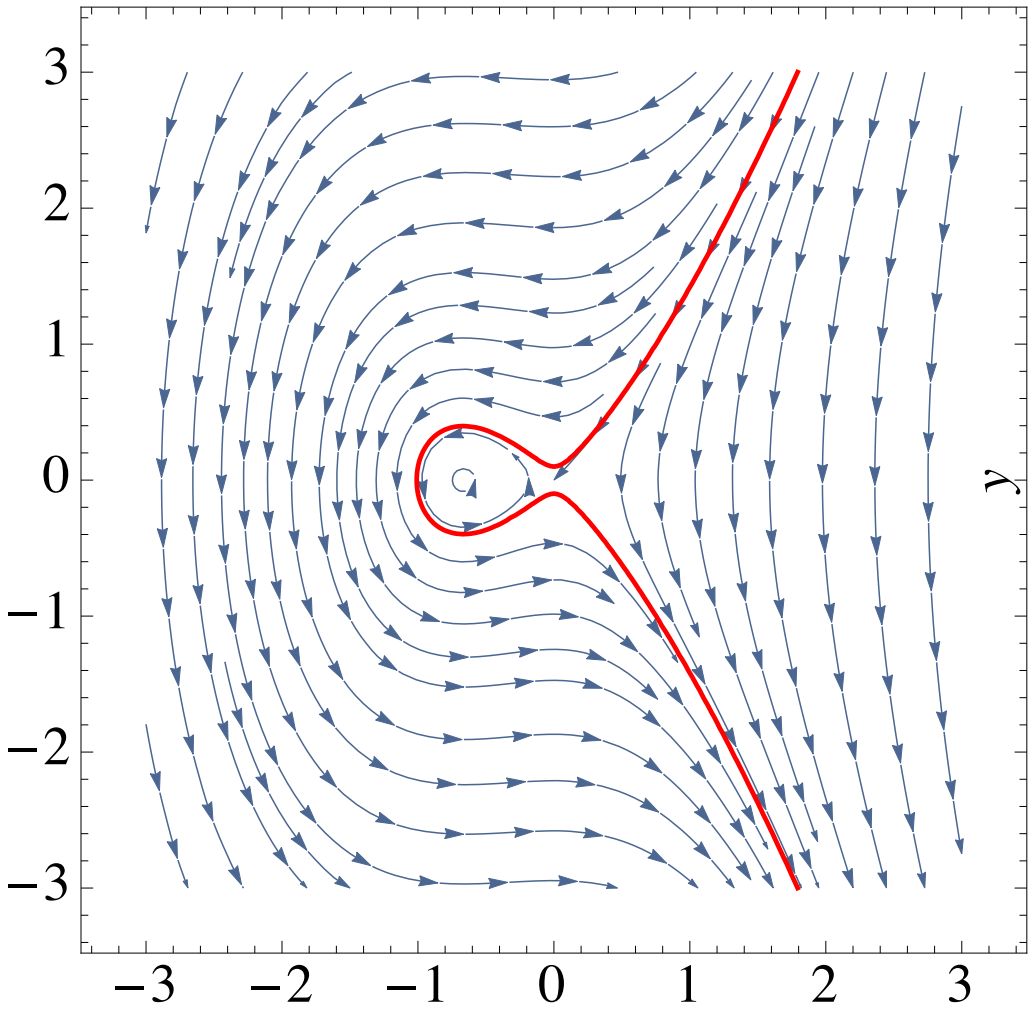
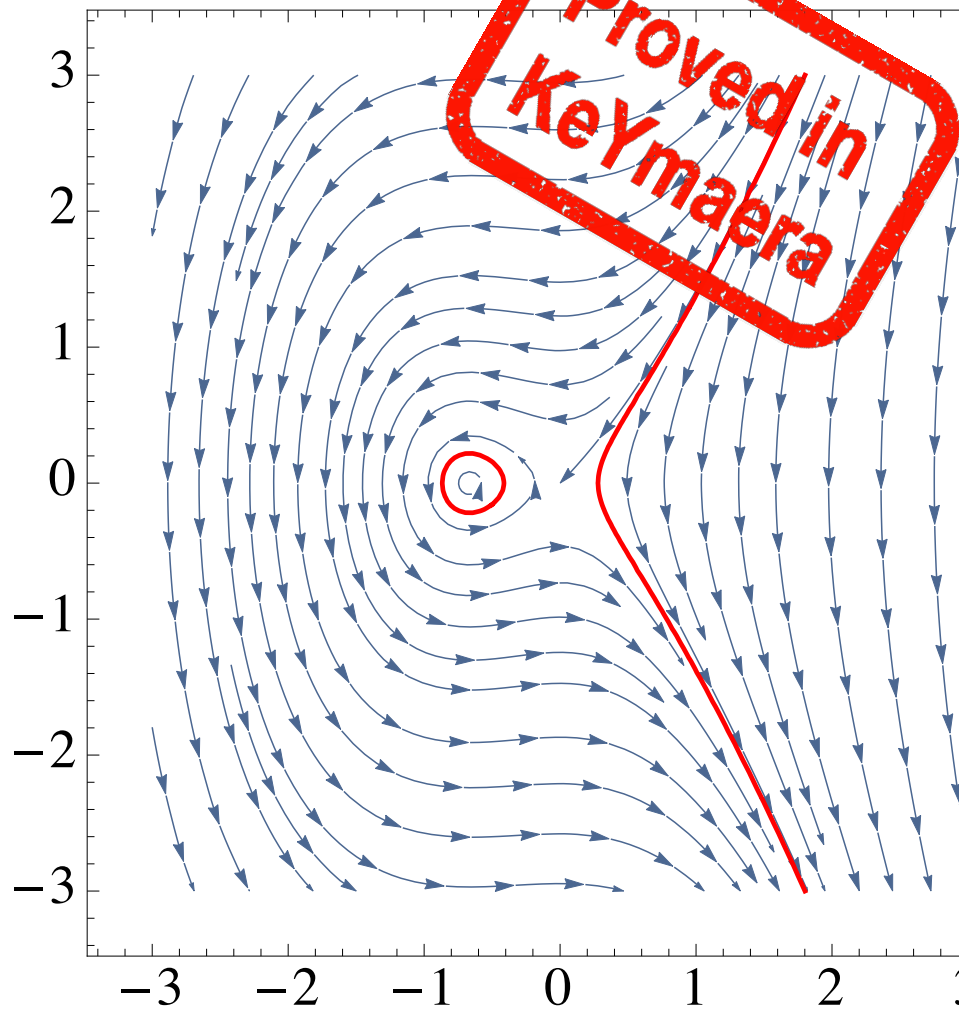




Differential Invariants for Differential Equations



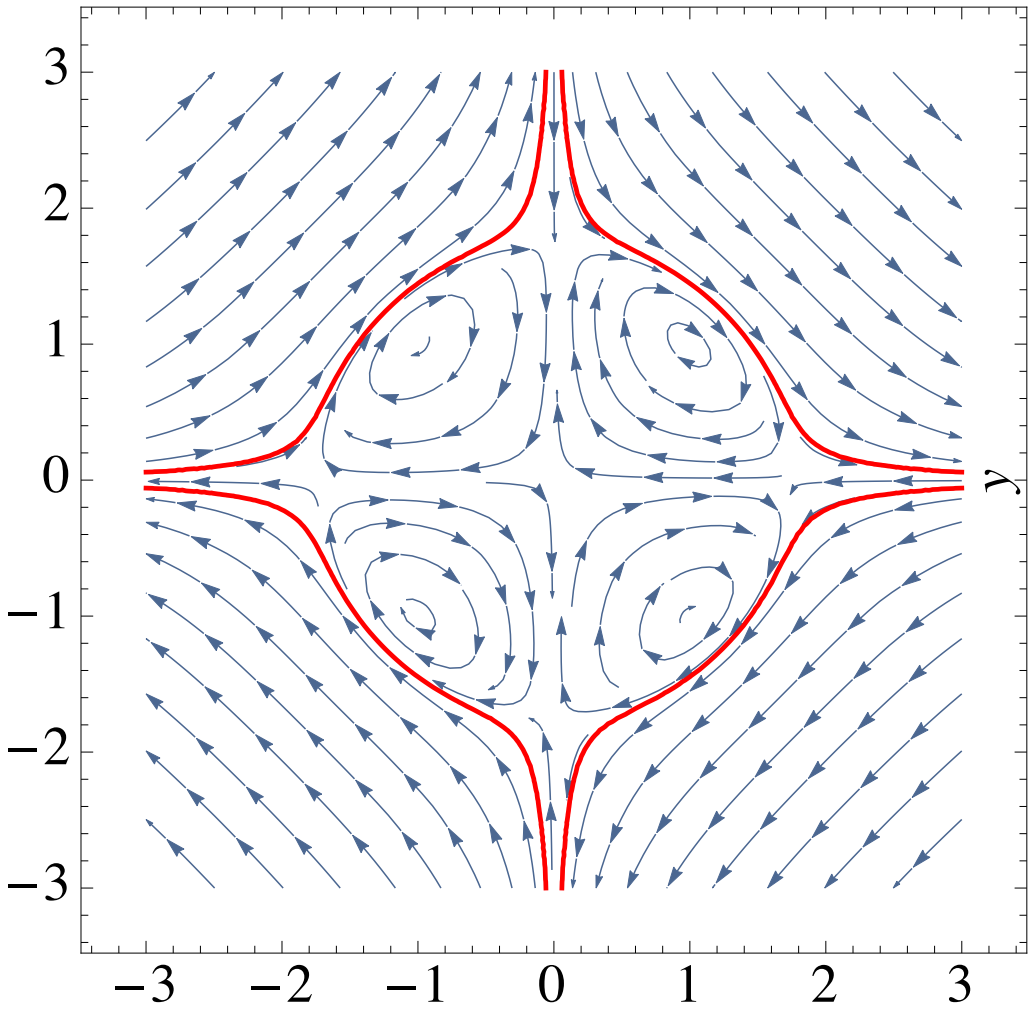


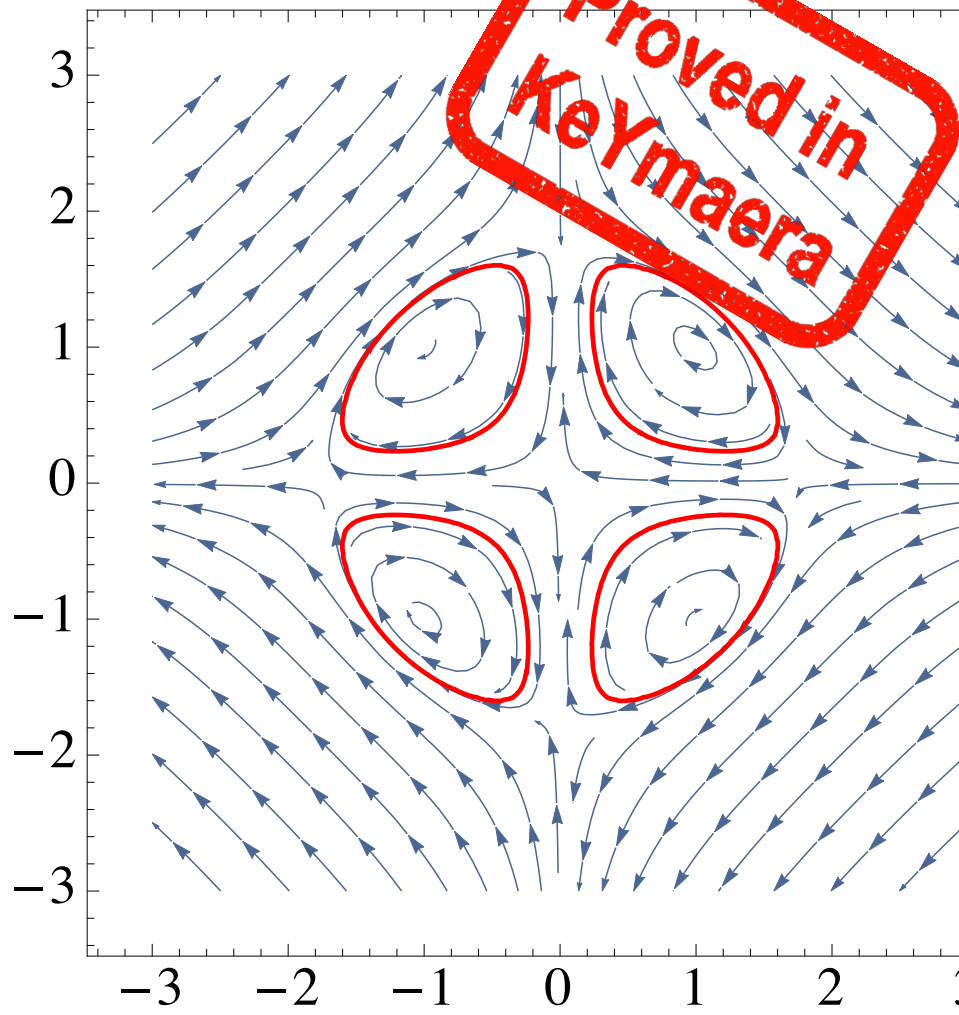
Proved in KeYmaera

$$x^2 + x^3 - y^2 - c = 0 \rightarrow [x' = -2y, y' = -2x - 3x^2]\, x^2 + x^3 - y^2 - c = 0$$

SAS'14

Differential Invariants for Differential Equations



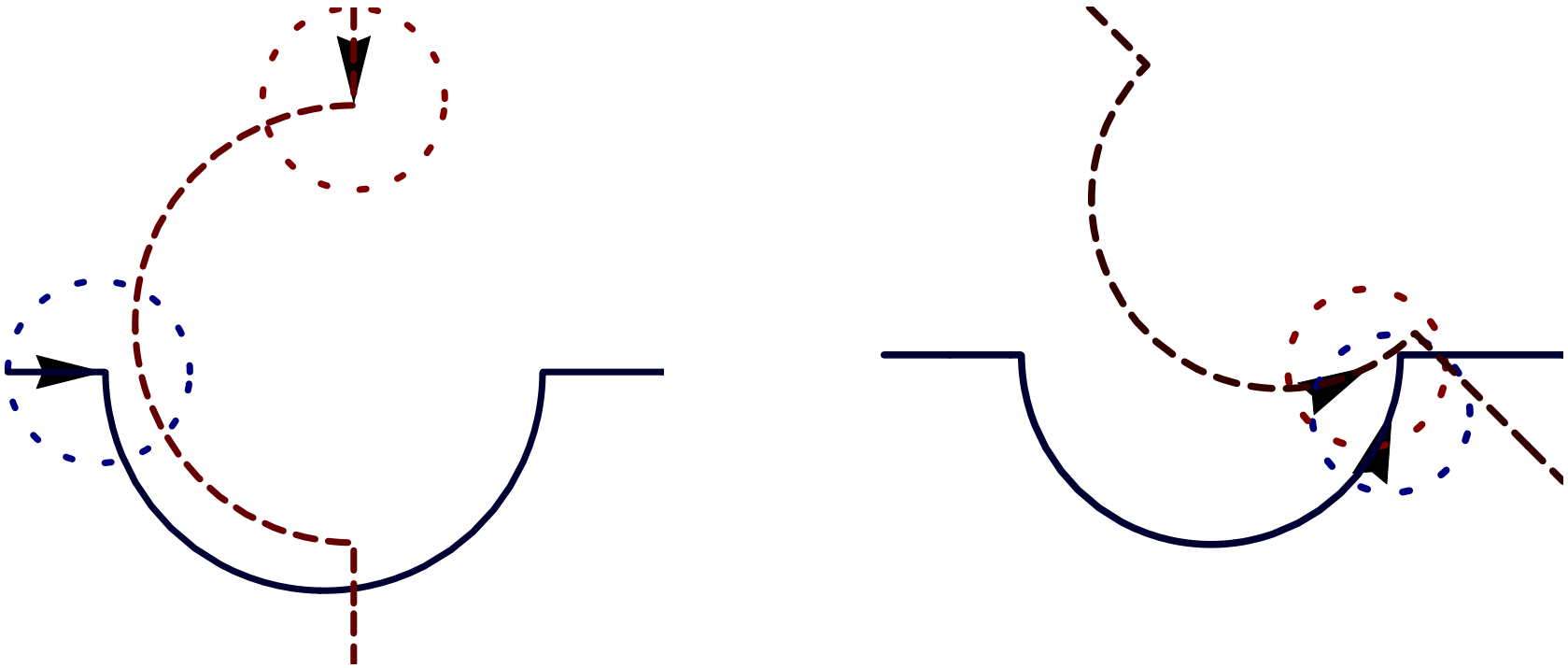


Proved in KeYmaera

$$[x' = 2x^4y+4x^2y^3-6x^2y, y' = -4x^3y^2-2xy^4+6xy^2]x^4y^2+x^2y^4-3x^2y^2 \leq c$$

SAS'14

Air Traffic Control

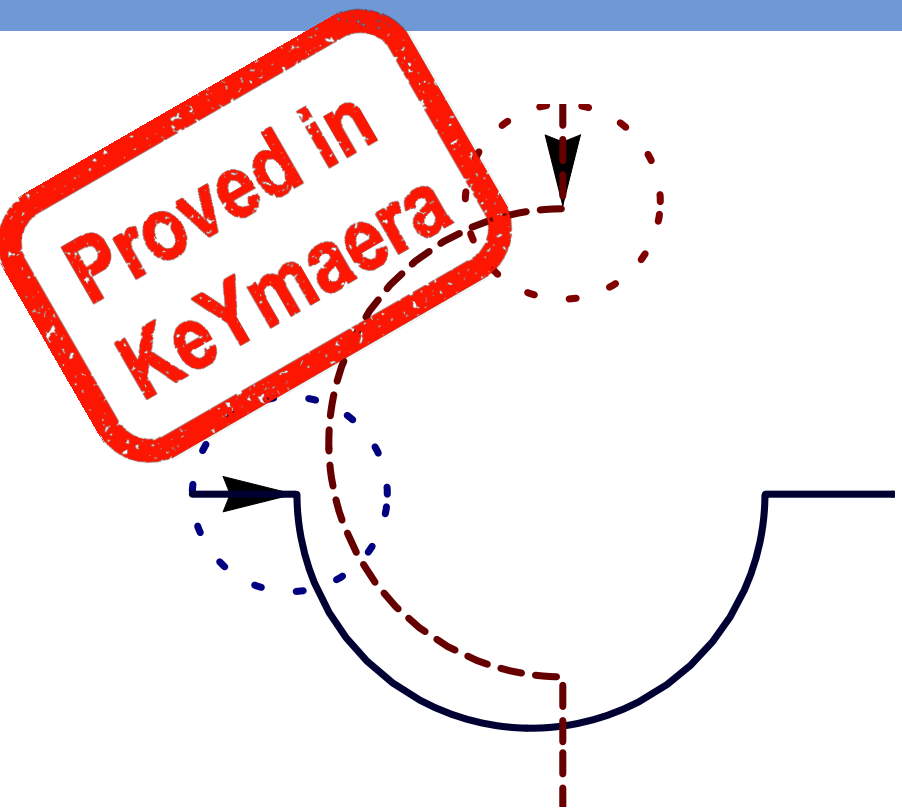


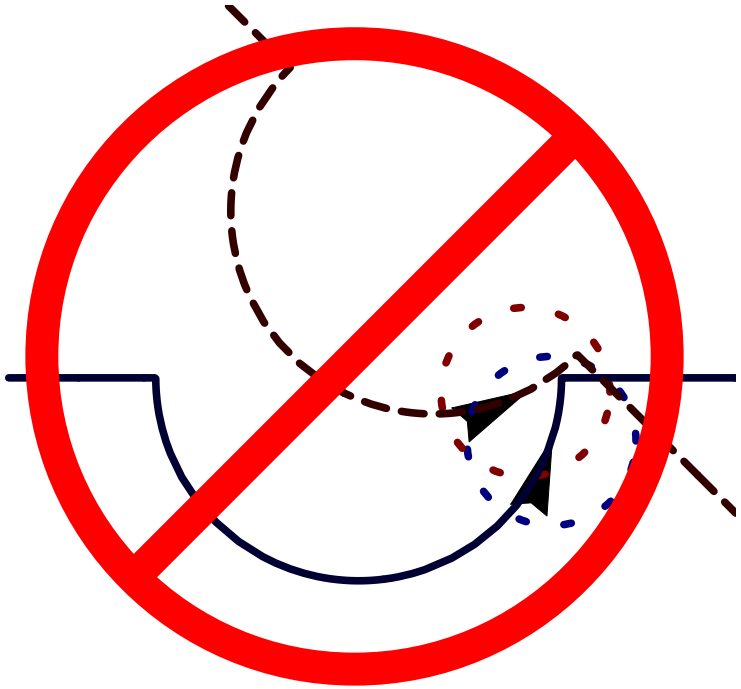
Verification

Simple proof distinguishes safe from unsafe flight maneuver

JLogComput'10

Air Traffic Control





Verification

Simple proof distinguishes safe from unsafe flight maneuver

JLogComput'10

Differential Radical Invariants

Theorem (Differential radical invariant characterization)

$$\frac{h = 0 \to \bigwedge_{i=0}^{N-1} (h^{(i)})^p_{x'} = 0}{h = 0 \to [x' = p]h = 0}$$

characterizes all algebraic invariants, where $N = ord\ \sqrt[\prime]{(h)}$, *i.e.*

$$(h^{(N)})^p_{x'} = \sum_{i=0}^{N-1} g_i (h^{(i)})^p_{x'} \quad (g_i \in \mathbb{R}[x])$$

Corollary (Algebraic Invariants Decidable)

Algebraic invariants of algebraic differential equations are decidable.

with Khalil Ghorbal TACAS'14

Case Study: Longitudinal Dynamics of an Airplane

Study (6th Order Longitudinal Flight Equations)

$$u' = \frac{X}{m} - g\sin(\theta) - qw \quad \text{axial velocity}$$

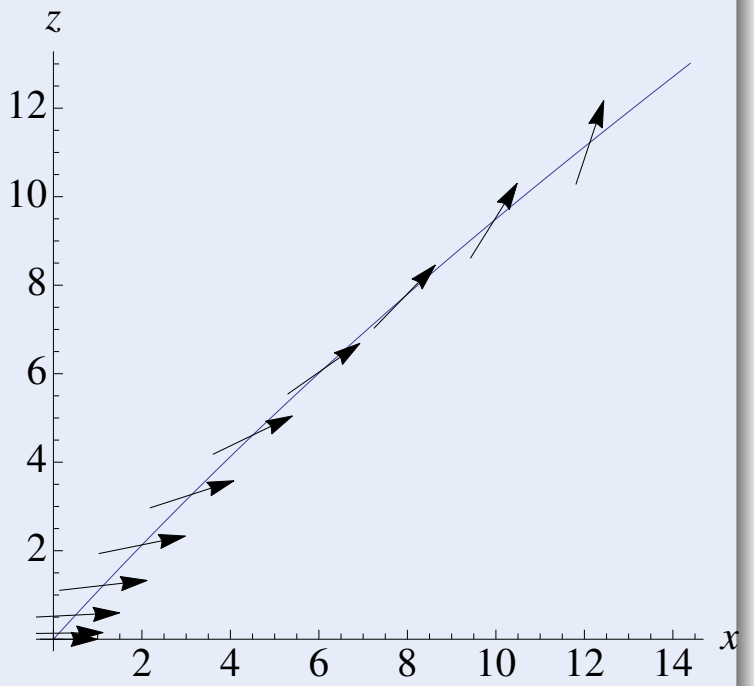
$$w' = \frac{Z}{m} + g\cos(\theta) + qu \quad \text{vertical velocity}$$

$$x' = \cos(\theta)u + \sin(\theta)w \quad \text{range}$$

$$z' = -\sin(\theta)u + \cos(\theta)w \quad \text{altitude}$$

$$\theta' = q \quad \text{pitch angle}$$

$$q' = \frac{M}{I_{yy}} \quad \text{pitch rate}$$



X : thrust along u Z : thrust along w M : thrust moment for w

g : gravity m : mass I_{yy} : inertia second diagonal

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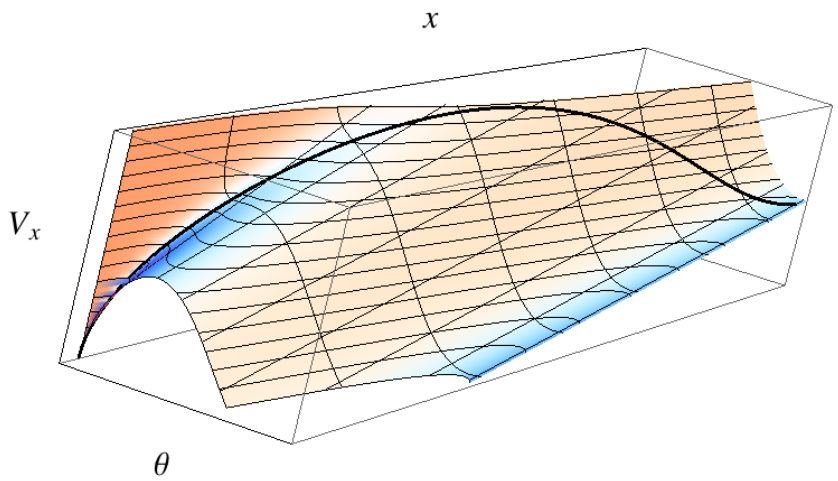
Case Study: Longitudinal Dynamics of an Airplane

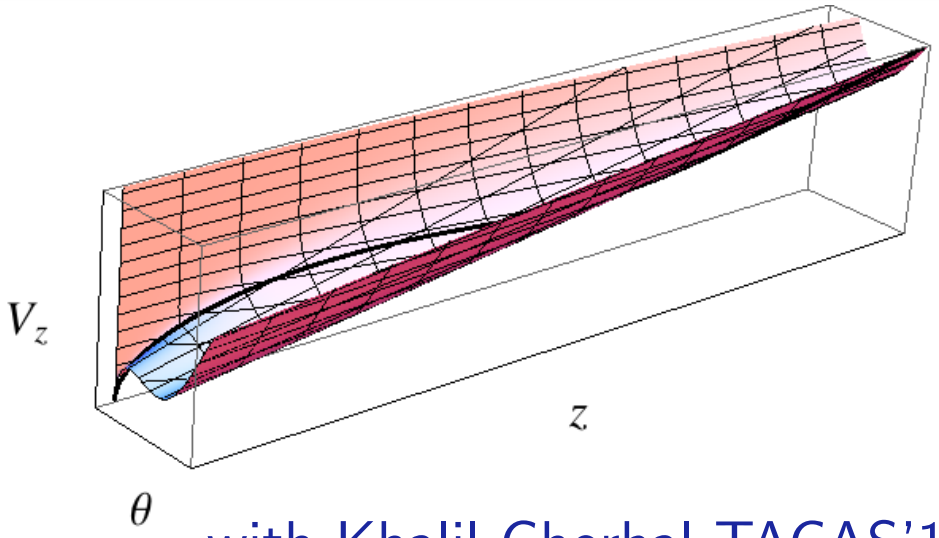
Result (DRI Automatically Generates Invariant Functions)

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$





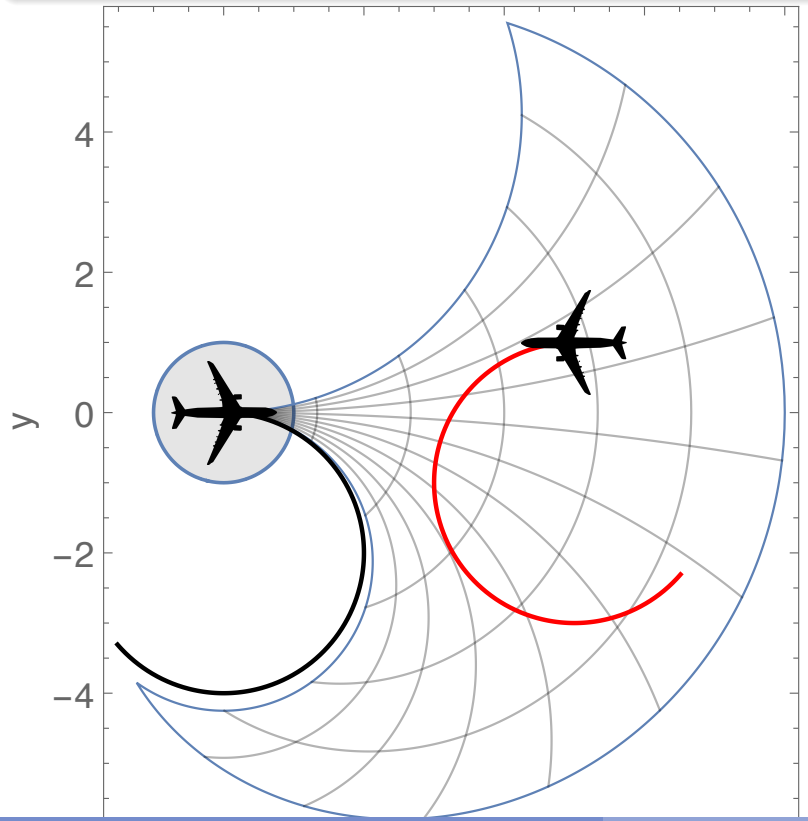
with Khalil Ghorbal TACAS'14

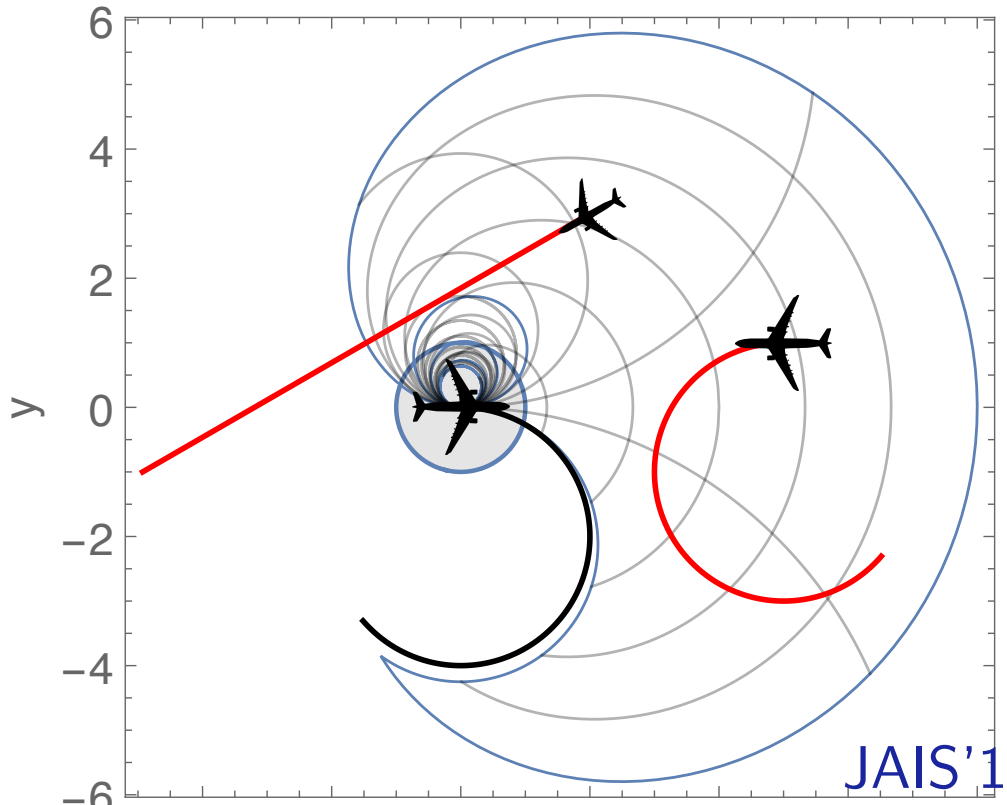
Case Study: Dubins Dynamics of 2 Airplanes

Result (DRI Automatically Generates Invariants)

$$\omega_1 = 0 \wedge \omega_2 = 0 \rightarrow v_2 \sin\vartheta x = (v_2 \cos\vartheta - v_1) y > p(v_1 + v_2)$$

$$\omega_1 \neq 0 \vee \omega_2 \neq 0 \rightarrow -\omega_1\omega_2(x^2 + y^2) + 2v_2\omega_1 \sin\vartheta x + 2(v_1\omega_2 - v_2\omega_1 \cos\vartheta) y + 2v_1 v_2 \cos\vartheta > 2v_1 v_2 + 2p(v_2|\omega_1| + v_1|\omega_2|) + p^2|\omega_1\omega_2|$$

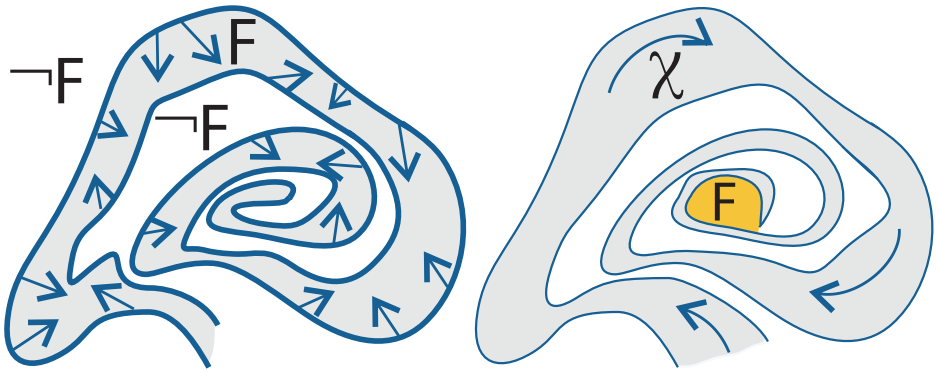




JAIS'14

Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
2. Dynamic Logic for Multi-Dynamical Systems
   - Syntax
   - Semantics
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Radical Invariants
5. Applications
6. Summary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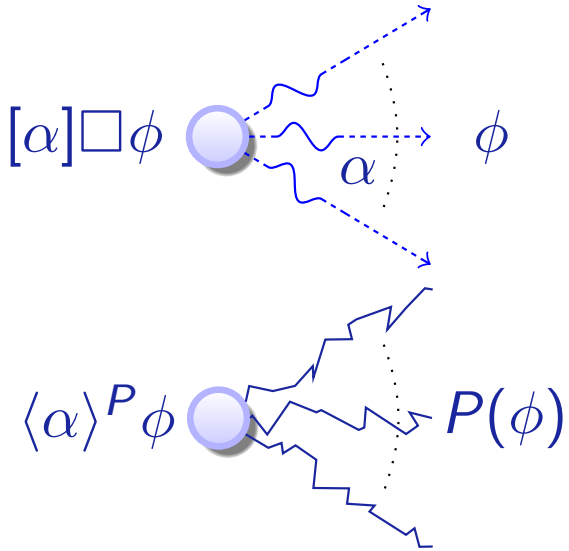
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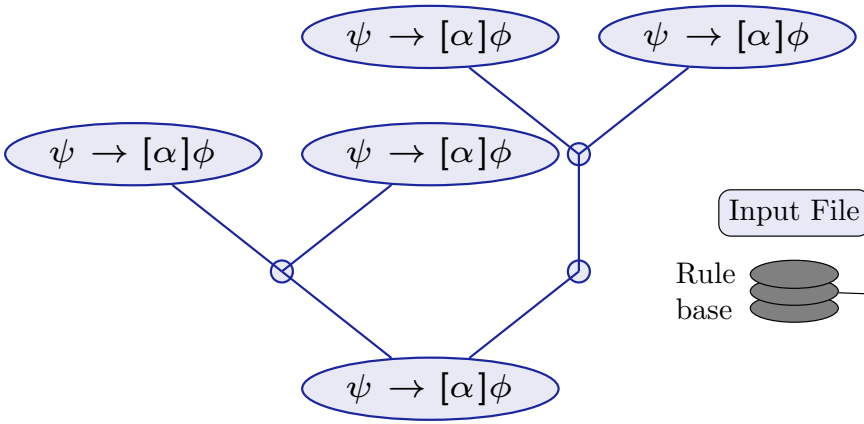
α
φ
⟨α⟩^P φ
P(φ)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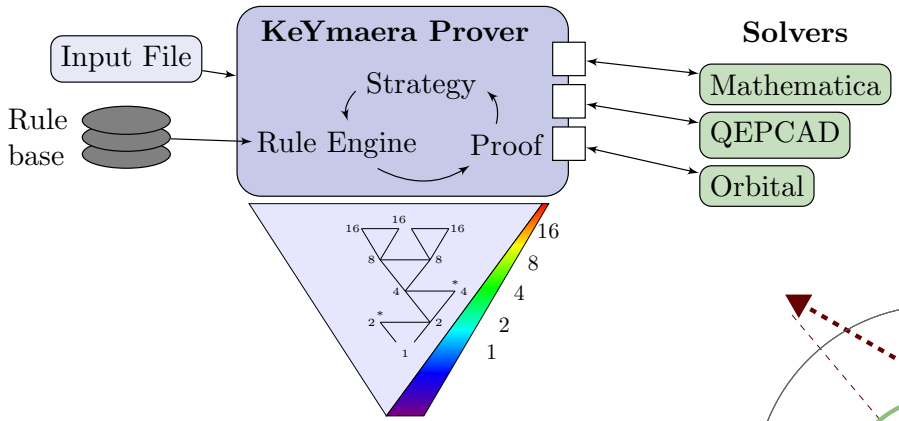
KeYmaera Prover
Input File
Strategy
Rule base
Rule Engine
Proof
Solvers
Mathematica
QEPCAD
Orbital
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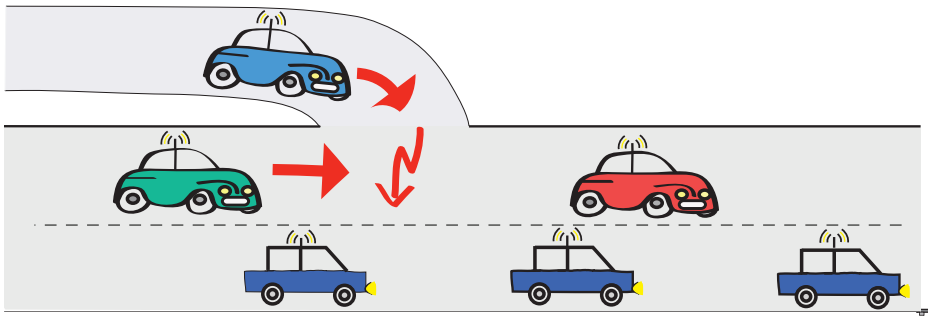
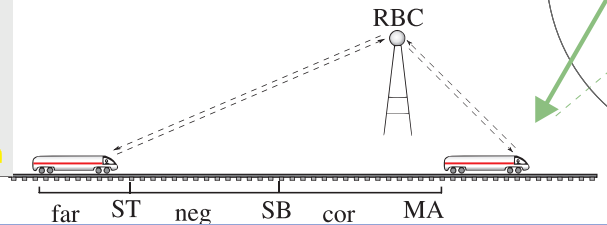
8
4
2
1




RBC
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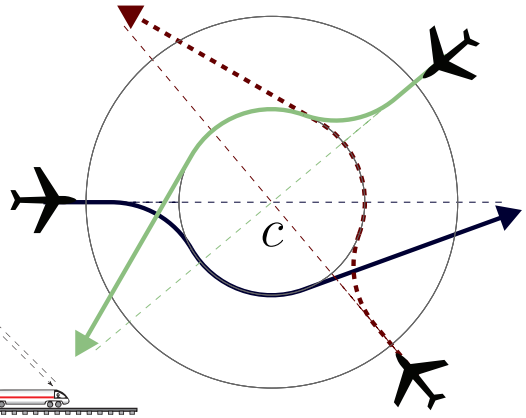
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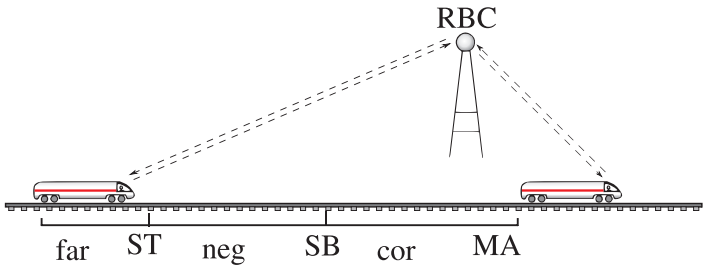
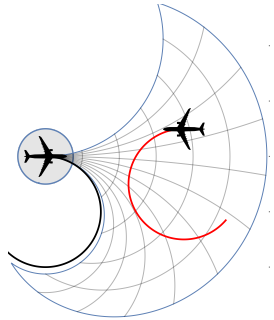
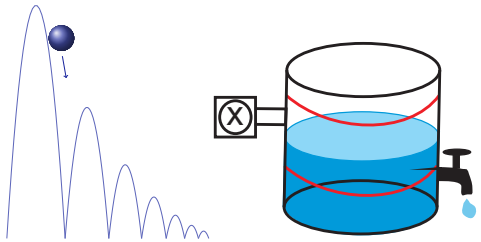
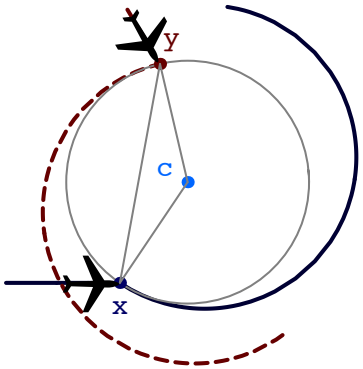
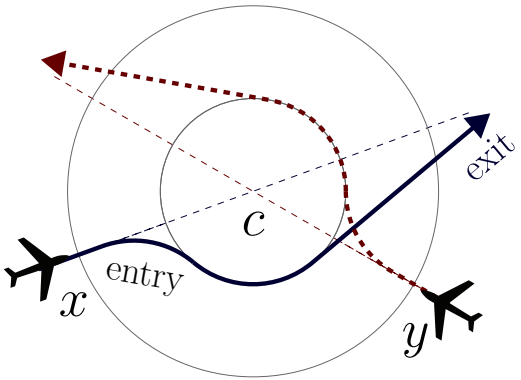
Successful CPS Proofs

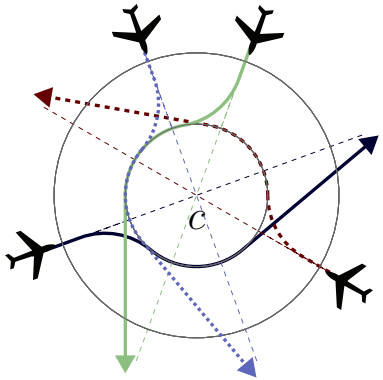


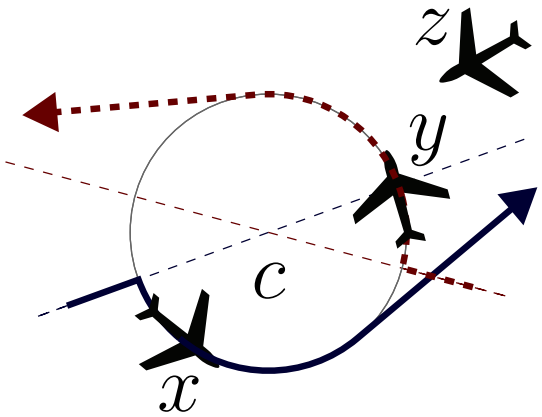


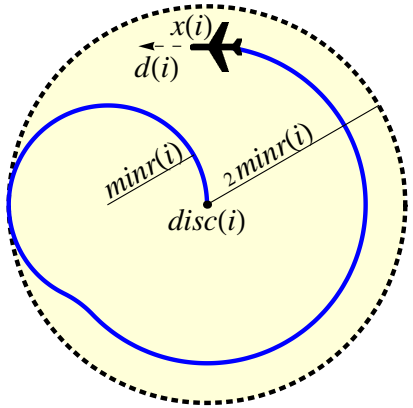


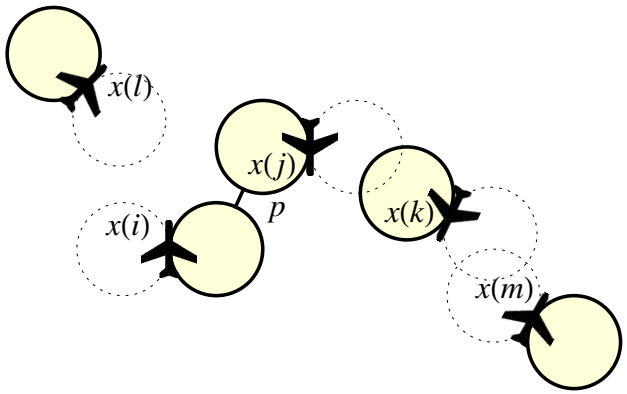






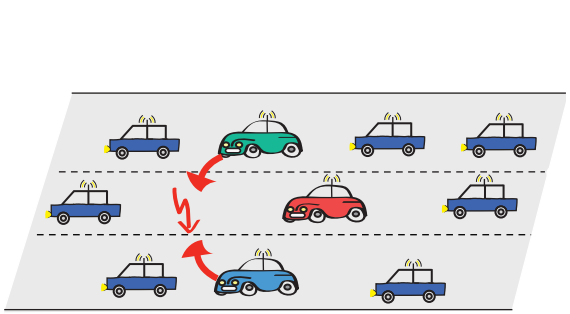


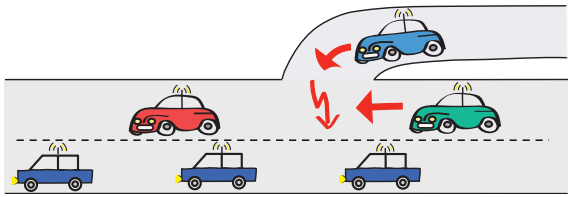


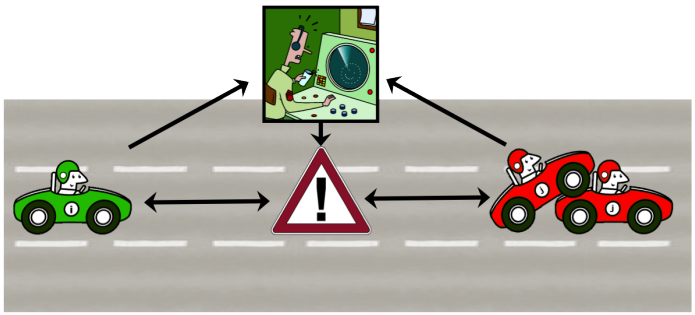


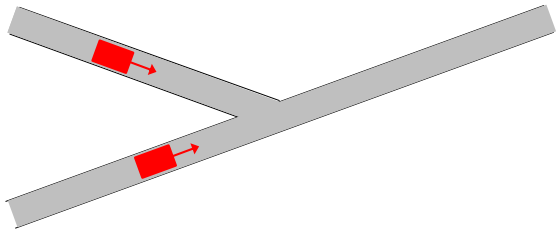
ICFEM'09,JAIS'14,CAV'08,FM'09,HSCC'11,HSCC'13

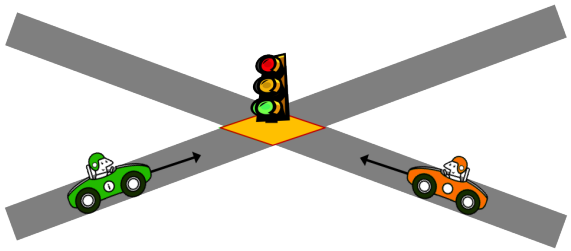
Successful CPS Proofs

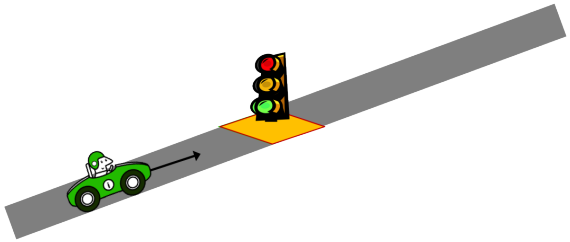


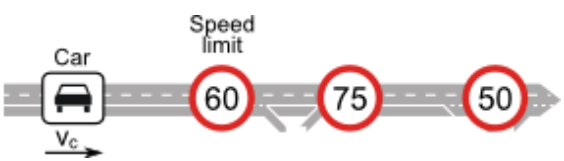


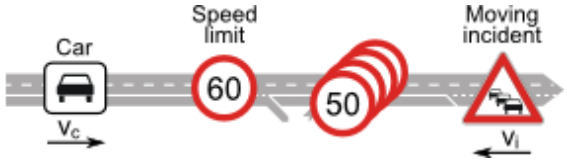


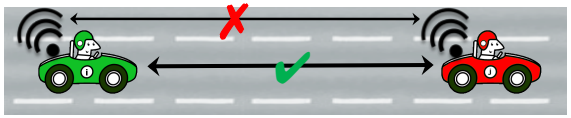


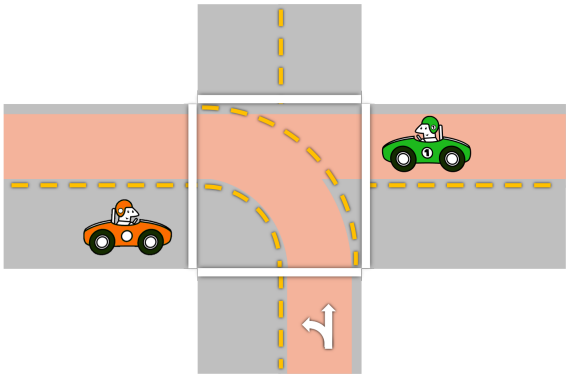


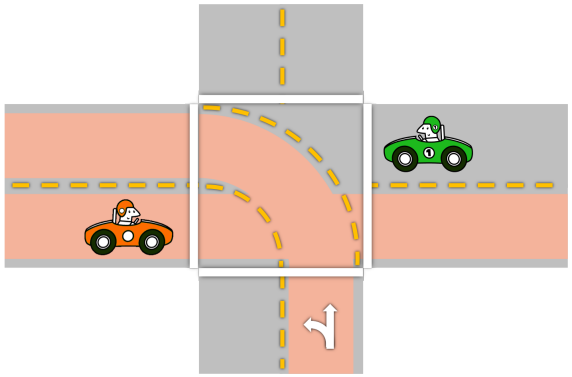


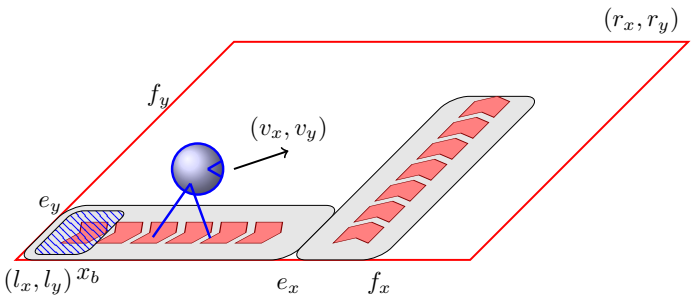






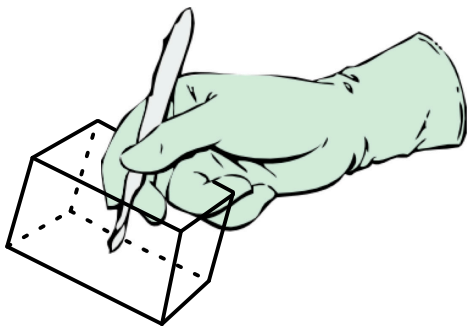


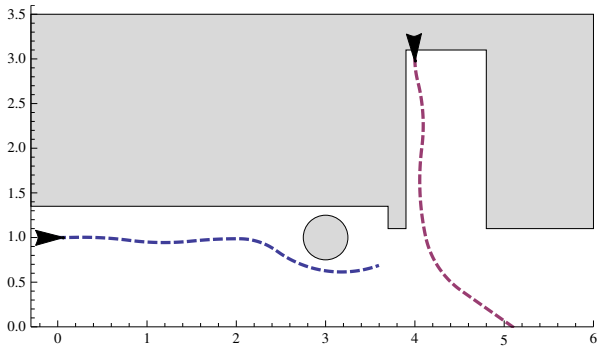


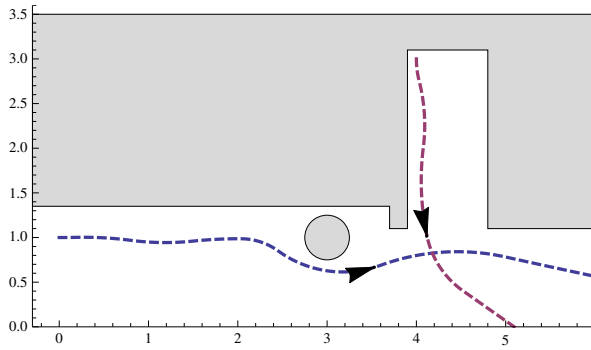


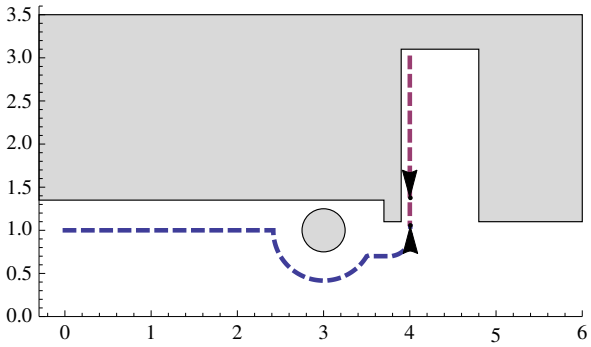
FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

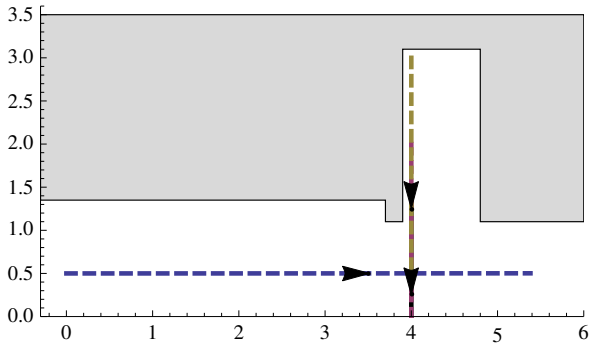
Successful CPS Proofs

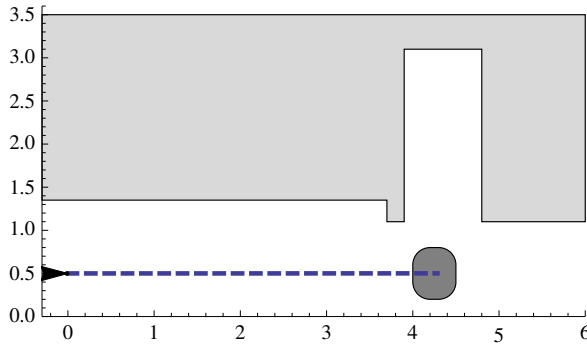


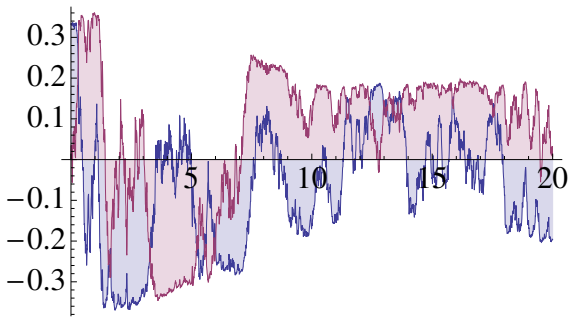


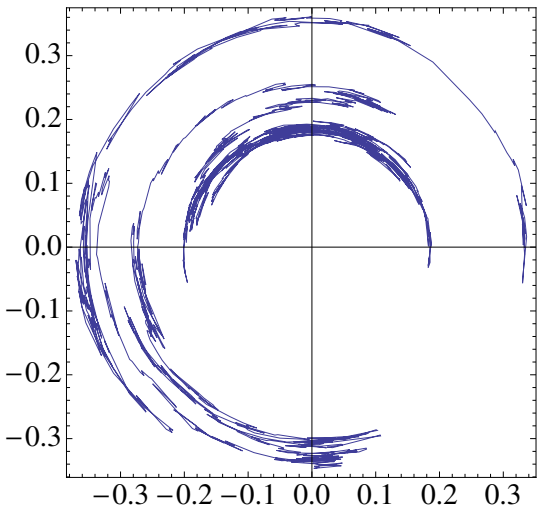


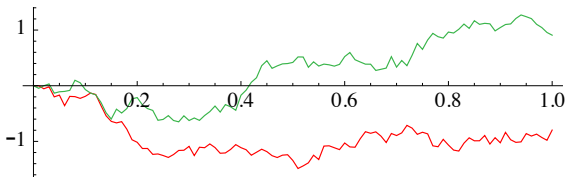






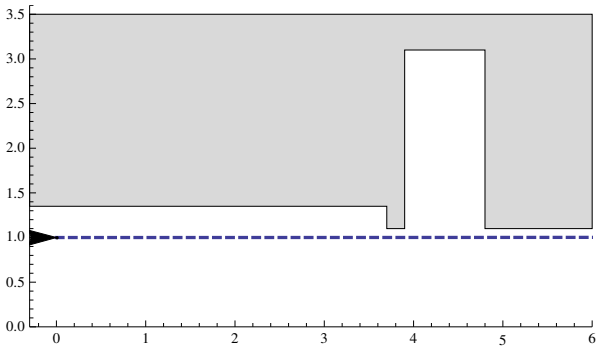
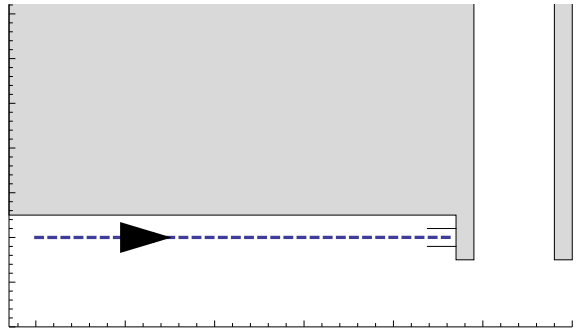
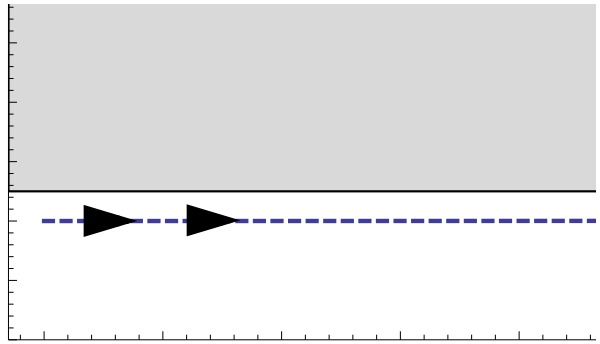
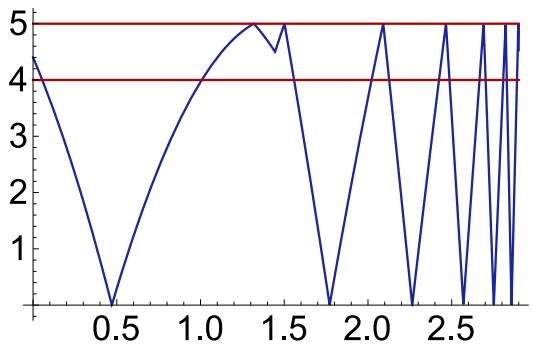
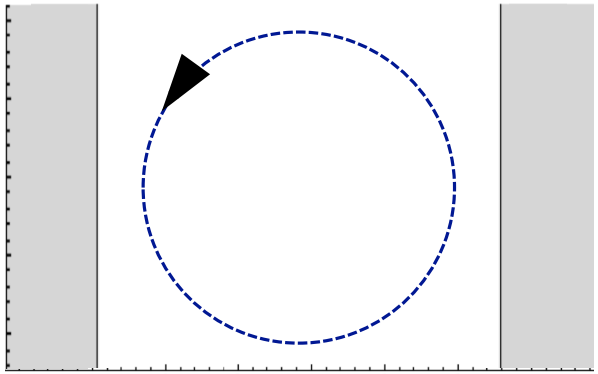
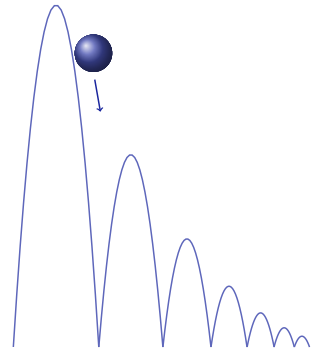
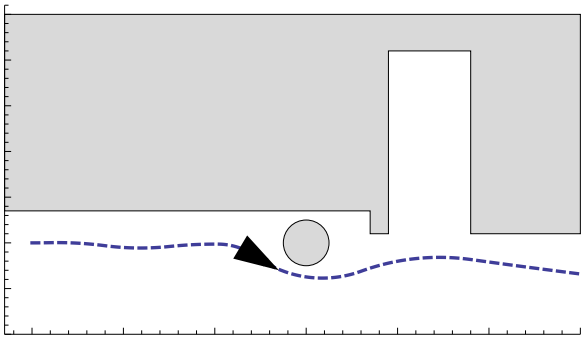
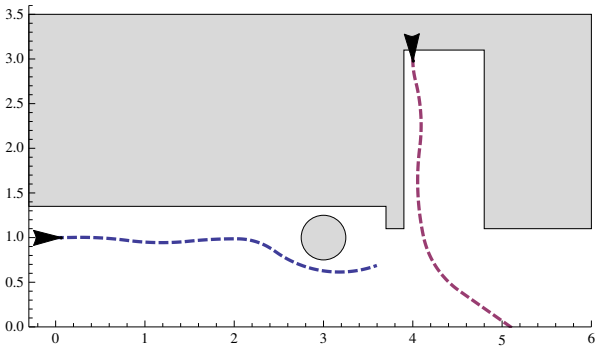
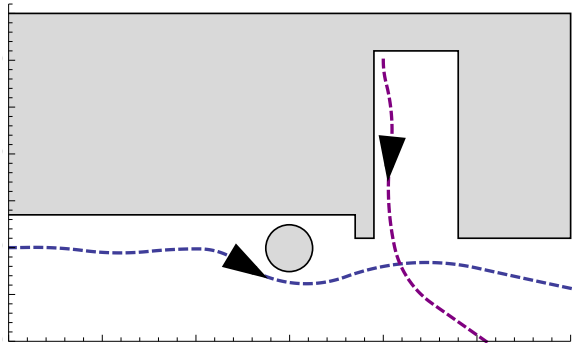






HSCC'13,RSS'13,CADE'12

Successful CPS Proofs By Undergrads

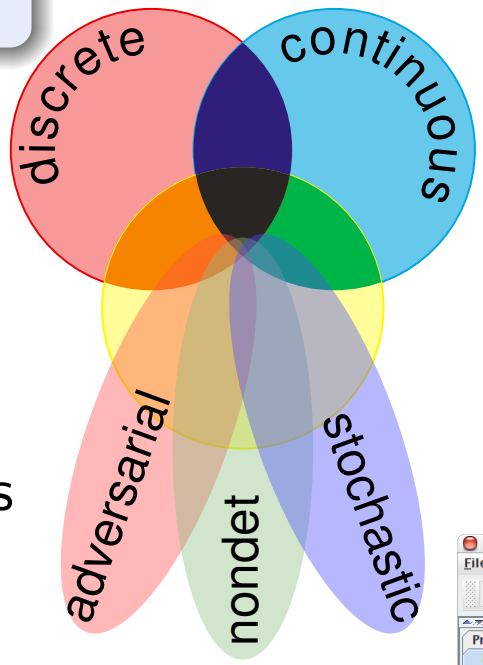
15-424/624 *Foundations of Cyber-Physical Systems* students

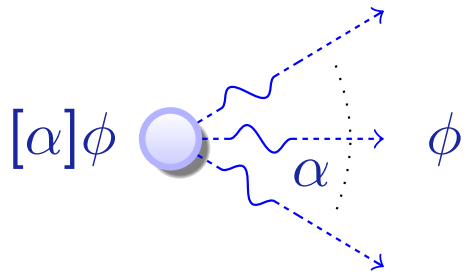
Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
2. Dynamic Logic for Multi-Dynamical Systems
   - Syntax
   - Semantics
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Radical Invariants
5. Applications
6. Summary

differential dynamic logic

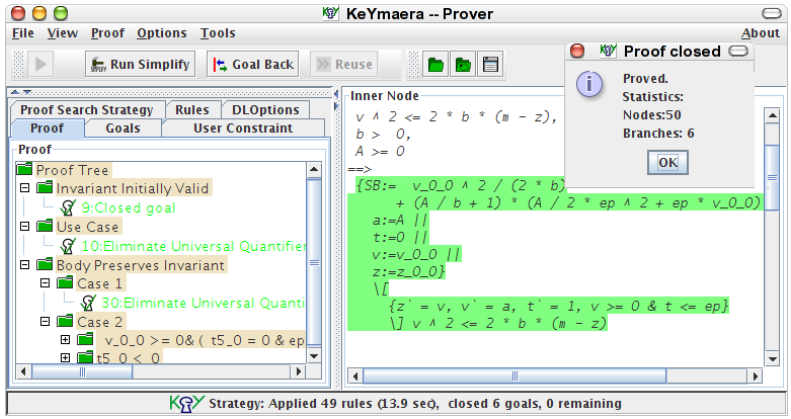
$$\mathrm{d}\mathcal{L} = \mathrm{DL} + \mathrm{HP}$$

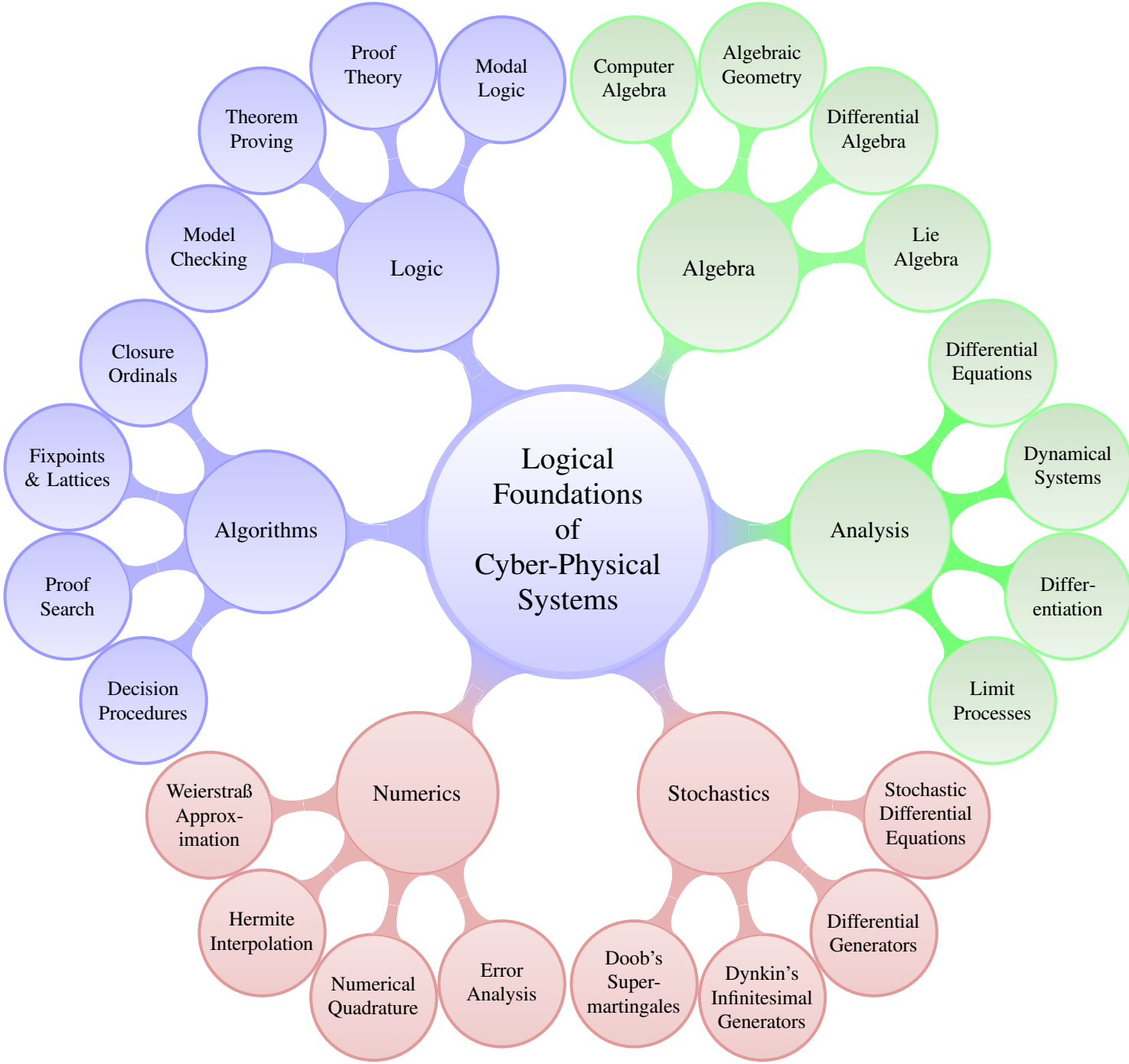




- Multi-dynamical systems
- Combine simple dynamics
- Tame complexity
- Logic & proofs for CPS
- Theory of CPS
- Applications
- Undergrad course 15-424

KeYmaera




Logical Foundations of Cyber-Physical Systems
Logic
Proof Theory
Modal Logic
Theorem Proving
Model Checking
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Lie Algebra
Analysis
Differential Equations
Dynamical Systems
Differ-entiation
Limit Processes
Algorithms
Closure Ordinals
Fixpoints & Lattices
Proof Search
Decision Procedures
Numerics
Weierstraß Approx-imation
Hermite Interpolation
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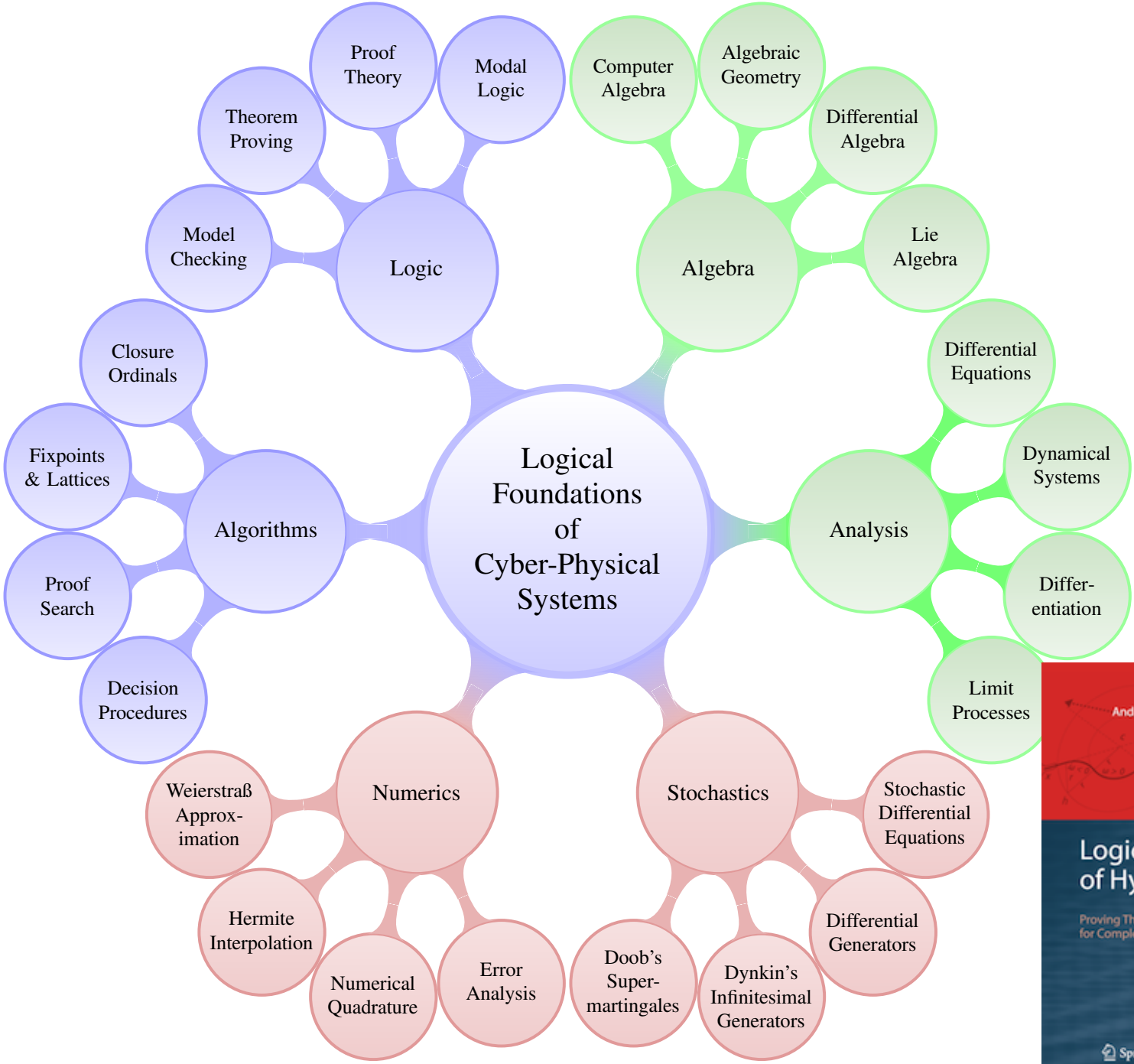
Error Analysis
Stochastics
Stochastic Differential Equations
Differential Generators
Doob's Super-martingales
Dynkin's Infinitesimal Generators


Logical Foundations of Cyber-Physical Systems
Logic
Proof Theory
Modal Logic
Theorem Proving
Model Checking
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Lie Algebra
Analysis
Differential Equations
Dynamical Systems
Differ-entiation
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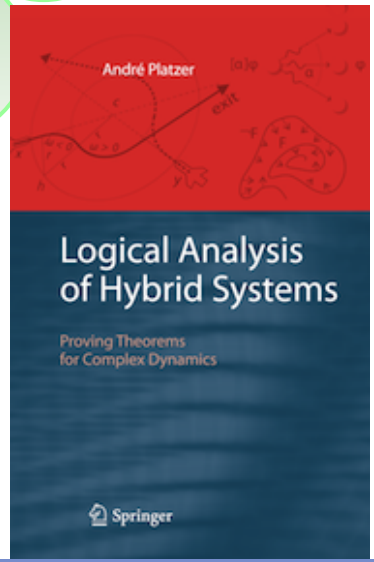
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   - Completeness Proof
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10. Deduction Modulo Real Algebraic and Computer Algebraic Constraints
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	Op	Par	T	CI	Tec	Aut	Cex	Dim	
HenzingerH94, HyTech	✓	×	✓	×	✓	✓	✓		LHA
LafferrierePY99	✓	×	✓	×	✓		✓		forgetful reset
Fränzle99	✓	×	✓	×	✓		✓	×	robust systems
CKrogh03, CheckMate	✓	×	✓	×	✓	✓	✓		polyhedral
Frehse05, PHAVer	✓	×	✓	×	✓	✓	✓	8	LHA (+affine)
MysorePM05	✓	×	✓	×	✓	●	✓	4	bounded prefix
TomlinPS98,MBT05	○	×	×	×	○	○	●	4	HJB numPDE
RatschanS07, HSolver	✓	×		×	✓	✓	×	4	interval
MannaS98, STeP	✓			×	✓	○	×	7	inv↦VCG, flat
ÁbrahámSH01, PVS	●			×	●	○	×	≈9	HA↪PVS, -"-
ZhouRH92, EDC	×	●	✓	..	×	×	×	×	no maths
DavorenN00, Lμ	×	×		✓	○	×	×	×	prop. H-semantics
RönkköRS03, HGC	✓	×	×	×	×	×	×	×	HGC↪HOL
SSManna04	●	○		×	✓		×	4/1	equational system
CTiwari05	●	○		×	✓		×	6/0	linear, -"-
PrajnaJP07, barrier	●	×		×	●		×	3	needs 10000-dim
d$\mathcal{L}$ & dTL	✓	✓	✓	✓	✓	●	×	28	expr., compos.

	Dom	Op	Base	Modal	Quant	Cmpl	Aut
DL	$\mathbb{N}$		$\text{FOL}_{(\mathbb{N})}$		FV+unify	$/\mathbb{N}$	
d$\mathcal{L}$	$\mathbb{R}$	x'	$\text{FOL}_{\mathbb{R}}$	ODE	FV+requant+QE	/ODE	IBC

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Proof (Soundness).

- $x' = f(x)$
- Side deductions
- Free variables & Skolemisation

Return

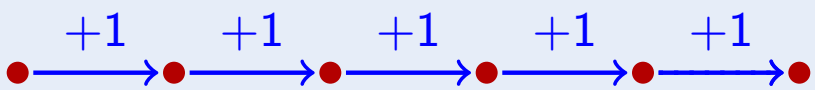
Theorem

Discrete fragment and continuous fragment of $\mathrm{d}\mathcal{L}$ *characterize* $\mathbb{N}$

Proof.

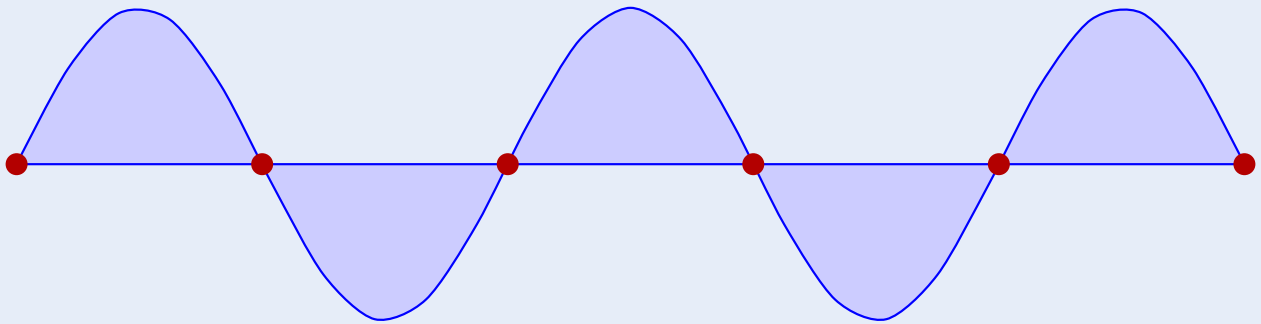
Discrete fragment:

$$\langle (x := x + 1)^* \rangle\, x = n$$



Continuous fragment:

$$\langle s'' = -s, \tau' = 1 \rangle (s = 0 \wedge \tau = n) \qquad \rightsquigarrow s = \sin$$



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Incomplete! But are we missing proof rules?

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Relativity

Cook,Harel: discrete-DL/data$_{\mathbb{N}}$ hybrid-d$\mathcal{L}$/data$_{\mathbb{R}}$??

Sources of Incompleteness

Sources of Incompleteness

continuous

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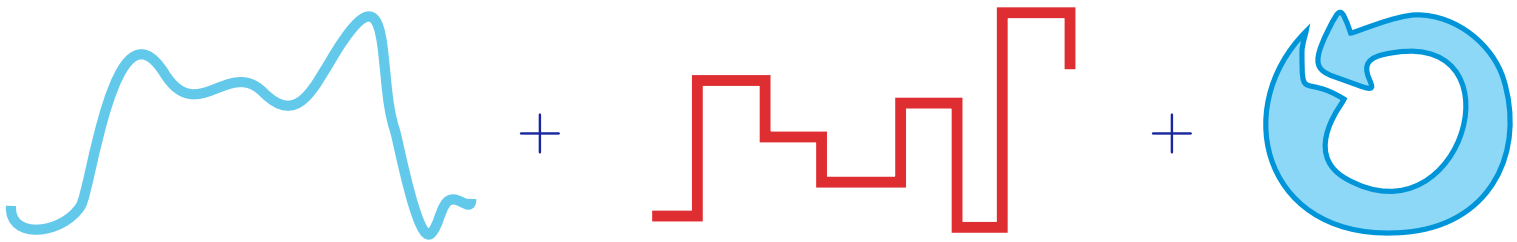

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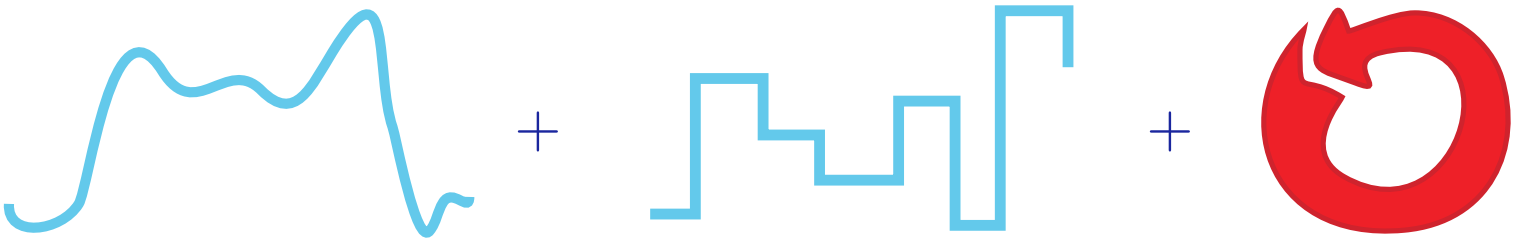

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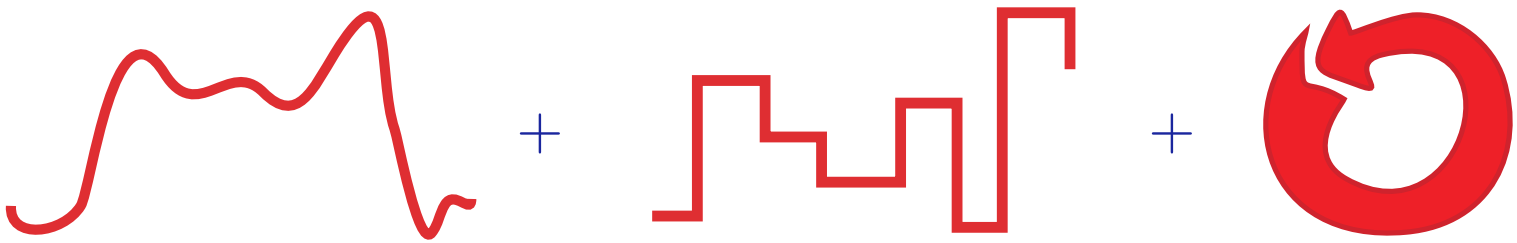

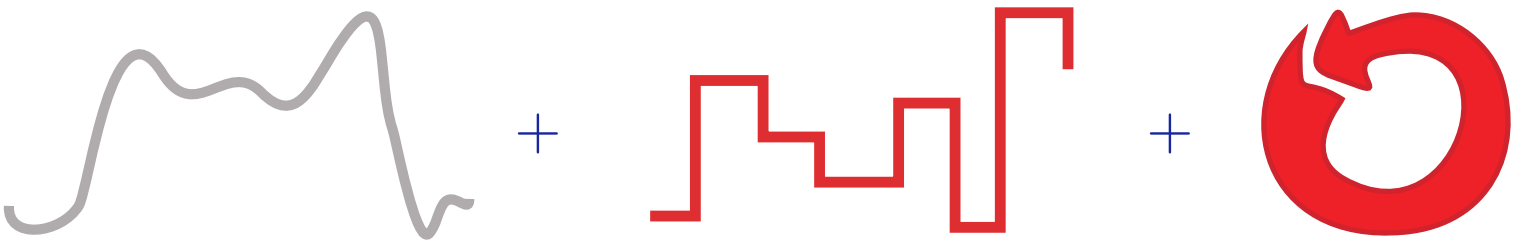
continuous
discrete
repeat

Relative Completeness

Theorem (Relative Completeness)

d$\mathcal{L}$ calculus is complete relative to first-order logic of differential equations.

$$\models \phi \quad \text{iff} \quad \mathit{Taut}_{\mathsf{FOD}} \vdash \phi$$

where $\mathsf{FOD} = \mathsf{FOL}_{\mathbb{R}} + [x_1' = \theta_1, \ldots, x_n' = \theta_n]F$

▸ Proof Outline 15p

continuous + discrete + repeat

$\Downarrow$

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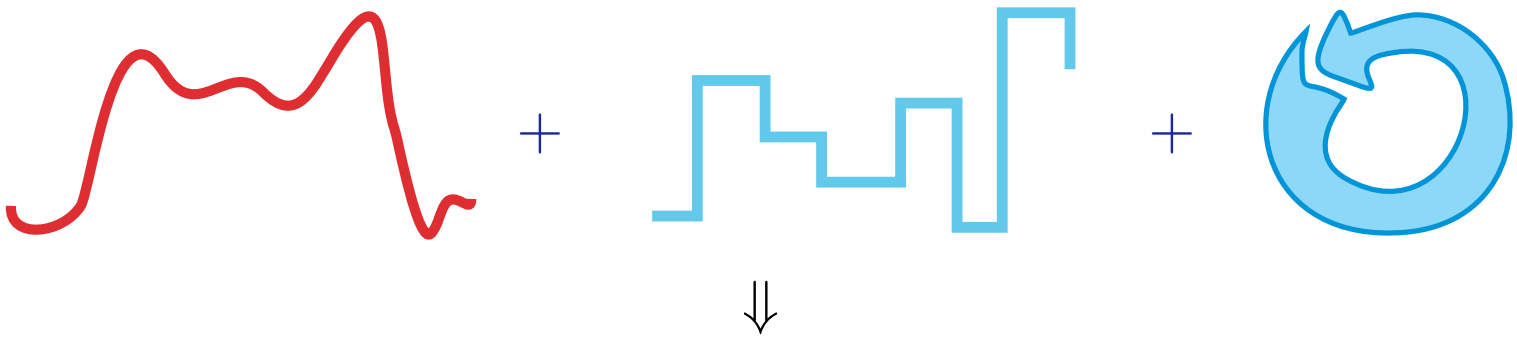

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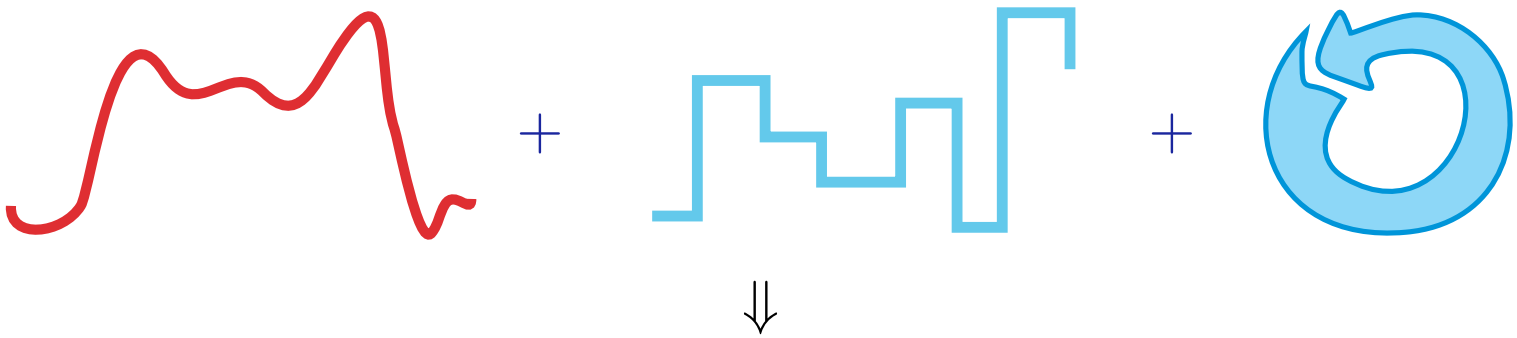

Corollary (Proof-theoretical Alignment)

verification of hybrid systems = verification of dynamical systems!

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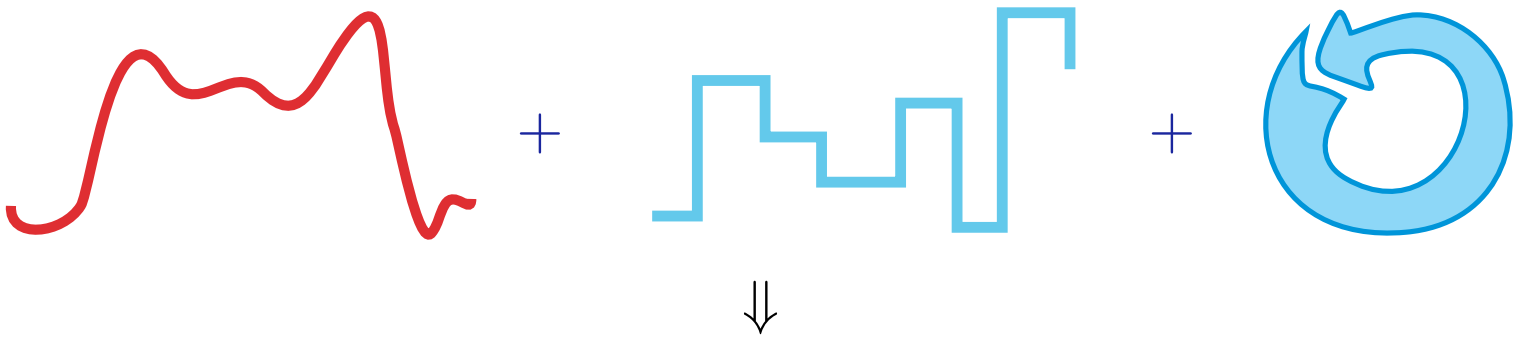

Corollary (Deductive Power)

d$\mathcal{L}$ calculus is *supremal hybrid* verification technique

Relative Completeness Proof

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Proof (Relative Completeness, 10 pages)

Return

1. Strong invariants and variants expressible in d$\mathcal{L}$
2. d$\mathcal{L}$ expressible in FOD
3. valid d$\mathcal{L}$ formulas d$\mathcal{L}$-derivable from corresponding FOD axioms
4. finite FOD formula characterising unbounded hybrid repetition
5. FOD characterises $\mathbb{R}$-Gödel encoding
6. First-order expressible & program rendition: $\forall\phi\ \exists F \in \mathsf{FOD} \quad \models \phi \leftrightarrow F$
7. Propositionally & first-order complete
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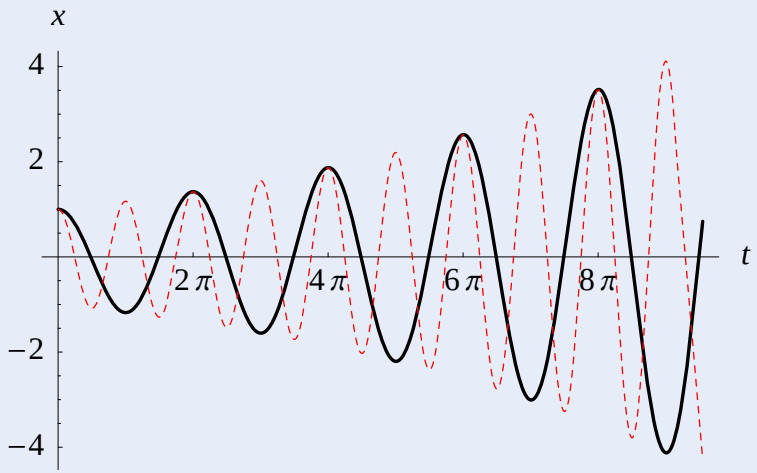
Proof ($\mathbb{R}$-Gödel encoding)

Return

FOD characterises constructive bijection $\mathbb{R} \to \mathbb{R}^2$

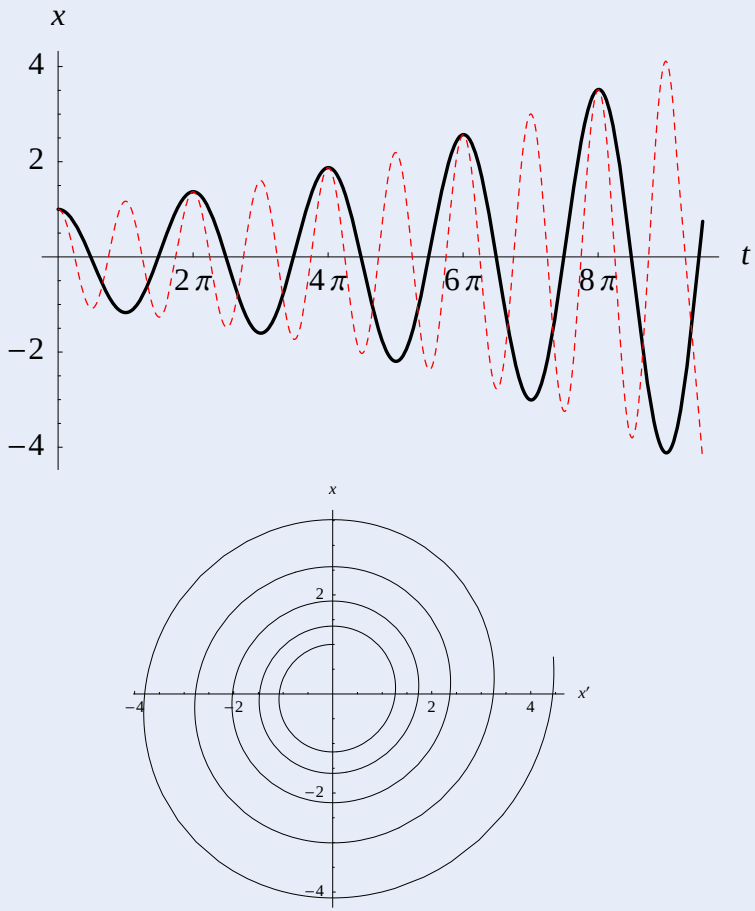


where $\mathsf{FOD} = \mathsf{FOL}_{\mathbb{R}} + [x_1' = \theta_1, \ldots, x_n' = \theta_n]F$

Proof ($\mathbb{R}$-Gödel encoding)

Return

FOD characterises constructive bijection $\mathbb{R} \to \mathbb{R}^2$



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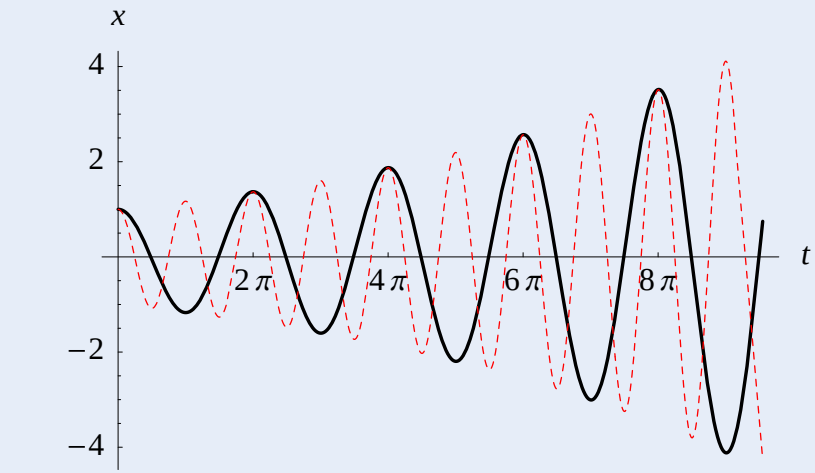

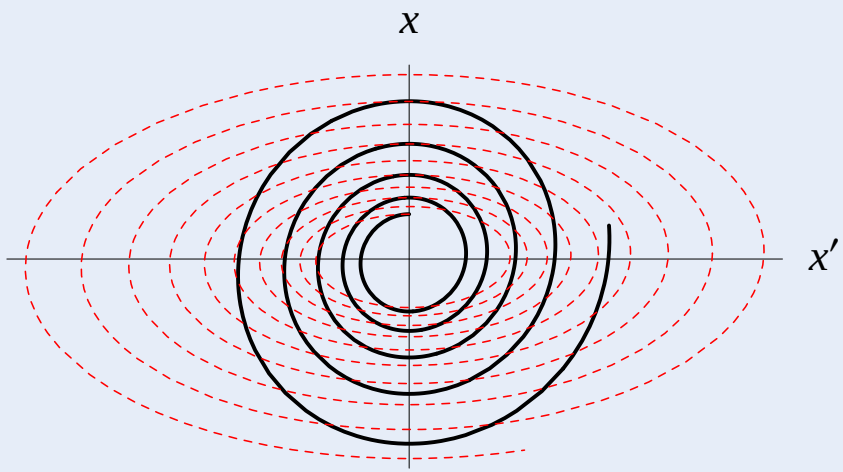

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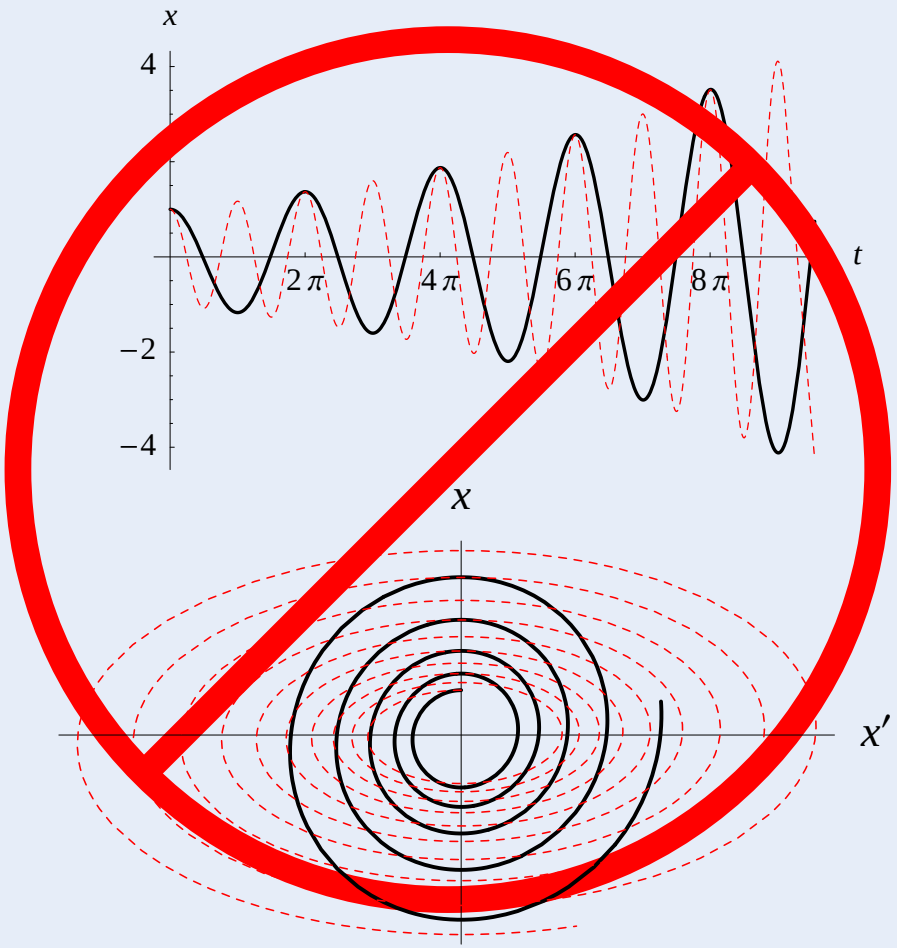
Proof ($\mathbb{R}$-Gödel encoding)

Return

FOD characterises constructive bijection $\mathbb{R} \to \mathbb{R}^2$ not differentiable!



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Proof ($\mathbb{R}$-Gödel encoding)

Return

FOD characterises constructive bijection $\mathbb{R} \to \mathbb{R}^2$

$$\sum_{i=1}^{\infty} \frac{a_i}{2^i} = 0.a_1 a_2 \ldots \qquad \sum_{i=1}^{\infty} \frac{b_i}{2^i} = 0.b_1 b_2 \ldots$$

$$\Leftrightarrow \quad \sum_{i=0}^{\infty} \left(\frac{a_i}{2^{2i+1}} + \frac{b_i}{2^{2i+2}} \right) = 0.a_1 b_1 a_2 b_2 \ldots$$

where $\text{FOD} = \text{FOL}_{\mathbb{R}} + [x_1' = \theta_1, \ldots, x_n' = \theta_n]F$

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$$\Rightarrow \sum_{i=0}^{\infty} \left(\frac{a_i}{2^{2i+1}} + \frac{b_i}{2^{2i+2}} \right) = 0.a_1 b_1 a_2 b_2 \ldots$$

$$2^n = z \quad \leftrightarrow \quad \langle x := 1; \tau := 0; x' = x \ln 2 \wedge \tau' = 1 \rangle (\tau = n \wedge x = z)$$

$$\ln 2 = z \quad \leftrightarrow \quad \langle x := 1; \tau := 0; x' = x \wedge \tau' = 1 \rangle (x = 2 \wedge \tau = z)$$

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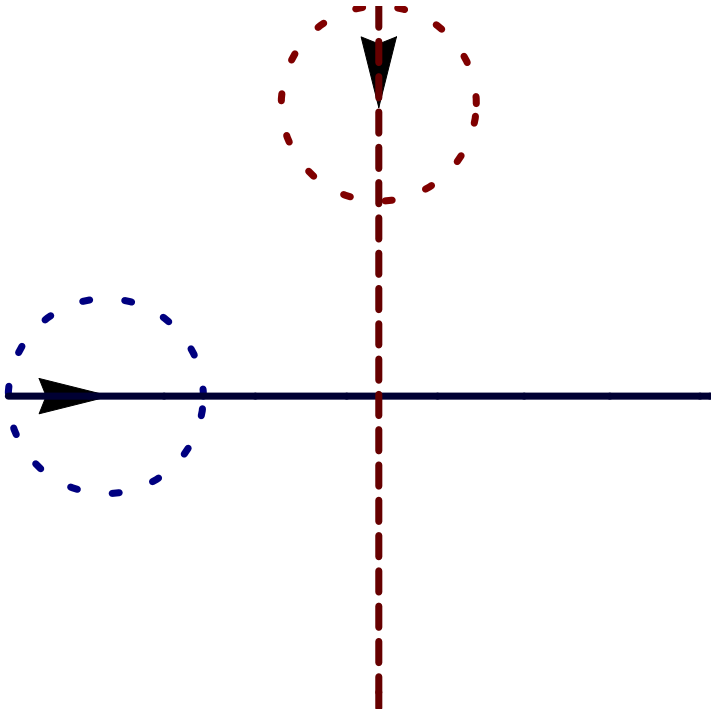
Air Traffic Control



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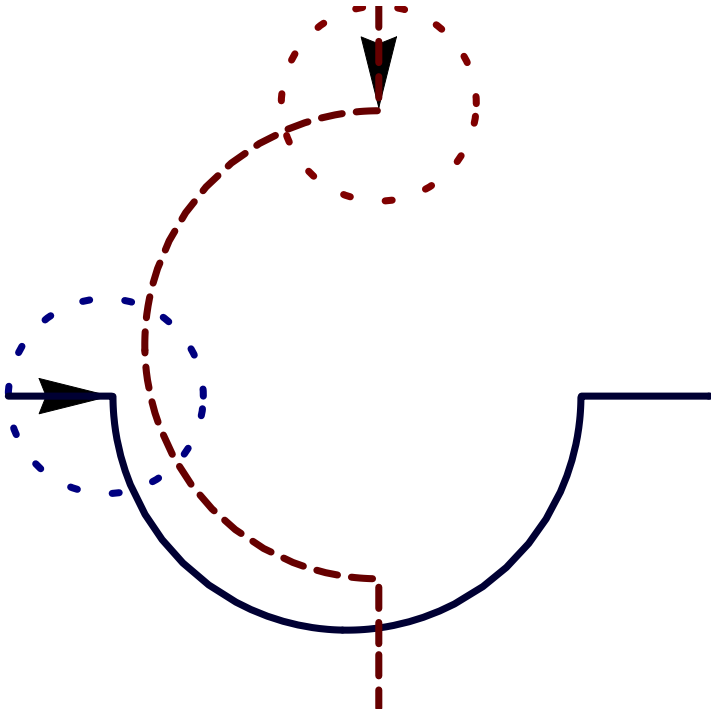

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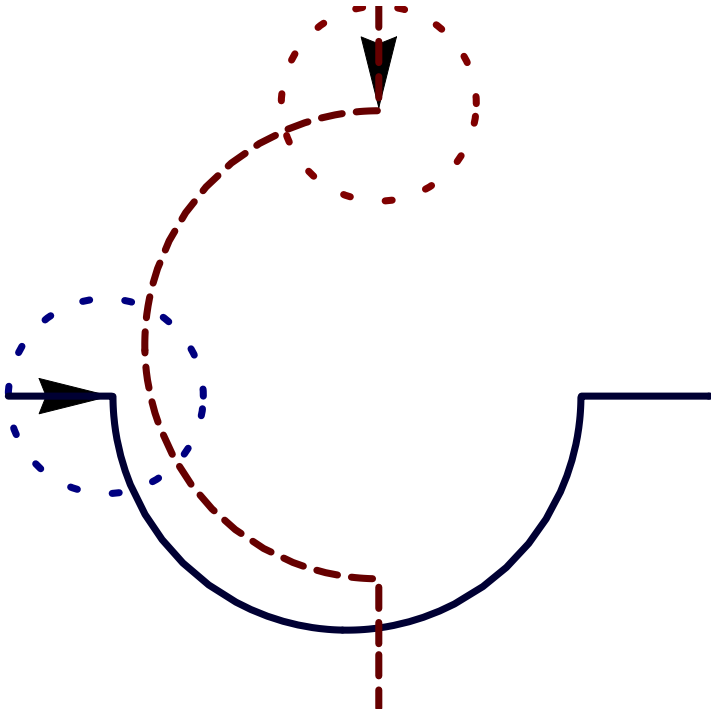
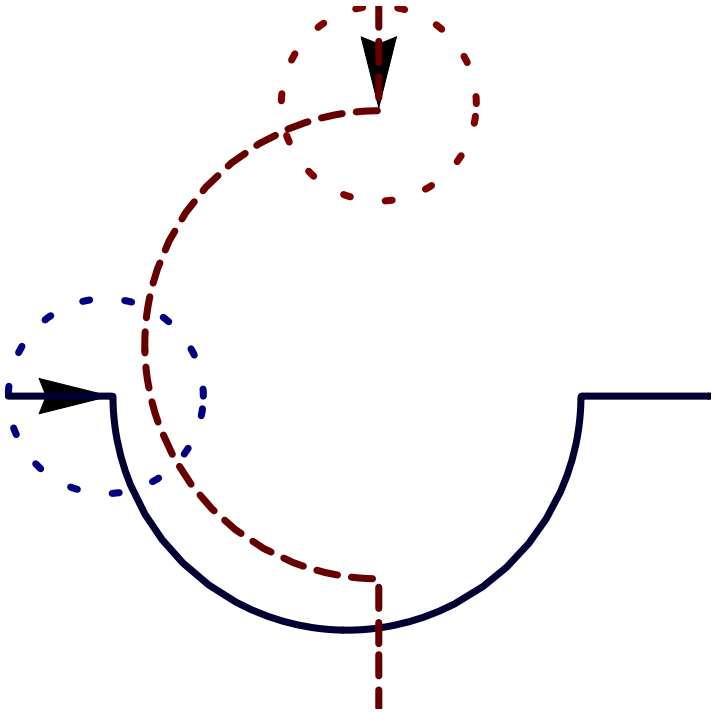
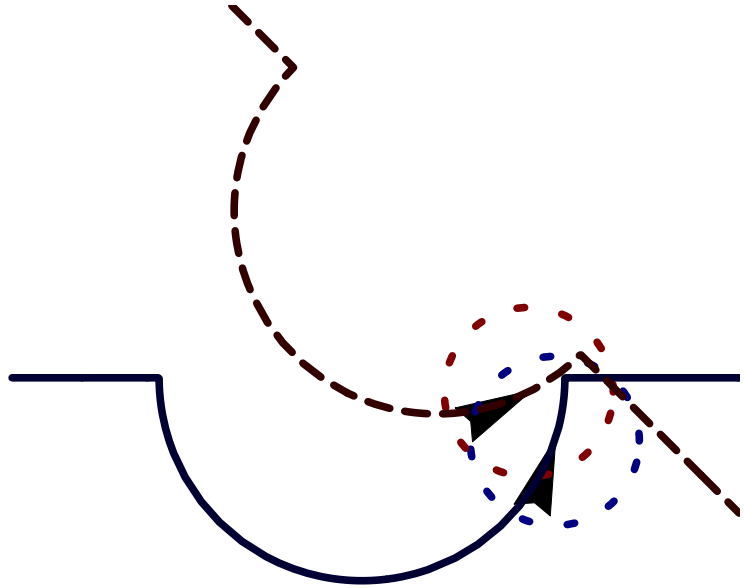


Verification?

looks correct

Air Traffic Control

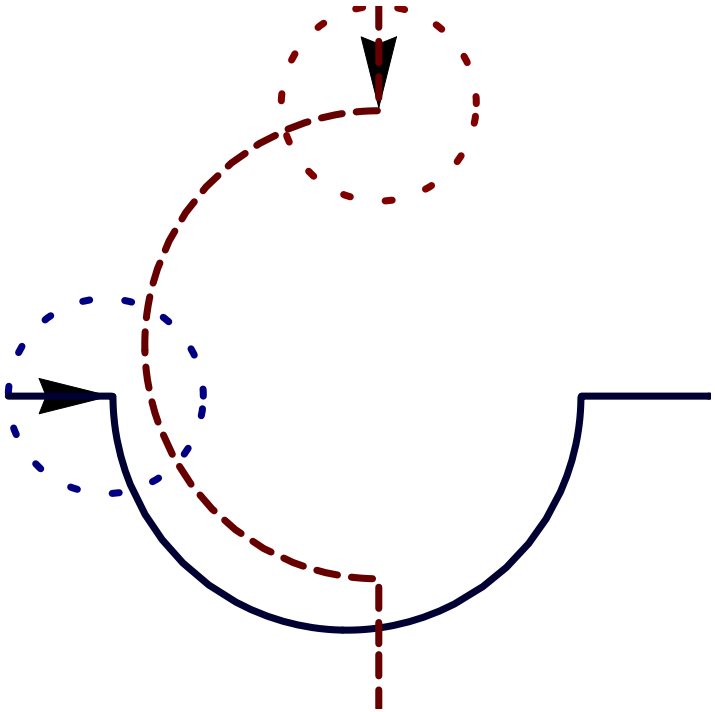


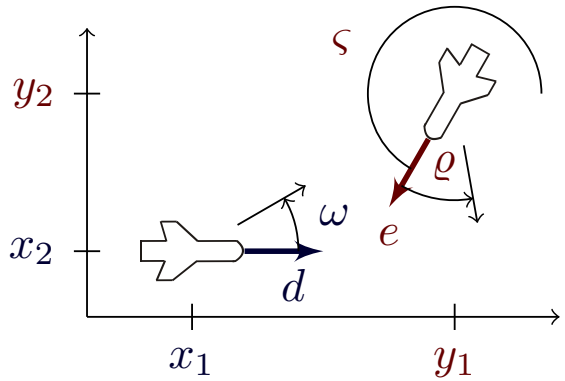


Verification?

looks correct NO!

Air Traffic Control





$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

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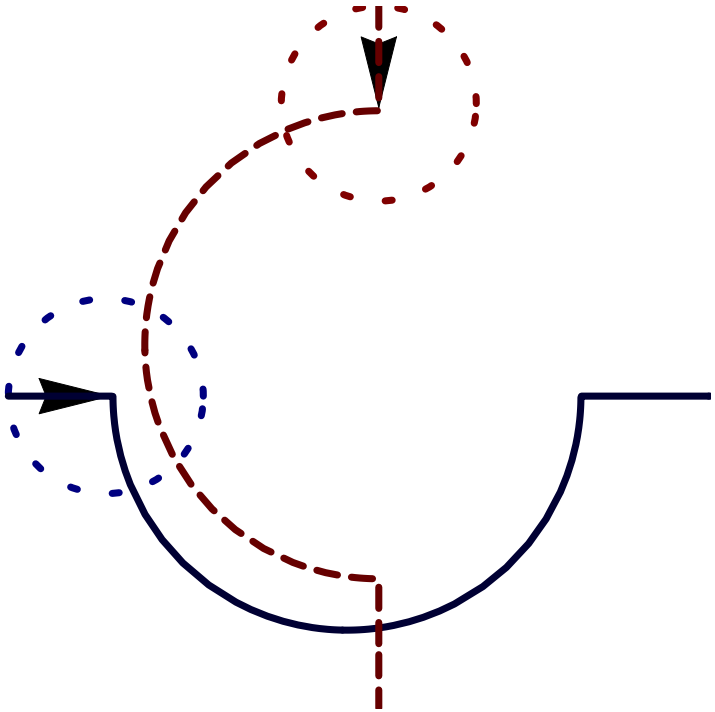
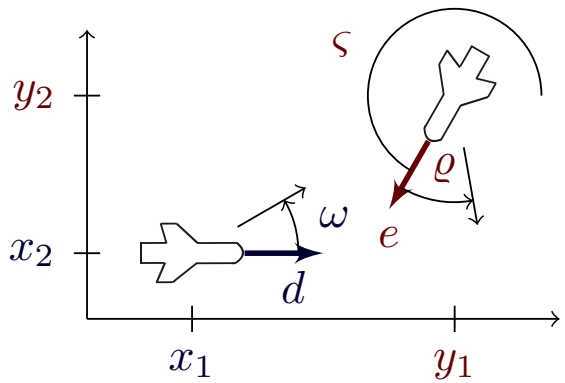
looks correct NO!





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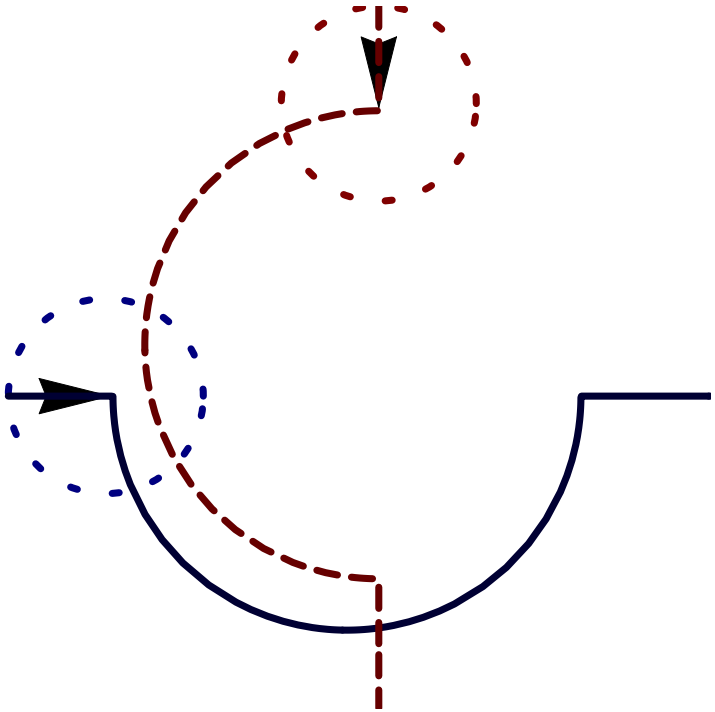
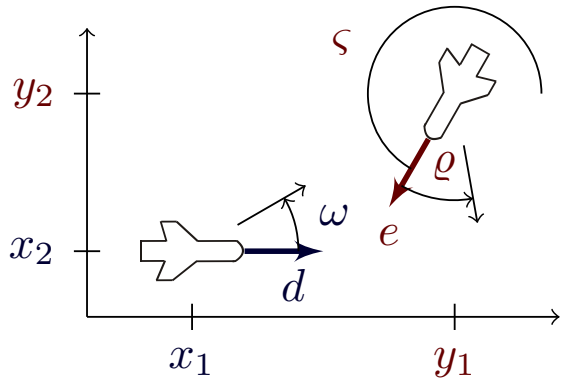
Example ("Solving" differential equations)

$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$

Air Traffic Control





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$$\forall t \geq 0 \quad \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
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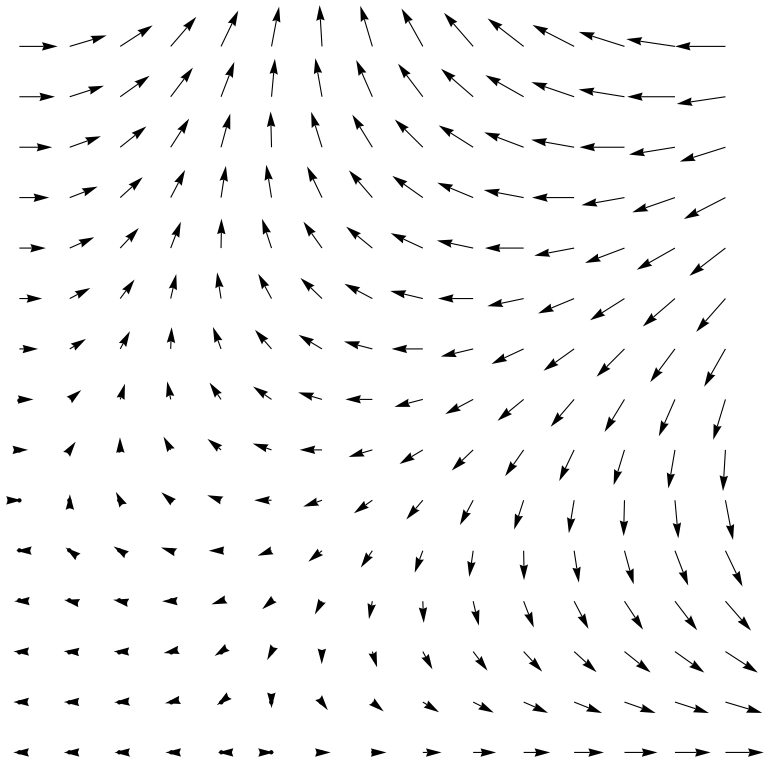
"Definition" (Differential Invariant)

"Formula that remains true in the direction of the dynamics"



JLogComput'10,CAV'08,FMSD'09,LMCS'12

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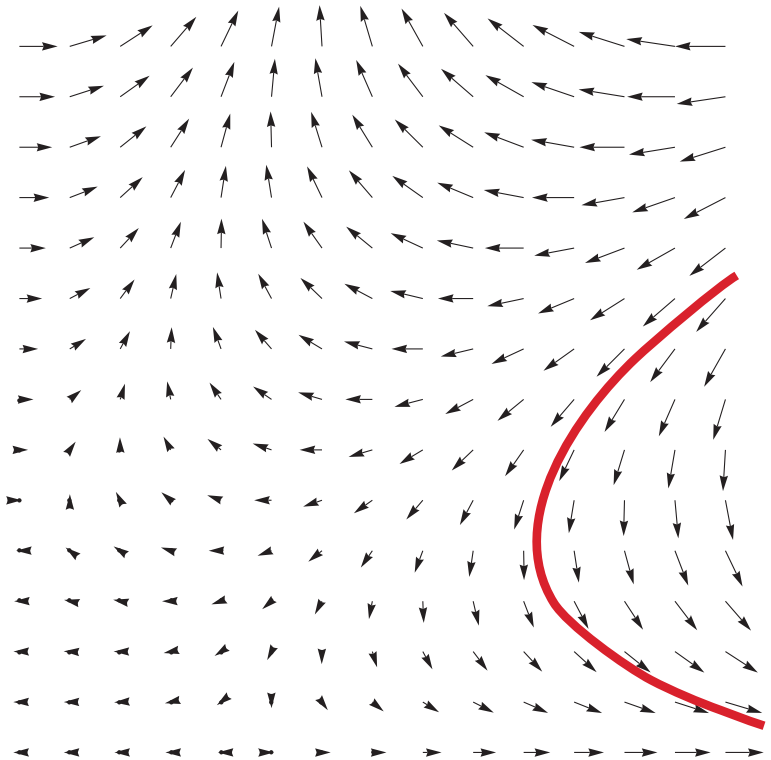

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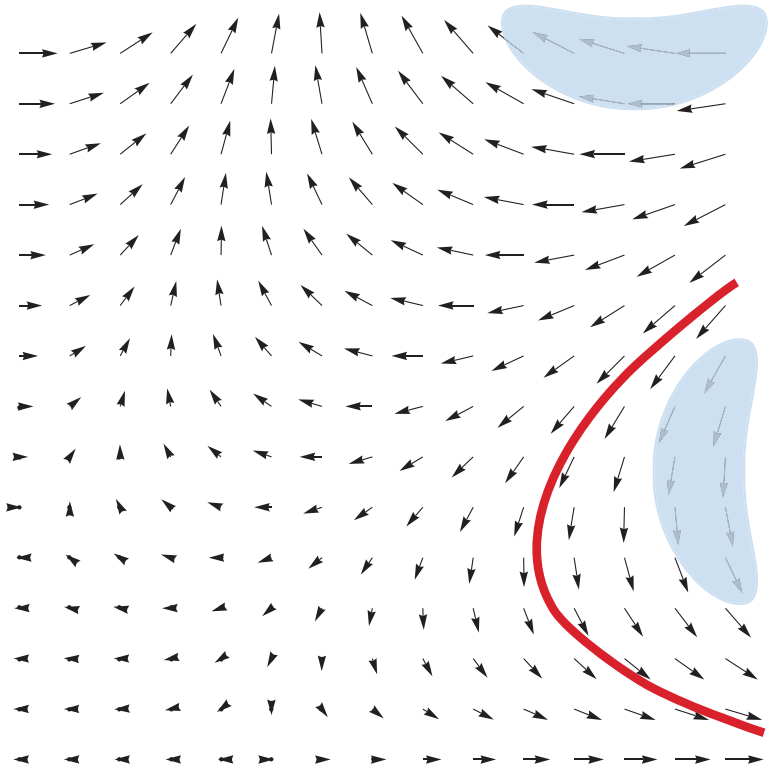

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Temporal Modalities + Dynamic Modalities

problem	technique	Op	Par	T	closed
$train \models z < M$	TL-MC	✓	×	✓	×
$\models (\mathrm{Ax}(train) \rightarrow z < M)$	TL-calculus	×	...	✓	...
$\models [train]\, z < M$	DL-calculus	✓	✓	×	✓
$\models [train]\Box\, z < M$	dTL-calculus	✓	✓	✓	✓

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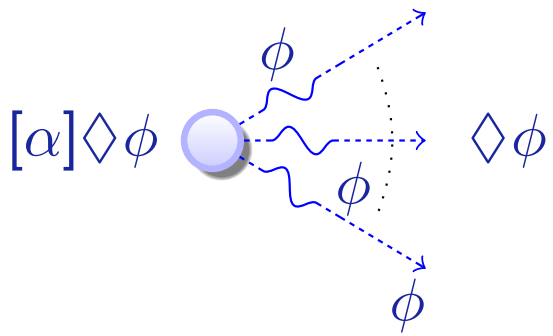
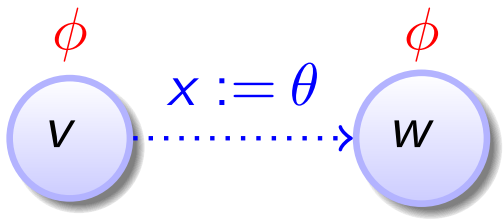
$\models [train]\Box\, z < M$	dTL-calculus	✓	✓	✓	✓

differential temporal dynamic logic

$$\mathrm{dTL} = \mathrm{TL} + \mathrm{DL} + \mathrm{HP}$$

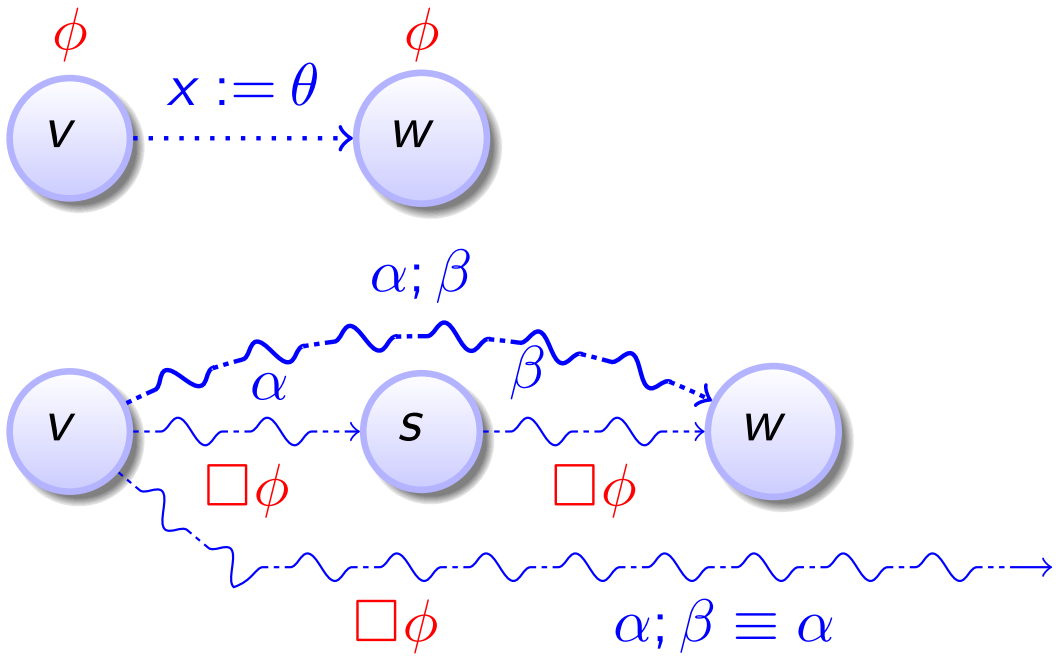


$$\frac{\phi \wedge [x := \theta]\phi}{[x := \theta]\Box\phi}$$



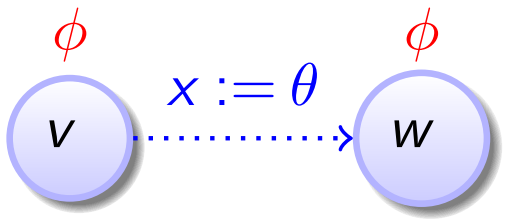
$$\frac{\phi \wedge [x := \theta]\phi}{[x := \theta]\Box\phi}$$

$$\frac{[\alpha]\Box\phi \wedge [\alpha][\beta]\Box\phi}{[\alpha;\beta]\Box\phi}$$

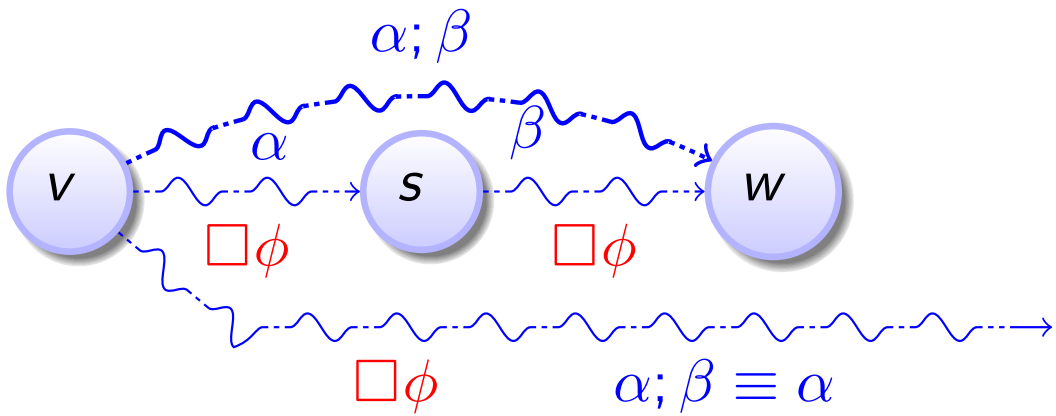


Proof Calculus for Temporal dTL

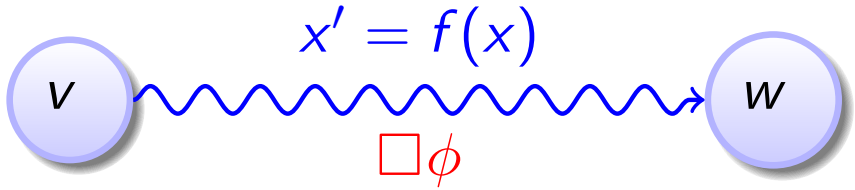
$$\frac{\phi \wedge [x := \theta]\phi}{[x := \theta]\Box\phi}$$



$$\frac{[\alpha]\Box\phi \wedge [\alpha][\beta]\Box\phi}{[\alpha;\beta]\Box\phi}$$



$$\frac{[x' = \theta]\phi}{[x' = \theta]\Box\phi}$$



Proof Calculus for Temporal dTL

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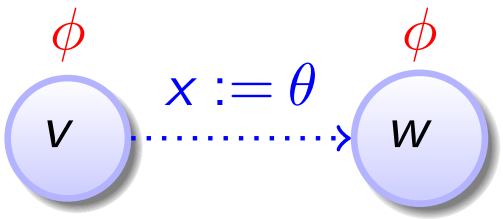
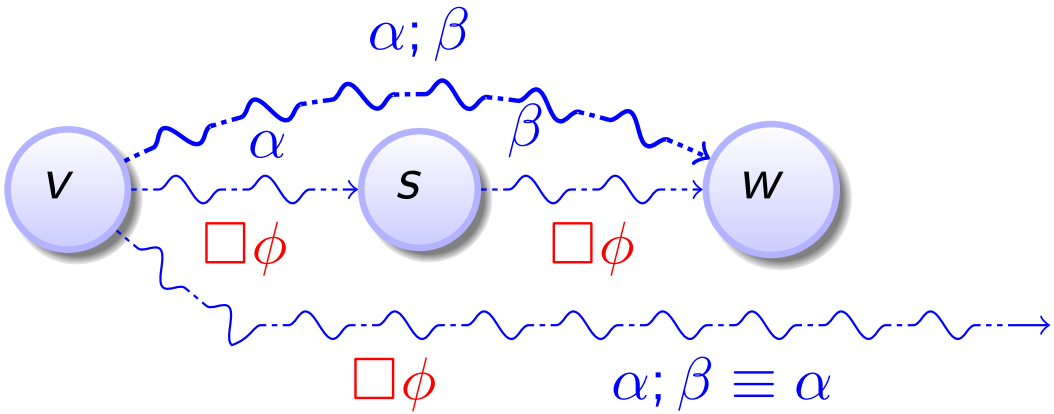
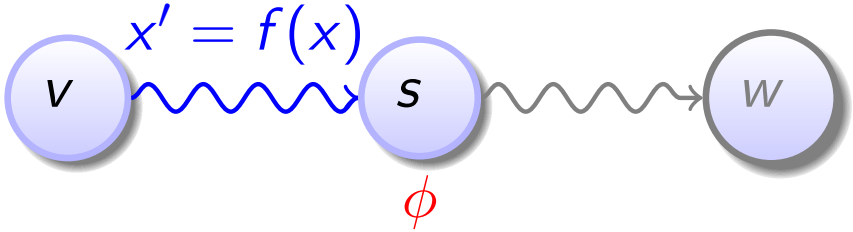


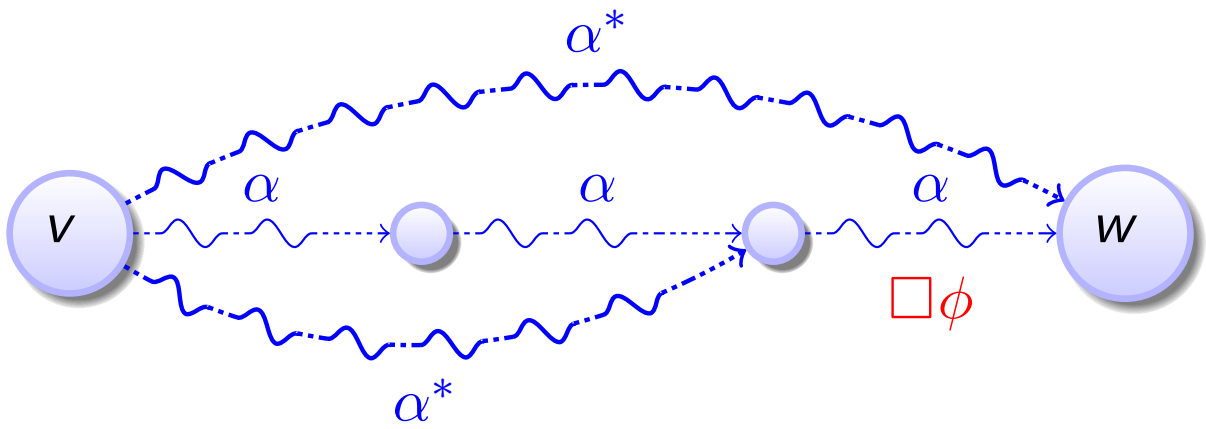
$$\frac{[\alpha]\Box\phi \wedge [\alpha][\beta]\Box\phi}{[\alpha;\beta]\Box\phi}$$



$$\frac{[x' = \theta]\phi}{[x' = \theta]\Box\phi}$$



$$\frac{[\alpha^*][\alpha]\Box\phi}{[\alpha^*]\Box\phi}$$



Theorem (Relative Completeness) (P. 2008)

dTL *calculus is a sound & complete axiomatization relative to* d$\mathcal{L}$.

Corollary (Continuous Relative Completeness)

dTL *calculus is a sound & complete axiomatization relative to differential equations.*

Corollary (Discrete Relative Completeness)

dTL *calculus is a sound & complete axiomatization relative to discrete systems.*

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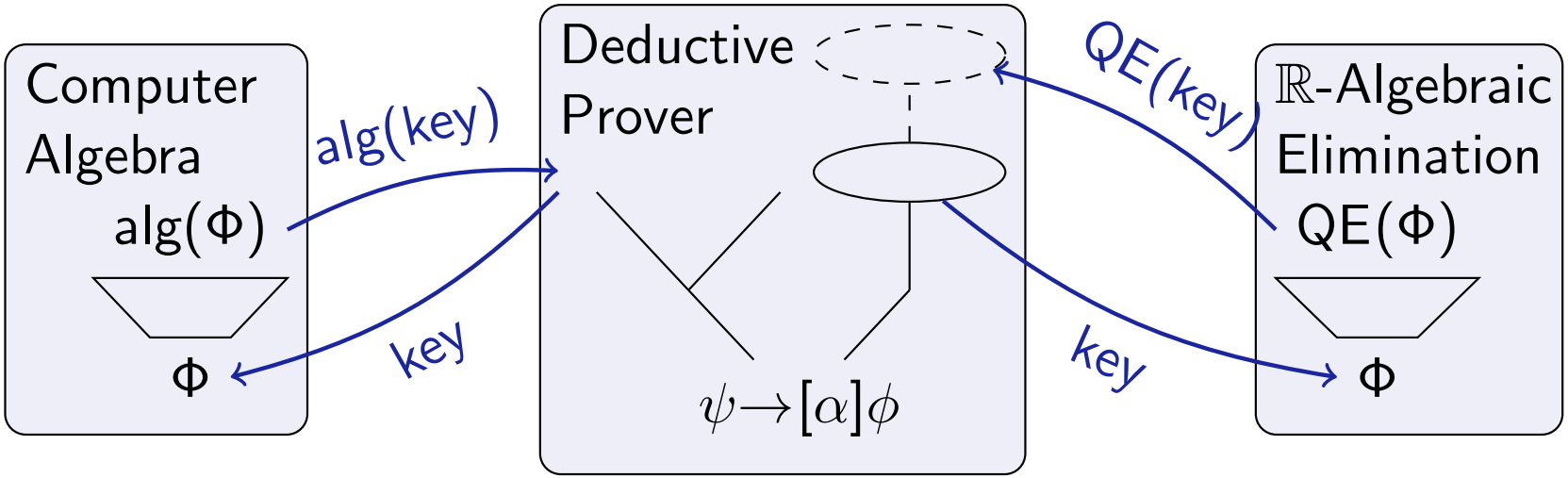
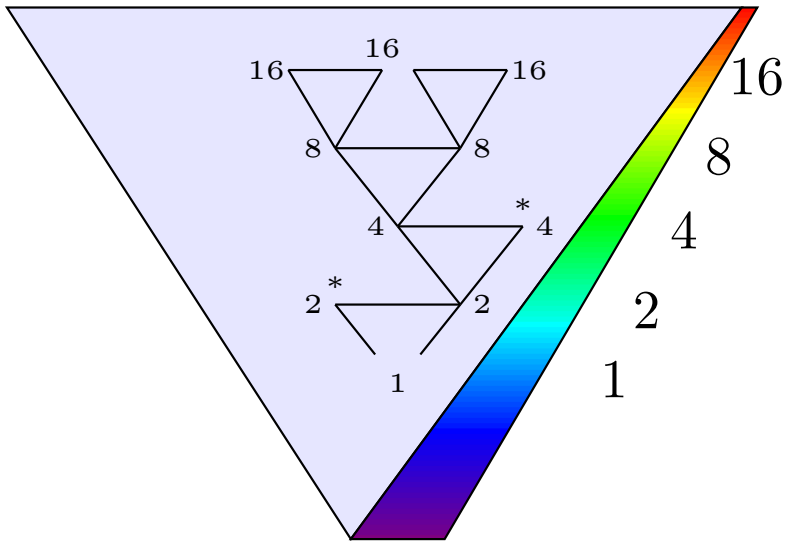
KeYmaera Verification Architecture

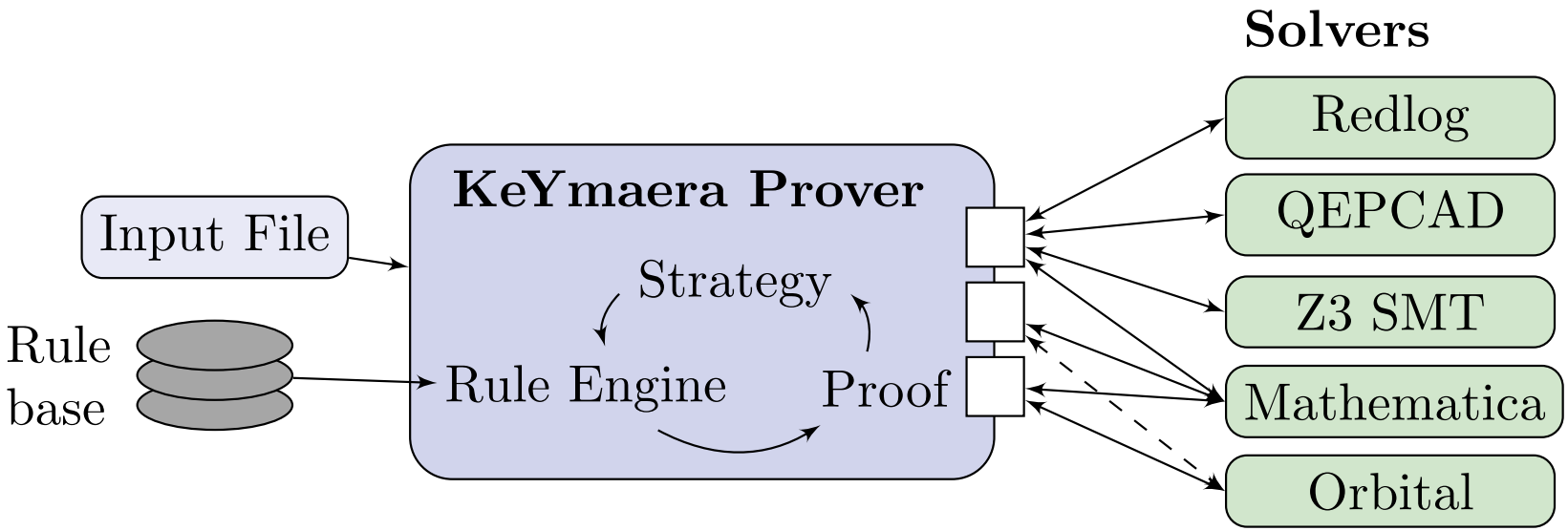


56 interactions?



0–1 interactions!

KeYmaera Prover Architecture



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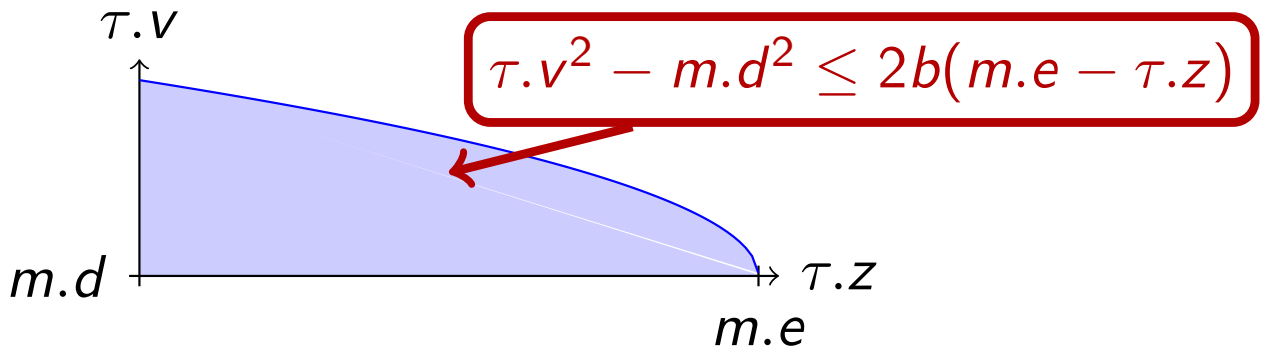
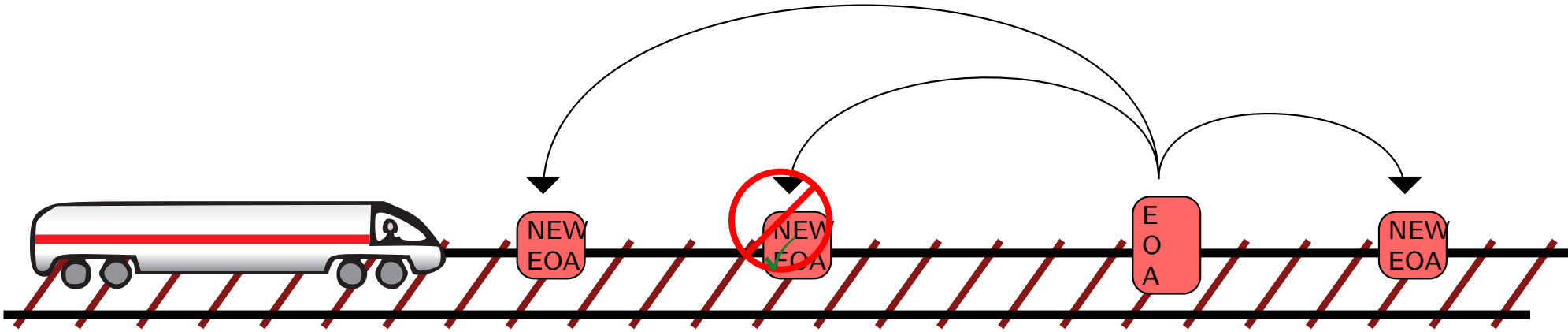


Proposition (Controllability)

$$[\tau.z' = \tau.v, \tau.v' = -b \,\&\, \tau.v \geq 0](\tau.z \geq m.e \rightarrow \tau.v \leq m.d)$$

$$\equiv \tau.v^2 - m.d^2 \leq 2b(m.e - \tau.z)$$

ETCS RBC Controllability

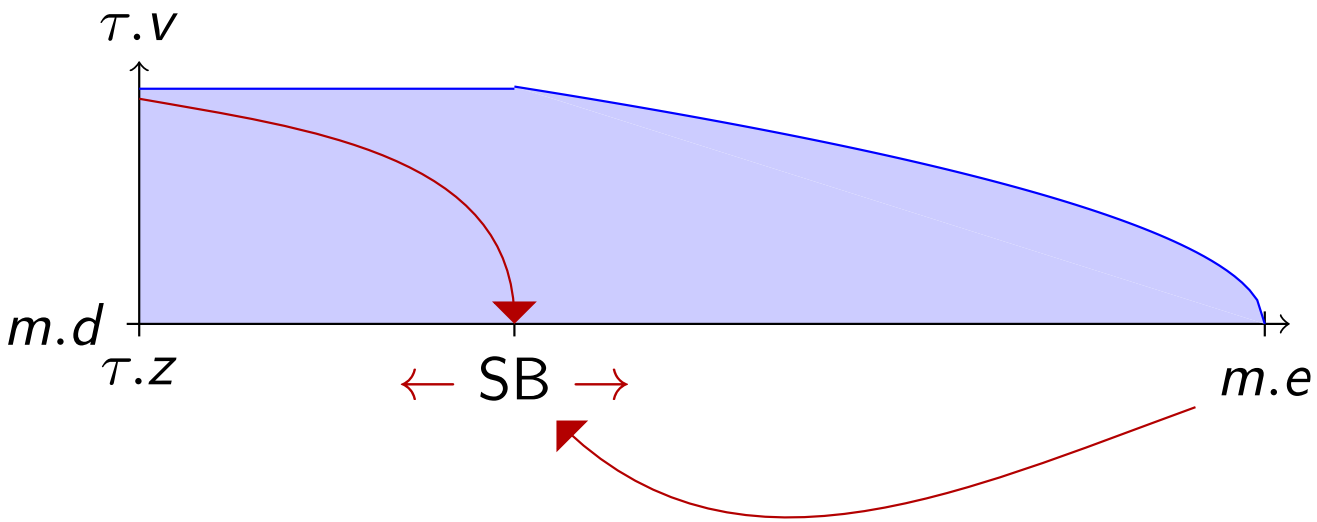


Proposition (RBC Controllability)

$$m.d \geq 0 \wedge b > 0 \rightarrow [m_0 := m;\ RBC] \Big($$

$$m_0.d^2 - m.d^2 \leq 2b(m.e - m_0.e) \wedge m_0.d \geq 0 \wedge m.d \geq 0 \leftrightarrow \forall \tau$$

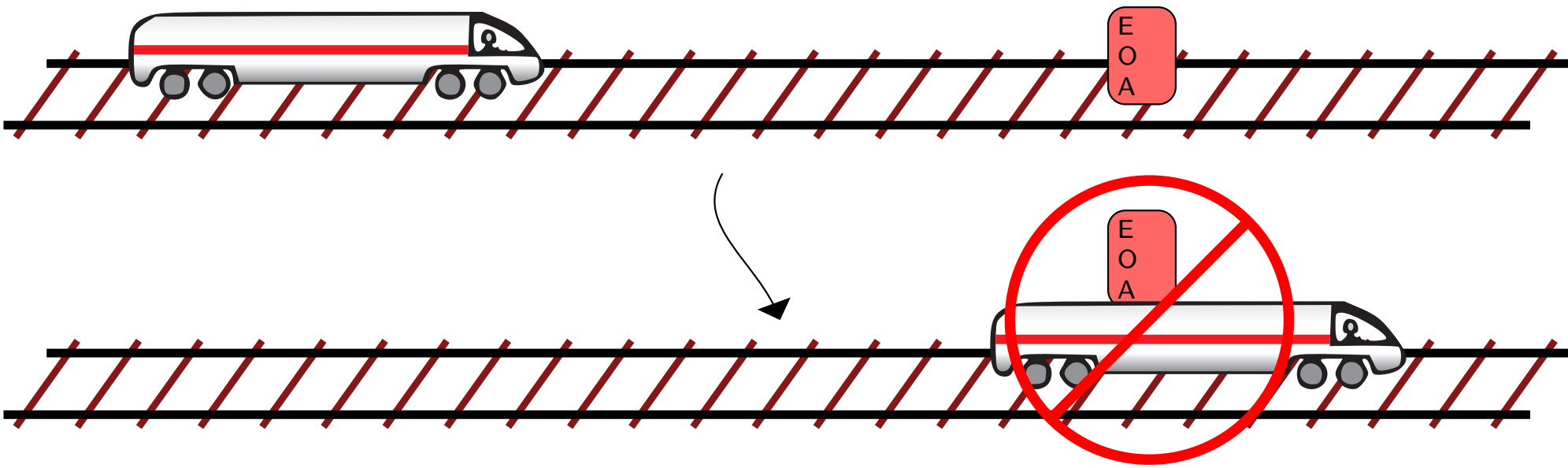
$$\big((\langle m := m_0 \rangle \tau.v^2 - m.d^2 \leq 2b(m.e - \tau.z)) \rightarrow \tau.v^2 - m.d^2 \leq 2b(m.e - \tau.z)\big)$$



Proposition (Reactivity)

$$\Big(\forall m.e\, \forall \tau.z \left(m.e - \tau.z \geq SB \land \tau.v^2 - m.d^2 \leq 2b(m.e - \tau.z) \rightarrow \right.$$
$$\left. [\tau.a := A;\ drive]\, \tau.v^2 - m.d^2 \leq 2b(m.e - \tau.z)\right)\Big)$$
$$\equiv SB \geq \frac{\tau.v^2 - m.d^2}{2b} + \left(\frac{A}{b} + 1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon\, \tau.v\right)$$

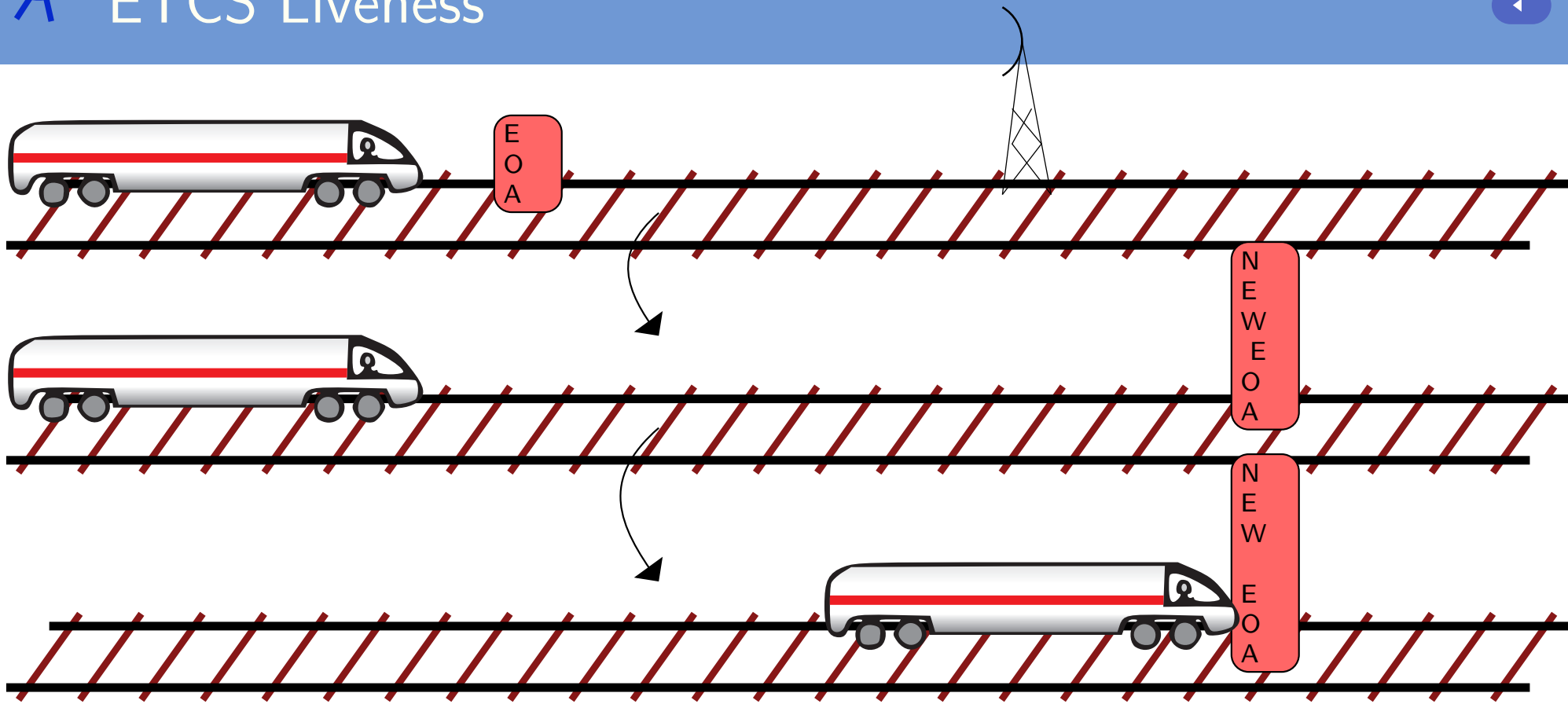
ETCS Safety



Proposition (Safety)

$$\tau.v^2 - m.d^2 \leq 2b(m.e - \tau.z) \rightarrow$$
$$[ETCS](\tau.z \geq m.e \rightarrow \tau.v \leq m.d)$$

ETCS Liveness



Proposition (Liveness)

$$\tau.v > 0 \wedge \varepsilon > 0 \rightarrow \forall P \langle ETCS \rangle \tau.z \geq P$$

Safety Despite Disturbances

So far: no wind, friction, etc.

Direct control of the acceleration

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Issue

This is unrealistic!

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Solution Take disturbances into account.

Theorem

ETCS is controllable, reactive, and safe in the presence of disturbances.

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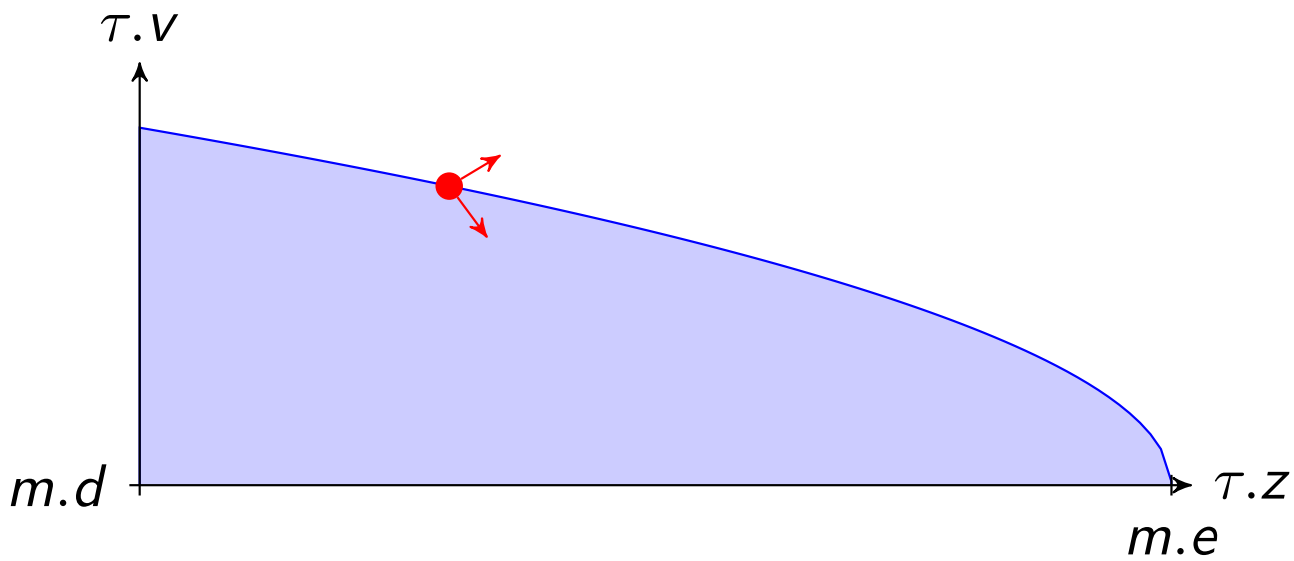

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So far: no wind, friction, etc.

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Theorem

ETCS is controllable, reactive, and safe in the presence of disturbances.

Proof sketch

The system now contains $\tau.a - l \leq \tau.v' \leq \tau.a + u$ instead of $\tau.v' = \tau.a$.

$\rightsquigarrow$ We cannot solve the differential equations anymore.

$\rightsquigarrow$ Use differential invariants for approximation. For details see paper.

Platzer, A.:
Differential-algebraic dynamic logic for differential-algebraic programs.
J. Log. Comput., 35(1): 309–352, 2010.

So far

Almost completely non-deterministic control.

Smooth Speed Control

So far

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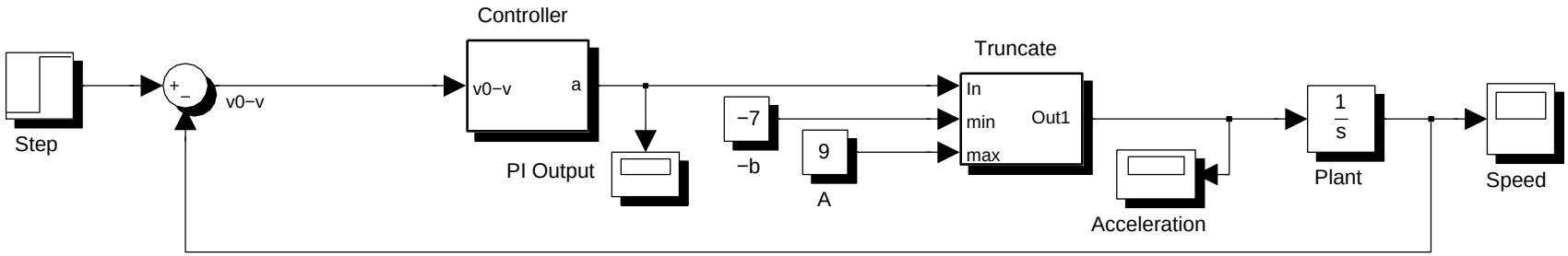
Solution Verify proportional-integral (PI) controllers used in trains.



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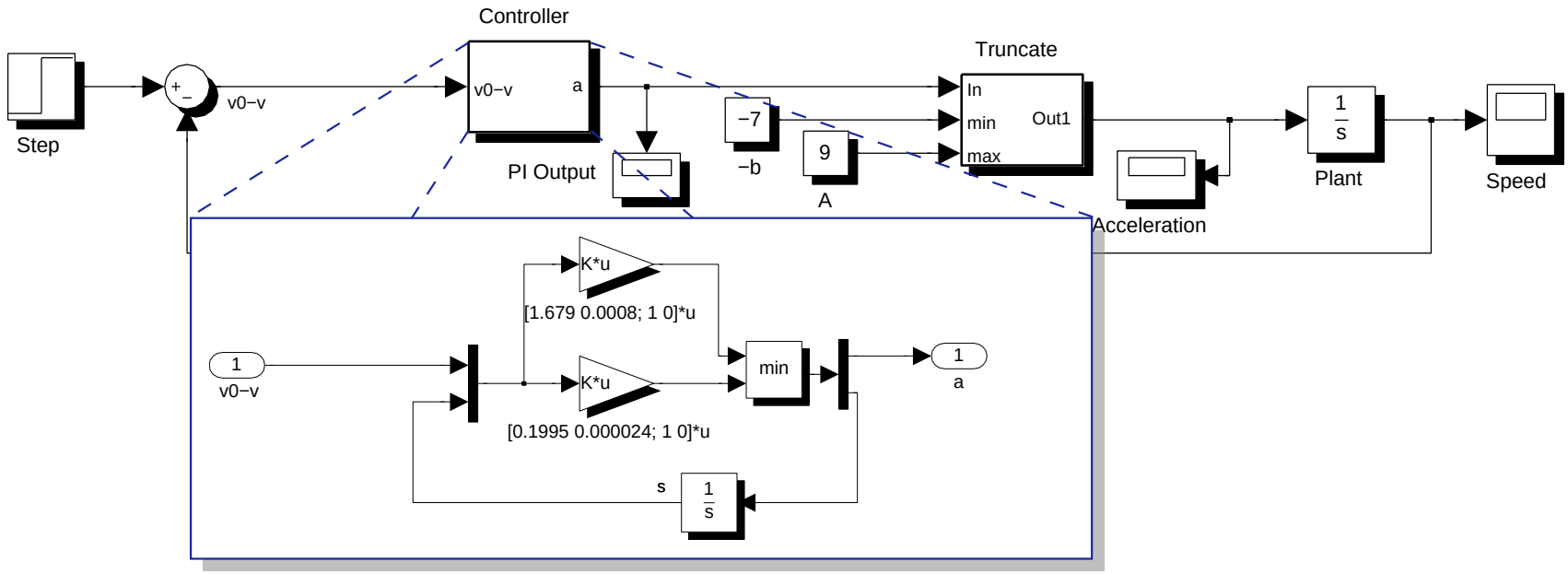

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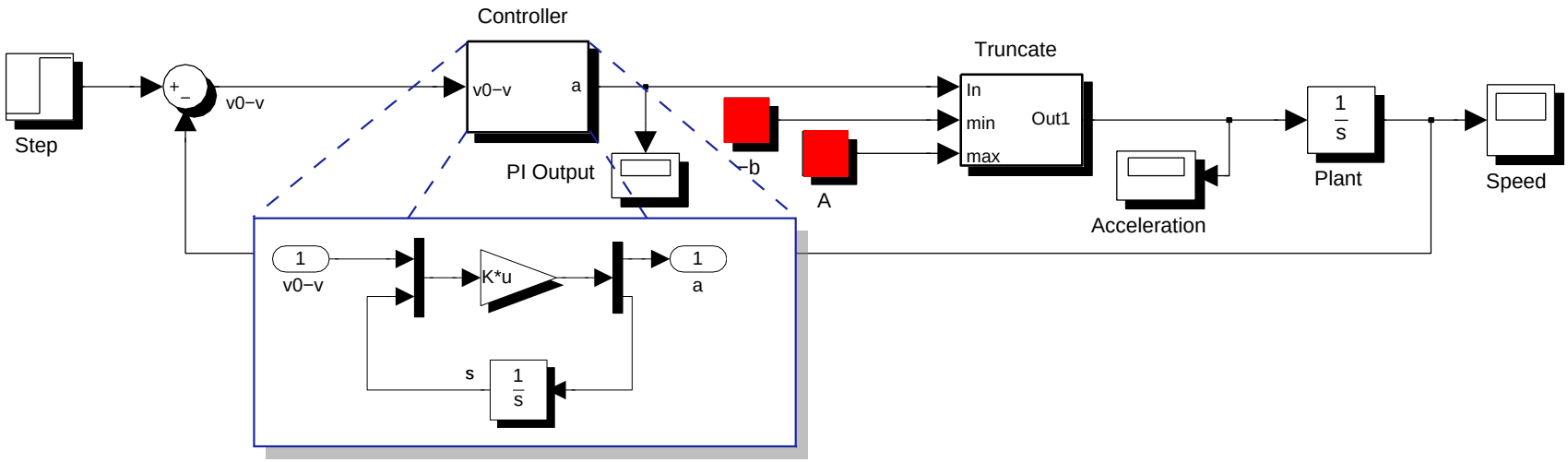

Differential equation system

$$\tau.v' = \min\Big(A, \max\big(-b,\ \ell(\tau.v - m.r) - i\,s - c\,m.r\big)\Big) \wedge s' = \tau.v - m.r$$

Smooth Speed Control

So far

Almost completely non-deterministic control.

Issue

This is unrealistic!

Solution Verify proportional-integral (PI) controllers used in trains.

Theorem

The ETCS system remains safe when speed is controlled by a PI controller.

Proof sketch

Cannot solve differential equations really. Use differential invariants! For details see paper.

Platzer, A.:
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Experimental Results (ETCS)

Case Study		Int	Time(s)	Mem(Mb)	Steps	Dim
controllability	train	0	0.6	6.9	14	5
controllability	RBC	0	0.5	6.4	42	12
controllability	RBC	0	0.9	6.5	82	12
reactivity		13	279.1	98.3	265	14
reactivity		0	103.9	61.7	47	14
safety		0	2052.4	204.3	153	14
liveness	essentials	4	35.2	92.2	62	10
liveness	simplified	6	9.6	23.5	134	13
controllability	disturbance	0	2.8	8.3	26	7
reactivity	disturbance	1	23.7	47.6	76	15
safety	disturbance	1	5805.2	34	218	16

European Train Control System (ETCS)

provable automatically!

$$
\begin{array}{lll}
\text{spec} & : & \tau.v^2 - \mathbf{m}.d^2 \leq 2b(\mathbf{m}.e - \tau.p) \wedge \tau.v \geq 0 \wedge \mathbf{m}.d \geq 0 \wedge b > 0 \\
& & \rightarrow\ [\text{ETCS}](\tau.p \geq \mathbf{m}.e \rightarrow \tau.v \leq \mathbf{m}.d) \\
\text{ETCS:} & & (\text{train} \cup \text{rbc})^* \\
\text{train} & : & \text{spd};\ \text{atp};\ \text{move} \\
\text{spd} & : & (?\tau.v \leq \mathbf{m}.r;\ \tau.a := *;\ ? - b \leq \tau.a \leq A) \\
& & \cup (?\tau.v \geq \mathbf{m}.r;\ \tau.a := *;\ ?0 > \tau.a \geq -b) \\
\text{atp} & : & SB := \frac{\tau.v^2 - \mathbf{m}.d^2}{2b} + \left(\frac{A}{b} + 1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon\,\tau.v\right); \\
& & (?(\mathbf{m}.e - \tau.p \leq SB \vee rbc.message = emergency);\ \tau.a := -b) \\
& & \cup (?\mathbf{m}.e - \tau.p \geq SB \wedge rbc.message \neq emergency) \\
\text{move} : & & t := 0;\ (\tau.p' = \tau.v, \tau.v' = \tau.a, t' = 1\,\&\,\tau.v \geq 0 \wedge t \leq \varepsilon) \\
\text{rbc} & : & (rbc.message := emergency) \\
& & \cup\ \big(\mathbf{m}_0 := \mathbf{m}; \mathbf{m} := *; \\
& & \quad ?\mathbf{m}.r \geq 0 \wedge \mathbf{m}.d \geq 0 \wedge \mathbf{m}_0.d^2 - \mathbf{m}.d^2 \leq 2b(\mathbf{m}.e - \mathbf{m}_0.e)\big)
\end{array}
$$

1 Branch of ETCS Proof

```
 state = 0,
 2 * b * (m - z) >= v ^ 2 - d ^ 2,
 v >= 0, d >= 0, v >= 0, ep >  0, b >  0, amax >  0, d >= 0
==>
    v <= vdes
 -> \forall R a_3;
      (   a_3 >= 0 & a_3 <= amax
       ->   (      m - z
              <= (amax / b + 1) * ep * v
               + (v ^ 2 - d ^ 2) / (2 * b)
               + (amax / b + 1) * amax * ep ^ 2 / 2
            -> \forall R t0;
                 (   t0 >= 0
                  -> \forall R ts0;  (0 <= ts0 & ts0 <= t0 -> -b * ts0 + v >= 0 & ts0 + 0 <= ep)
                  ->      2 * b * (m - 1 / 2 * (-b * t0 ^ 2 + 2 * t0 * v + 2 * z))
                       >= (-b * t0 + v) ^ 2
                        - d ^ 2
                     & -b * t0 + v >= 0
                     & d >= 0))
         & (      m - z
              >  (amax / b + 1) * ep * v
               + (v ^ 2 - d ^ 2) / (2 * b)
               + (amax / b + 1) * amax * ep ^ 2 / 2
            -> \forall R t2;
                 (   t2 >= 0
                  -> \forall R ts2;  (0 <= ts2 & ts2 <= t2 -> a_3 * ts2 + v >= 0 & ts2 + 0 <= ep)
                  ->      2 * b * (m - 1 / 2 * (a_3 * t2 ^ 2 + 2 * t2 * v + 2 * z))
                       >= (a_3 * t2 + v) ^ 2
                        - d ^ 2
                     & a_3 * t2 + v >= 0
                     & d >= 0)))
```

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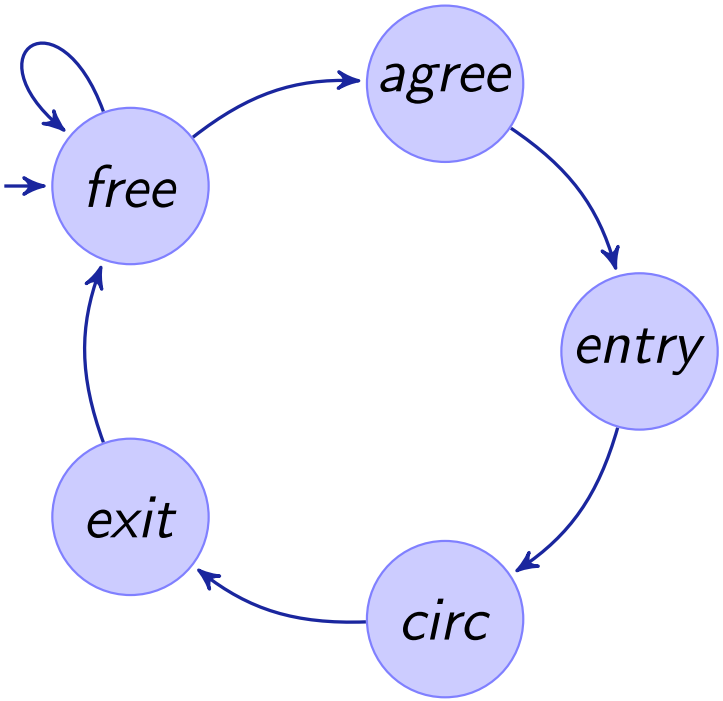
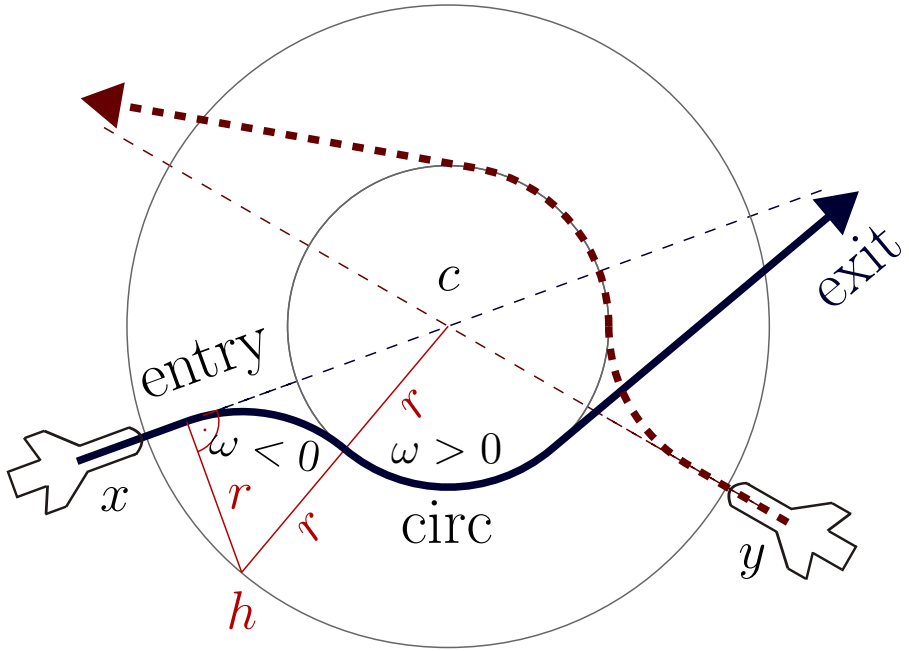
Flyable Roundabout Maneuver: Overview





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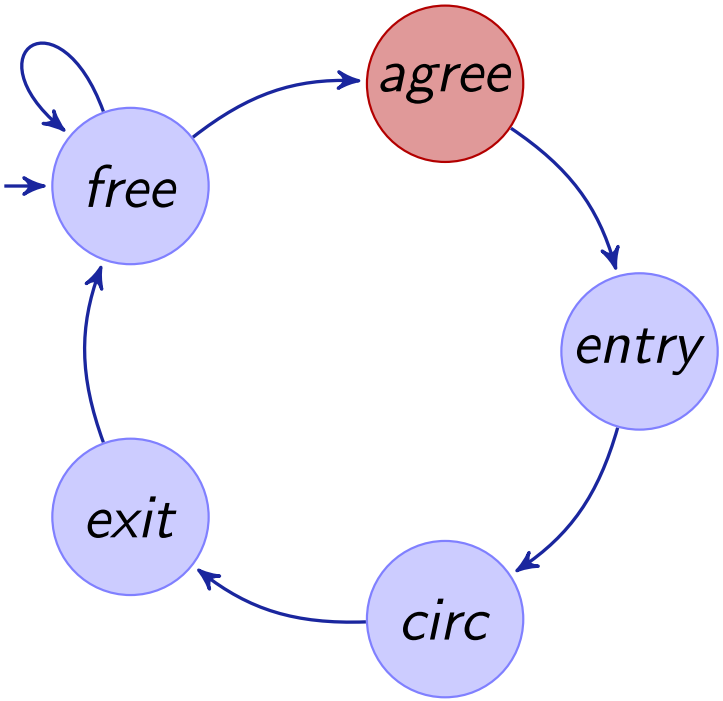

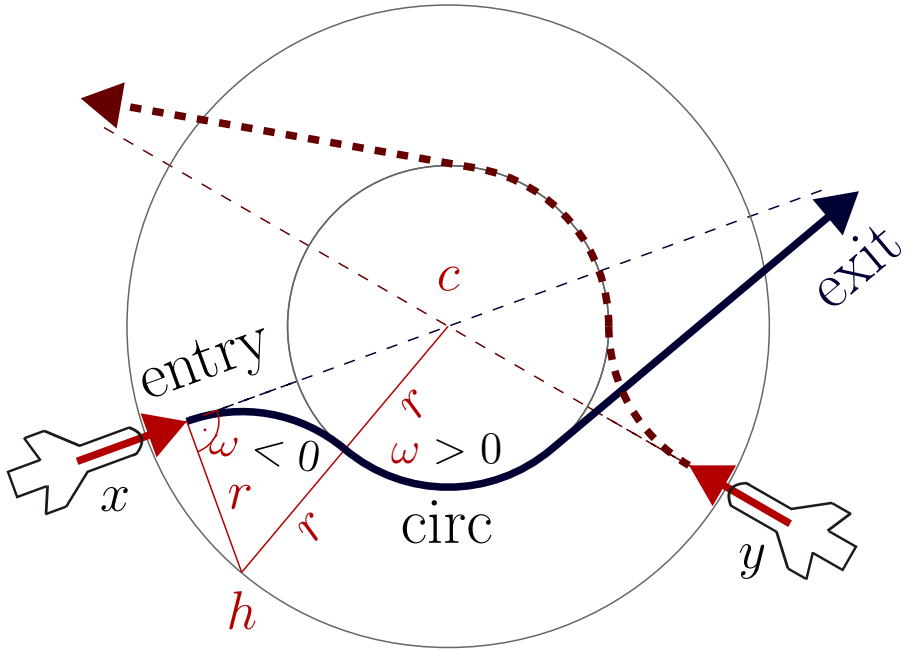

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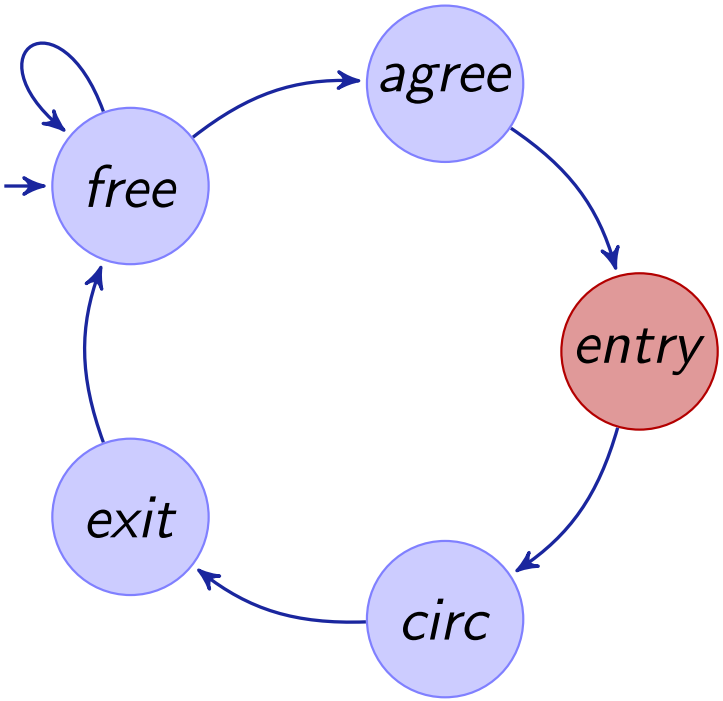

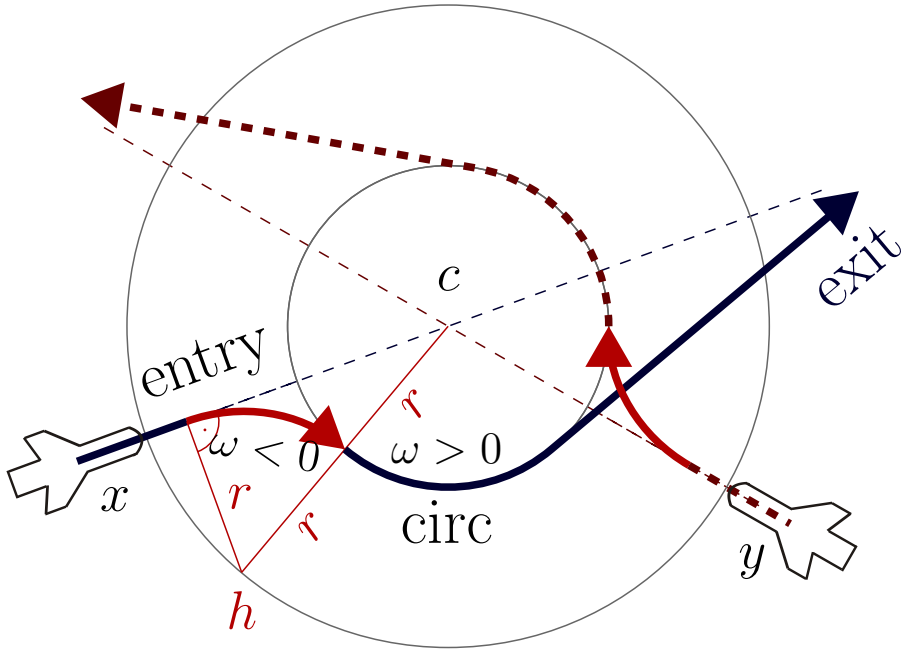

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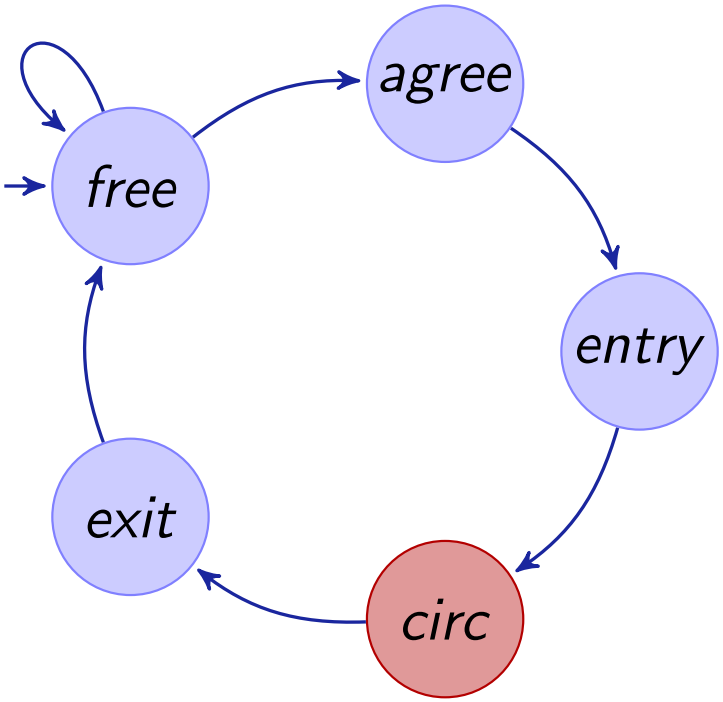

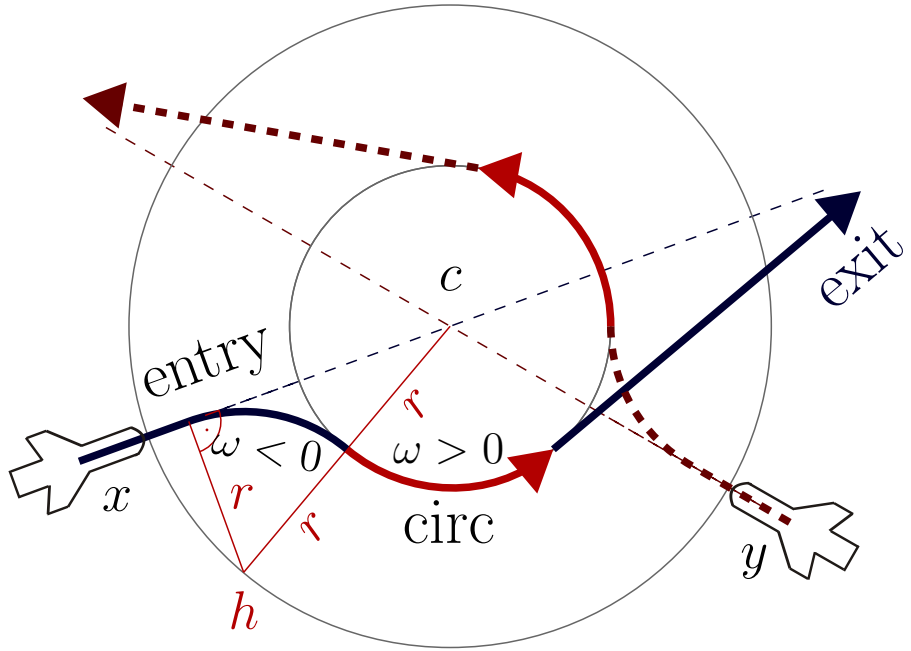

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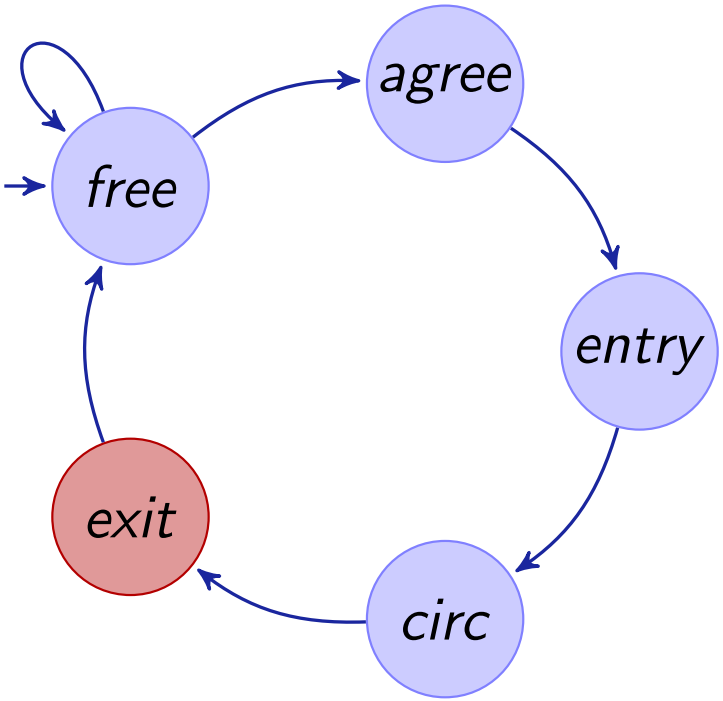

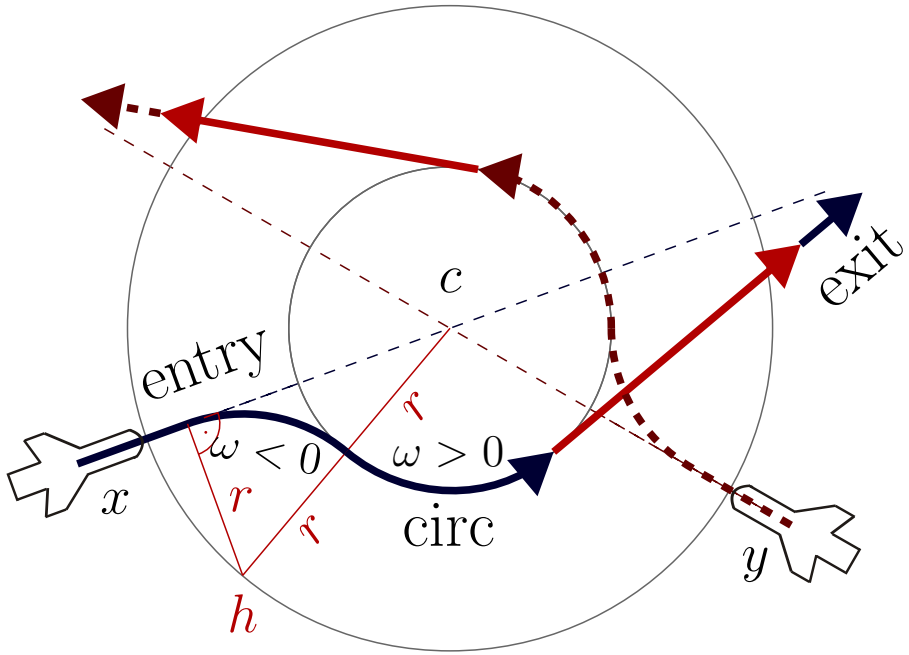

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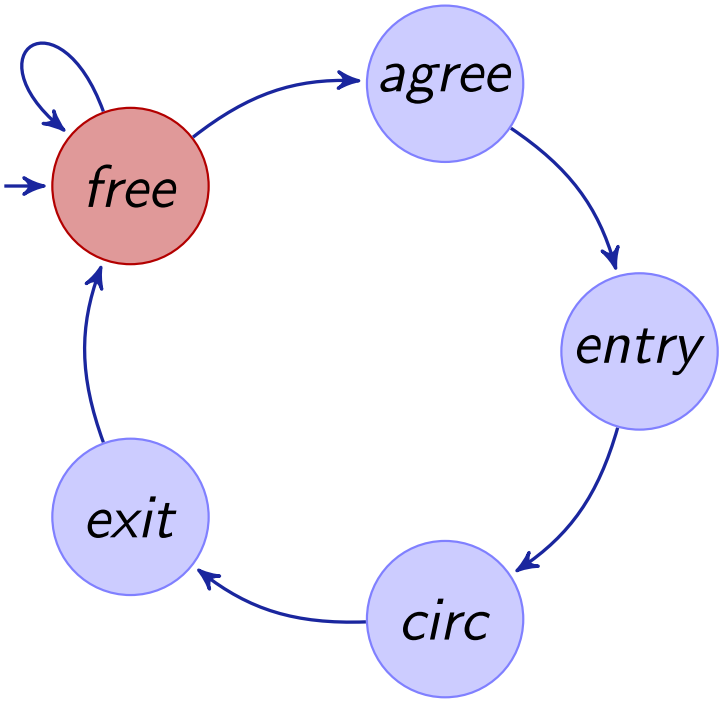

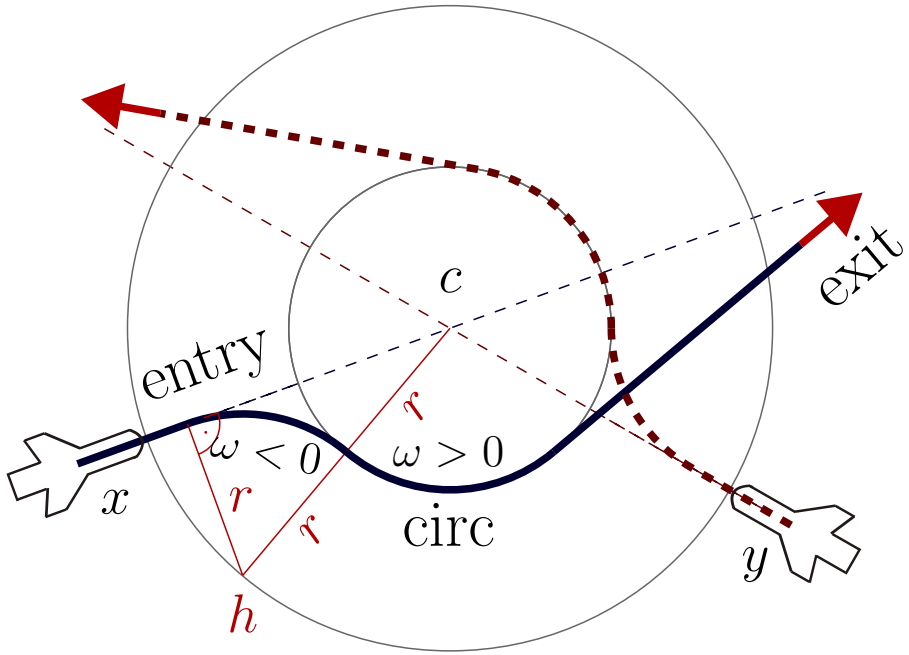

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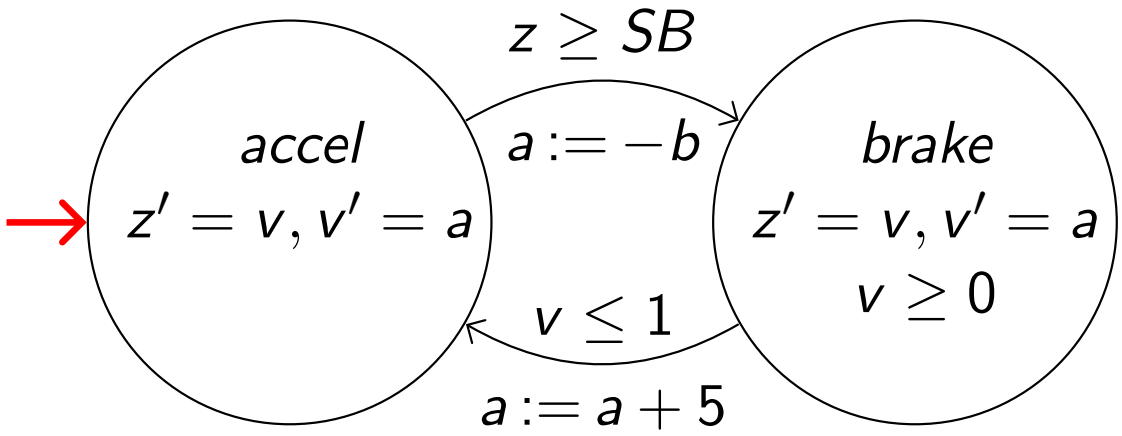
Embedding Hybrid Automata as Hybrid Programs



$$
\begin{array}{l}
q := accel; \\
\big(\quad (?q = accel; \quad z' = v, v' = a) \\
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\cup \ (?q = brake; \quad z' = v, v' = a \,\&\, v \geq 0) \\
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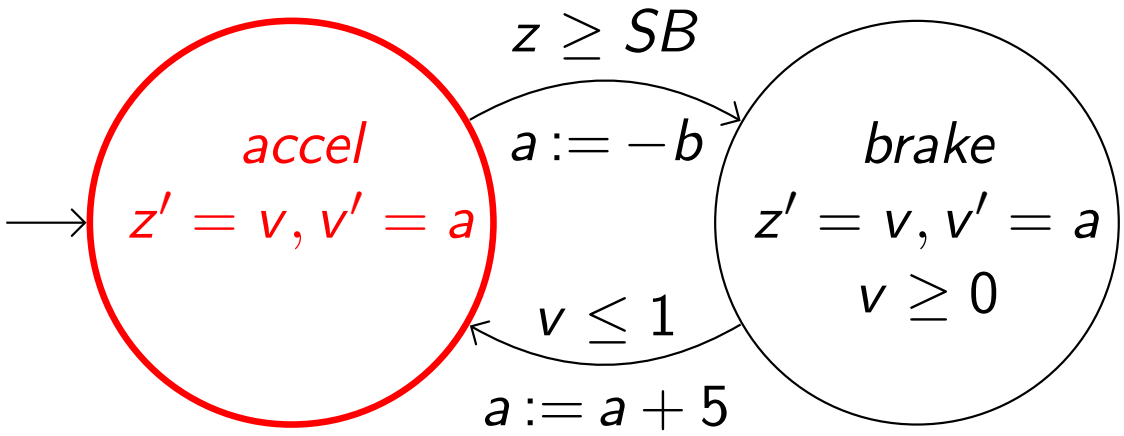

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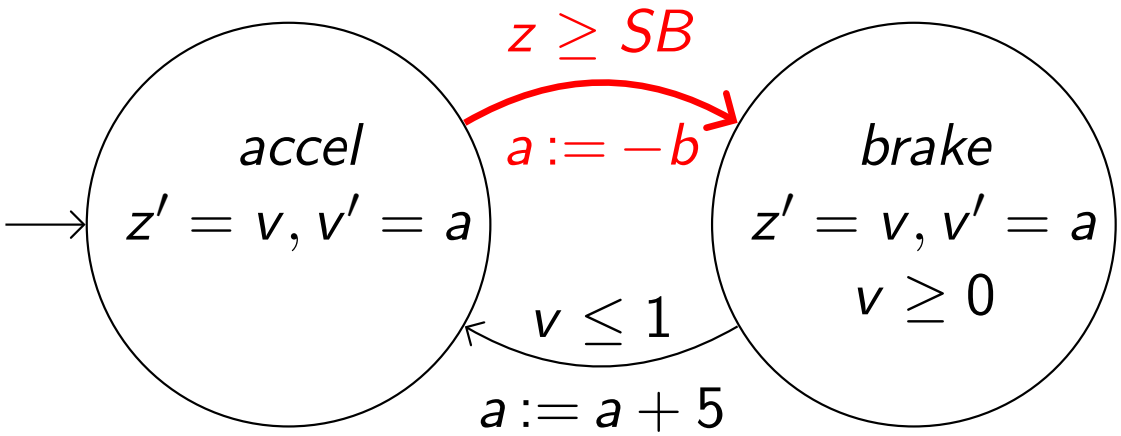

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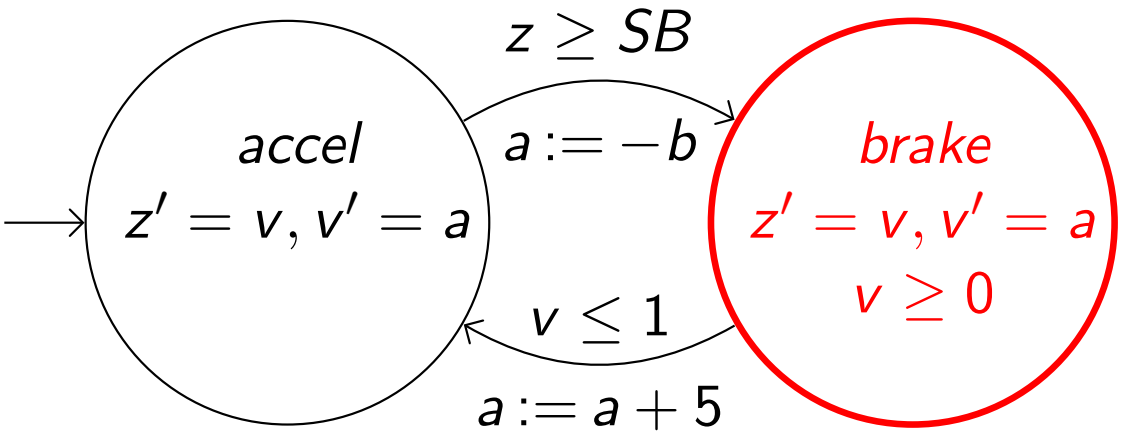

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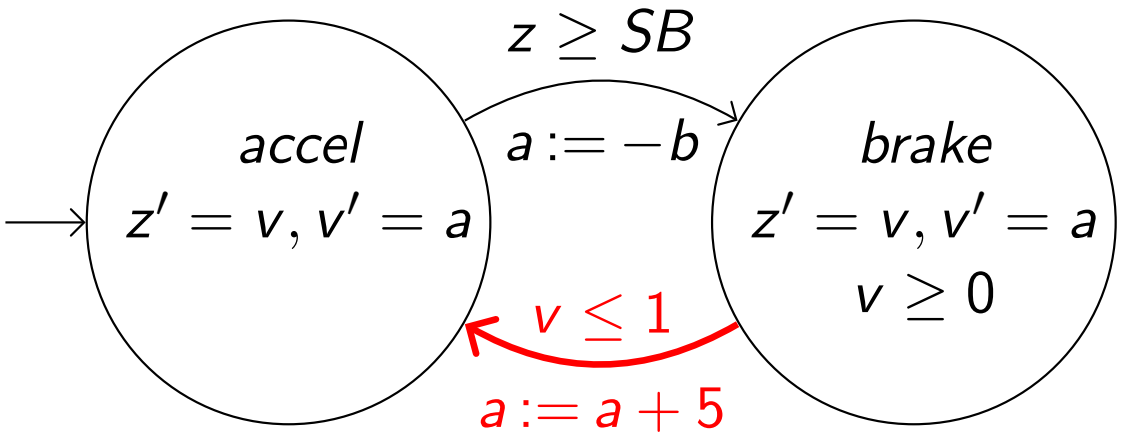

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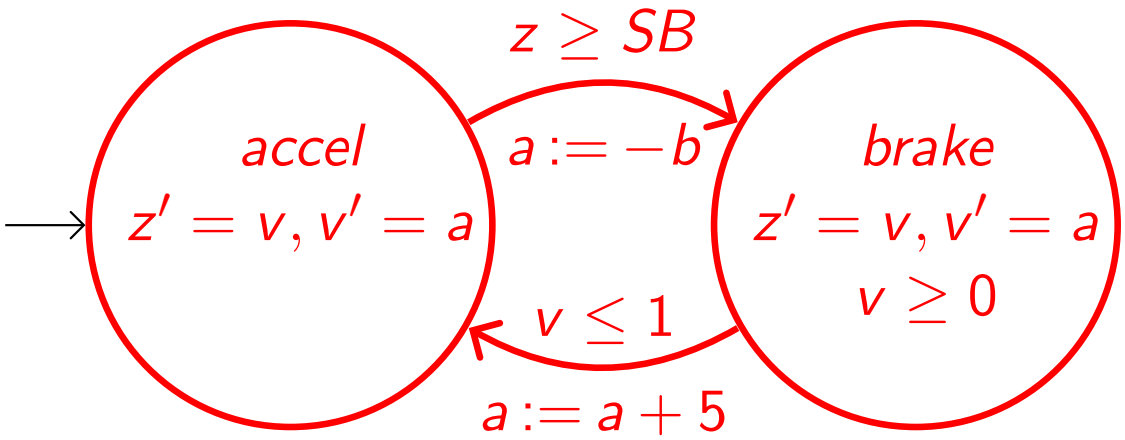

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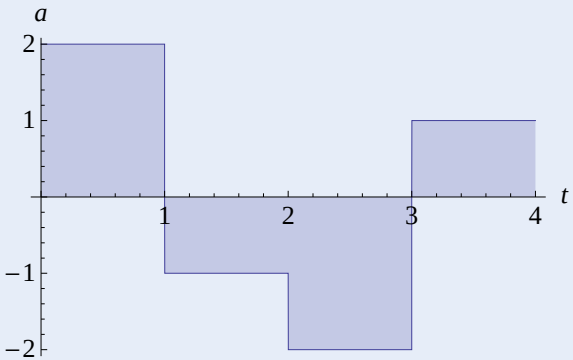
Q: I want to verify my car

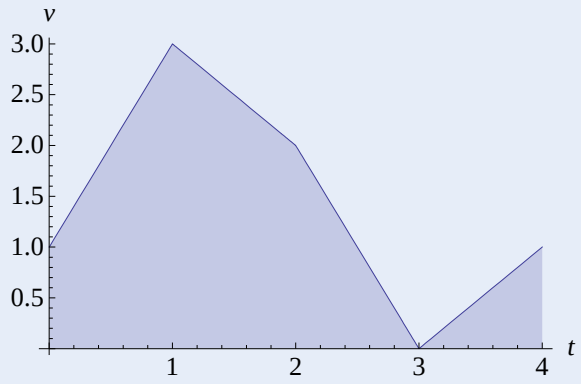
Challenge

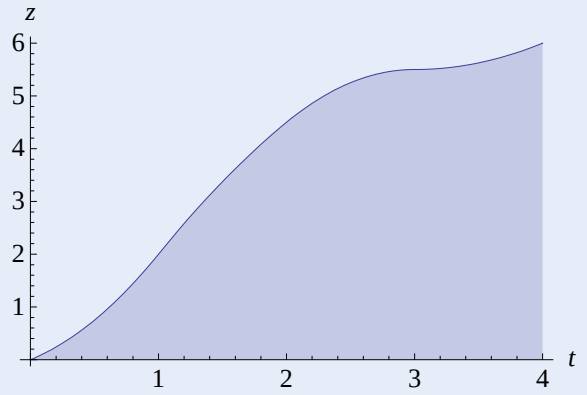
Q: I want to verify my car A: Hybrid systems

Challenge (Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)



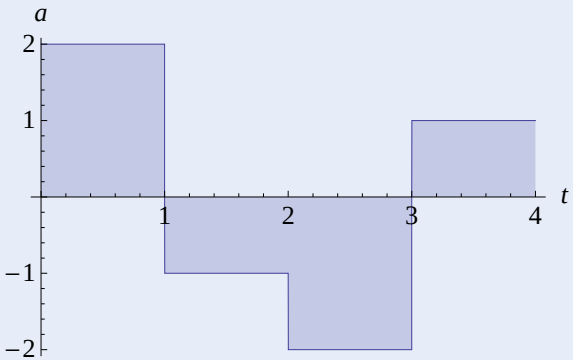


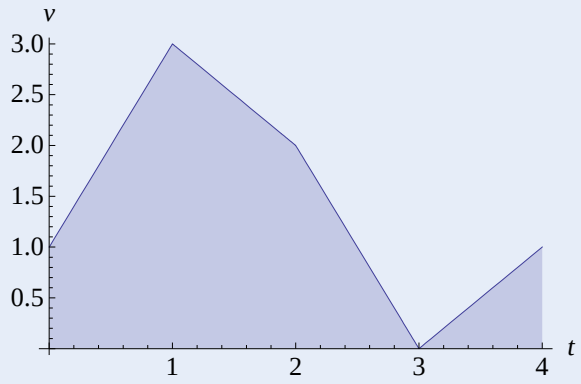


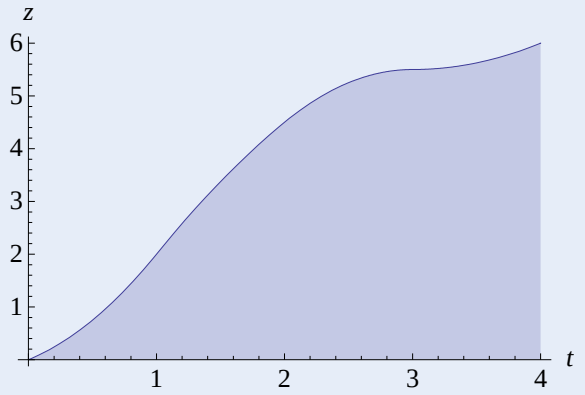
Q: I want to verify my car A: Hybrid systems Q: But there's a lot of cars!

Challenge (Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)

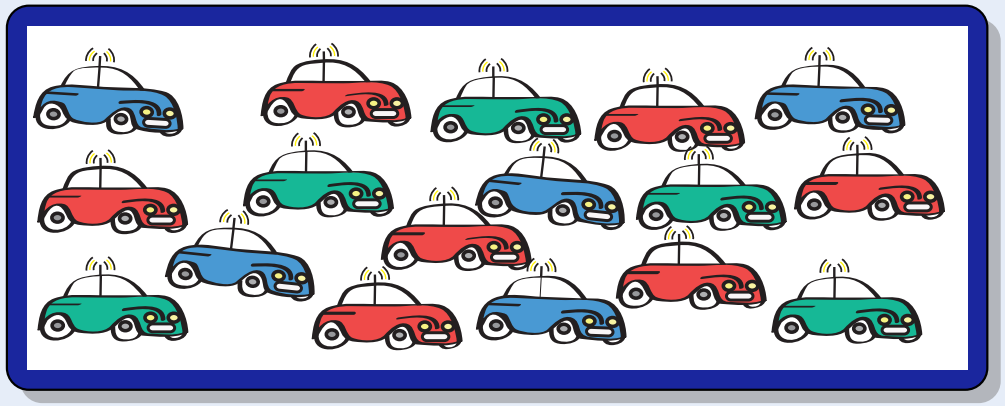






Q: I want to verify a lot of cars

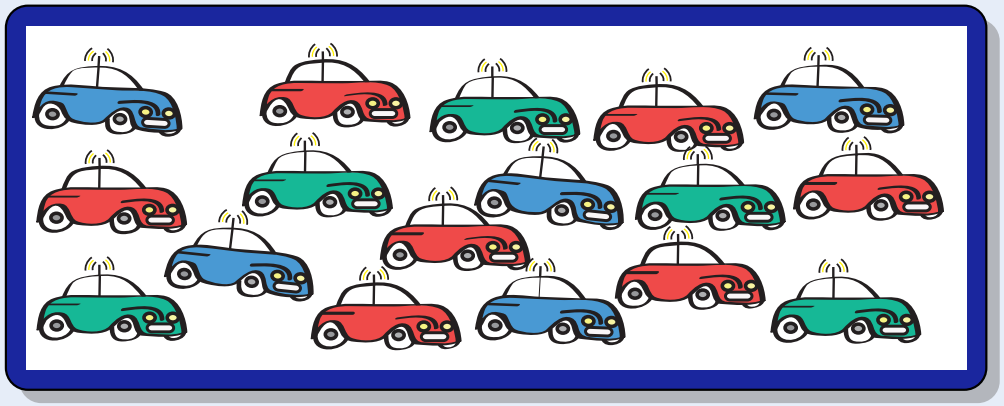
Challenge

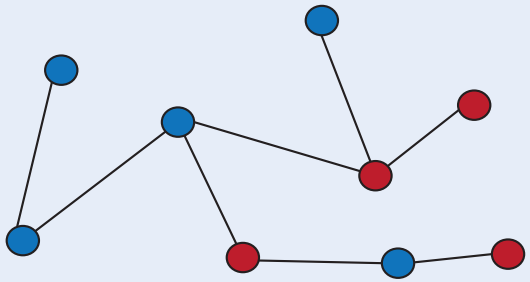


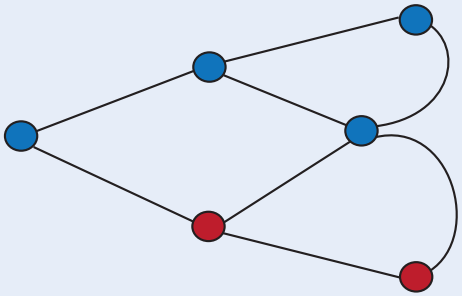
Q: I want to verify a lot of cars A: Distributed systems

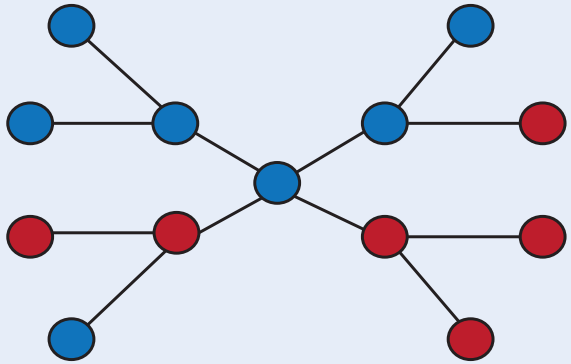
Challenge (Distributed Systems)

- Local computation (finite state automaton)
- Remote communication (network graph)





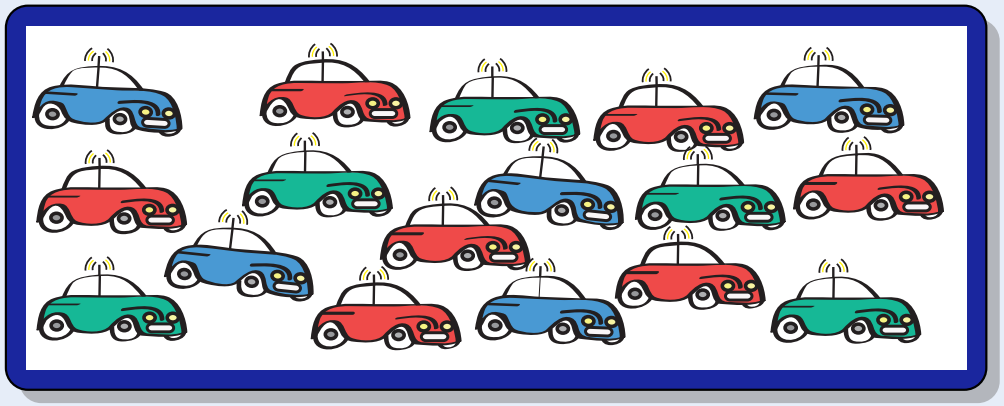


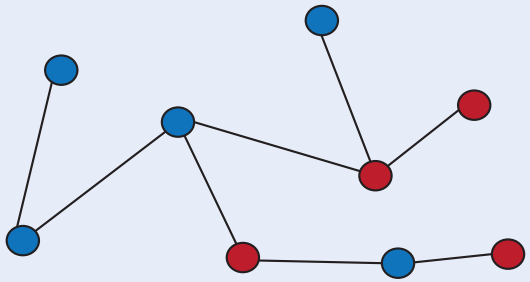


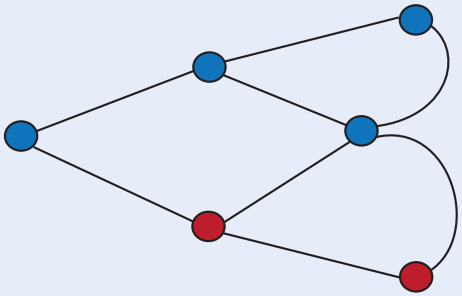
Q: I want to verify a lot of cars A: Distributed systems Q: But they move!

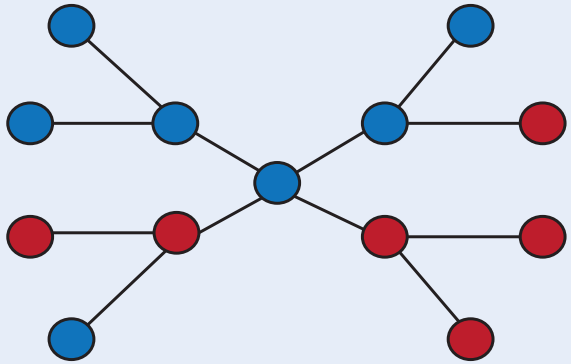
Challenge (Distributed Systems)

- Local computation (finite state automaton)
- Remote communication (network graph)



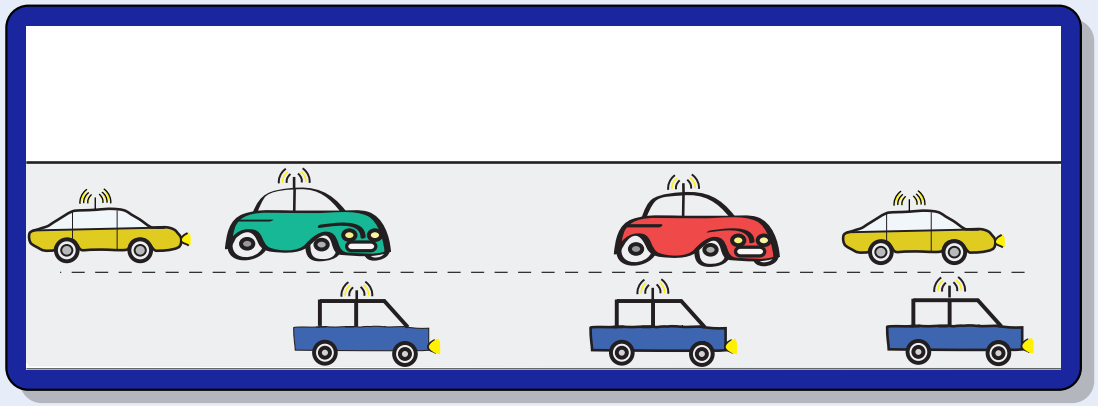






Q: I want to verify lots of moving cars

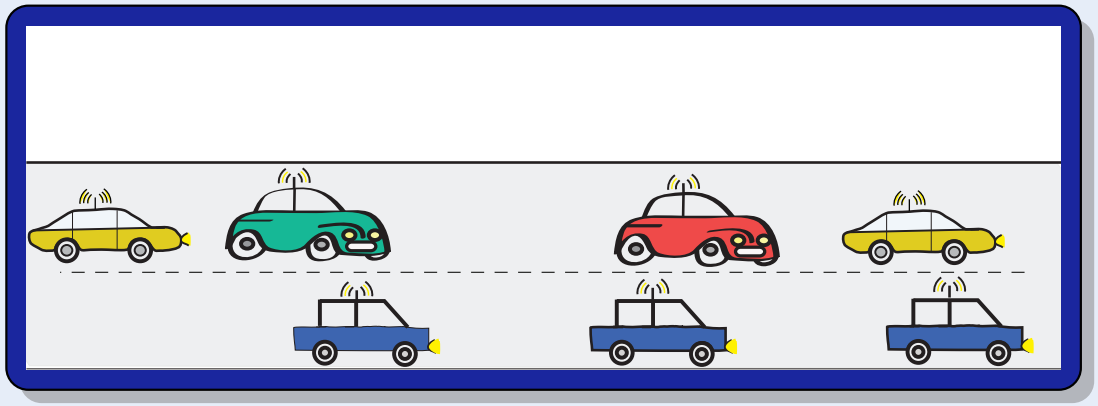
Challenge



Q: I want to verify lots of moving cars A: Distributed hybrid systems

Challenge (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Structural dynamics (communication/coupling)



Q: I want to verify lots of moving cars A: Distributed hybrid systems

Challenge (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
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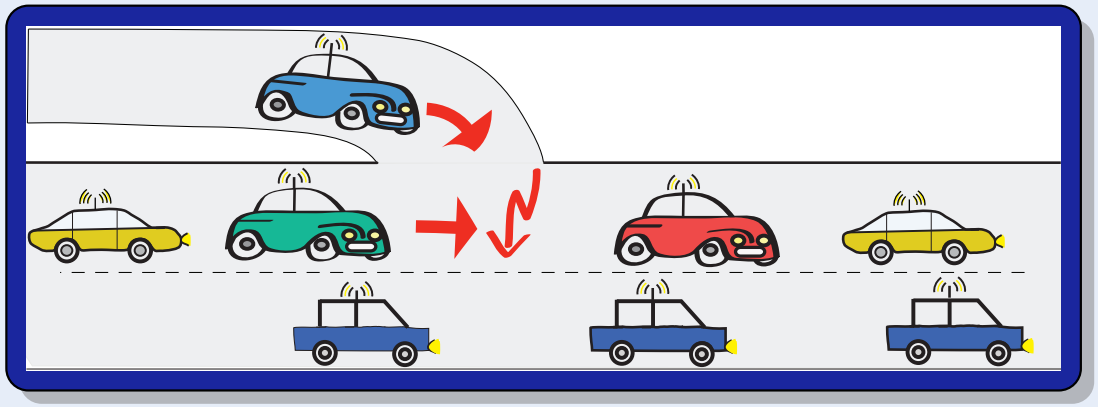
- Dimensional dynamics (appearance)



Q: I want to verify lots of moving cars A: Distributed hybrid systems Q: How?

Challenge (Distributed Hybrid Systems)

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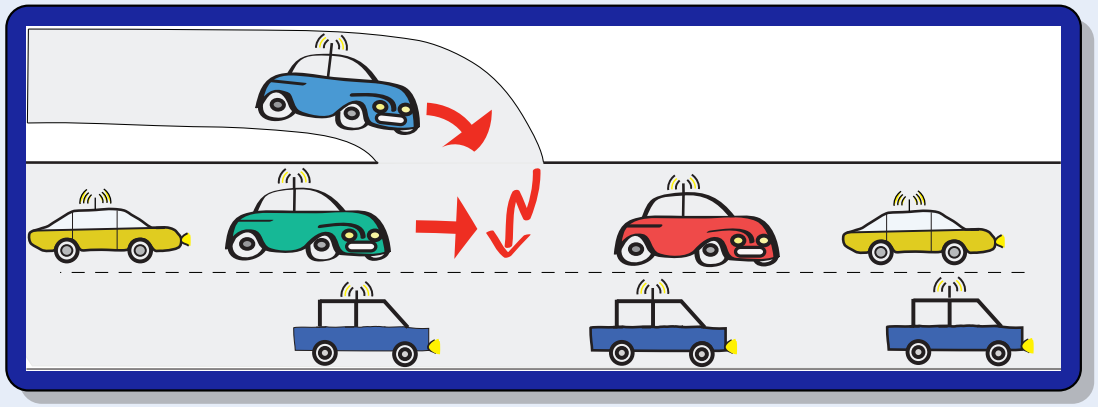

Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
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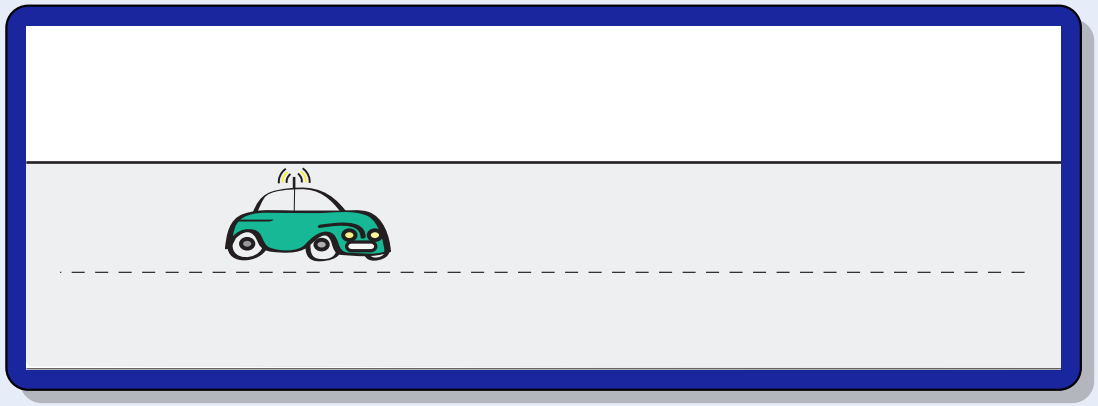
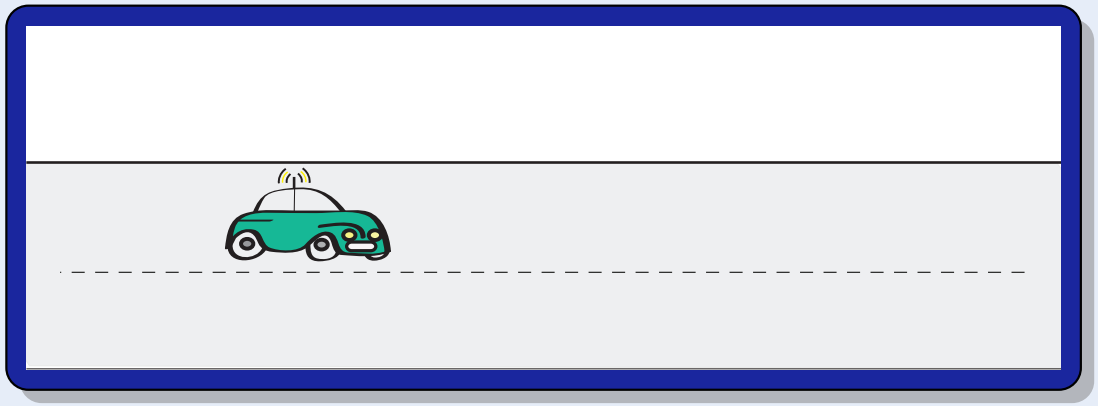


Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  $$x'' = a$$
- Discrete dynamics (control decisions)
- Structural dynamics (communication/coupling)



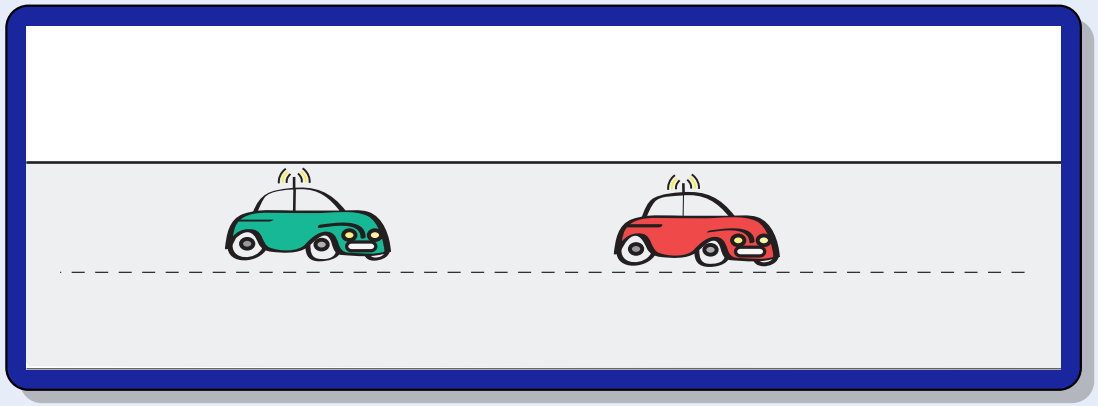
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
$$x'' = a$$
- Discrete dynamics (control decisions)

$a := \texttt{if}\ .. \texttt{then}\ a \texttt{else} - b \texttt{fi}$

- Structural dynamics (communication/coupling)



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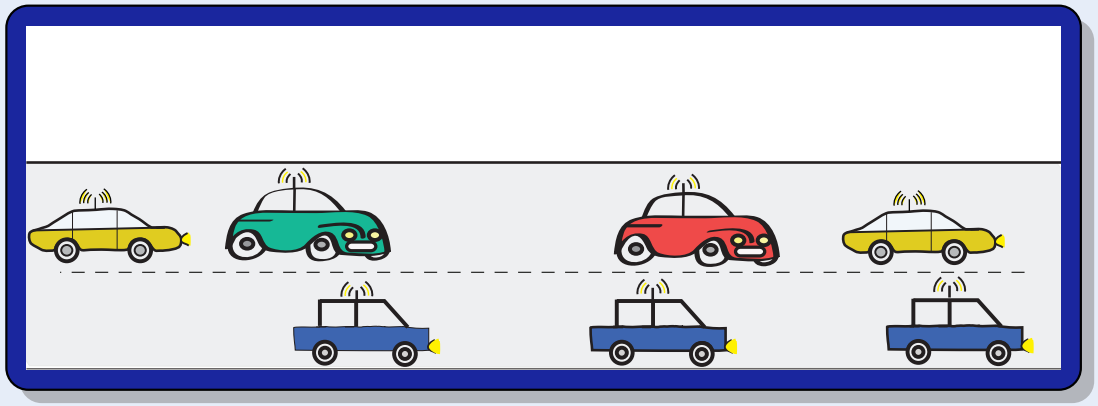

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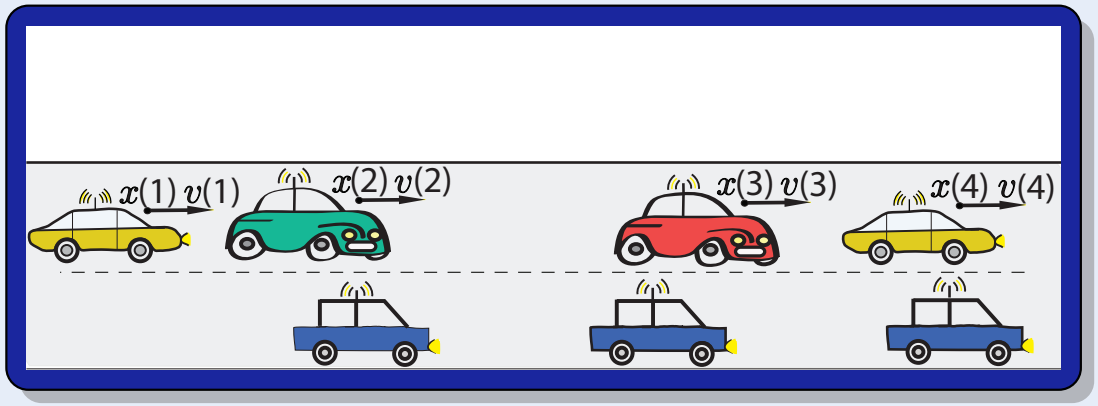


Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

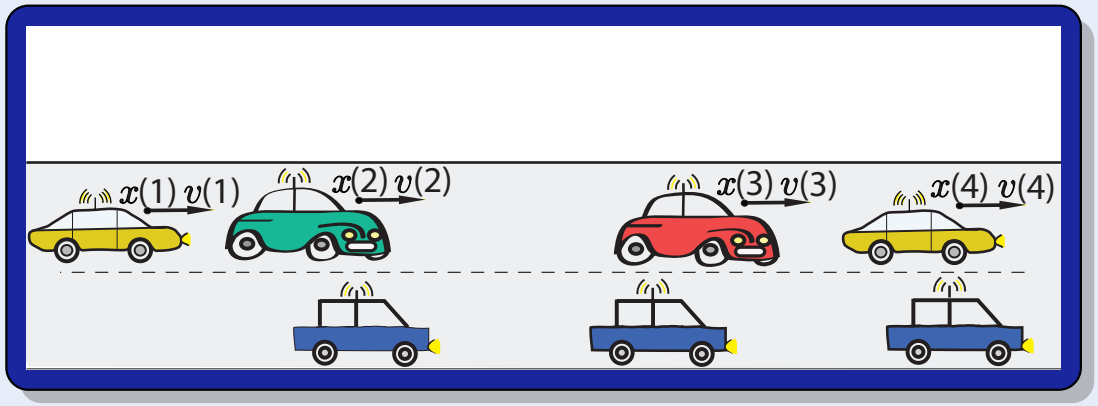
- Continuous dynamics (differential equations)

$$x(i)'' = a(i)$$

- Discrete dynamics (control decisions)

$a(i) := \texttt{if}\ ..\ \texttt{then}\ a\ \texttt{else}\ -b\ \texttt{fi}$

- Structural dynamics (communication/coupling)



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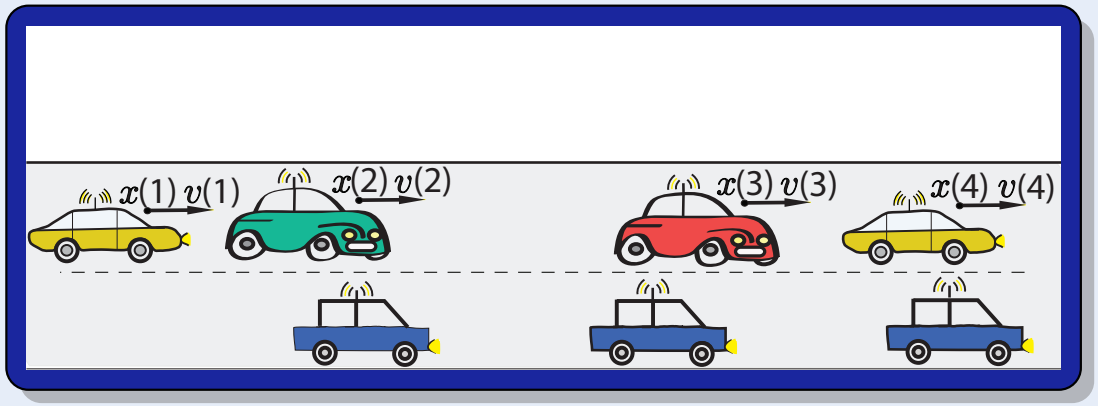

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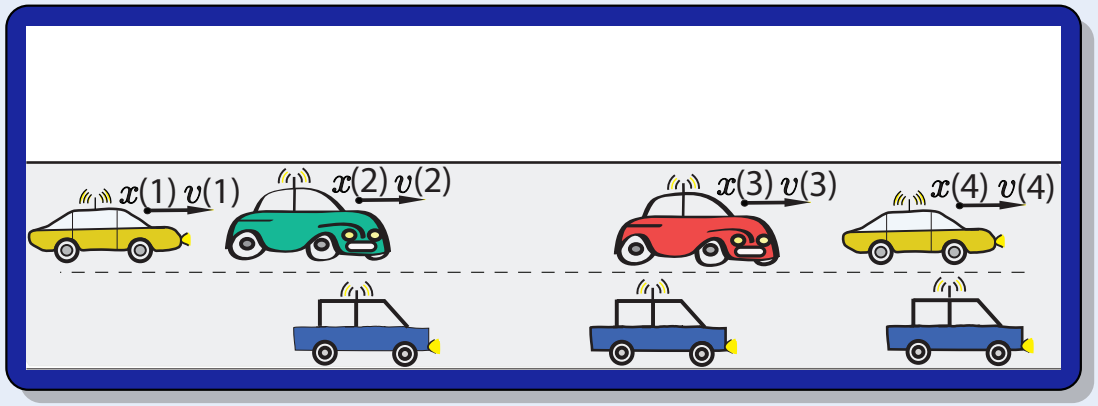
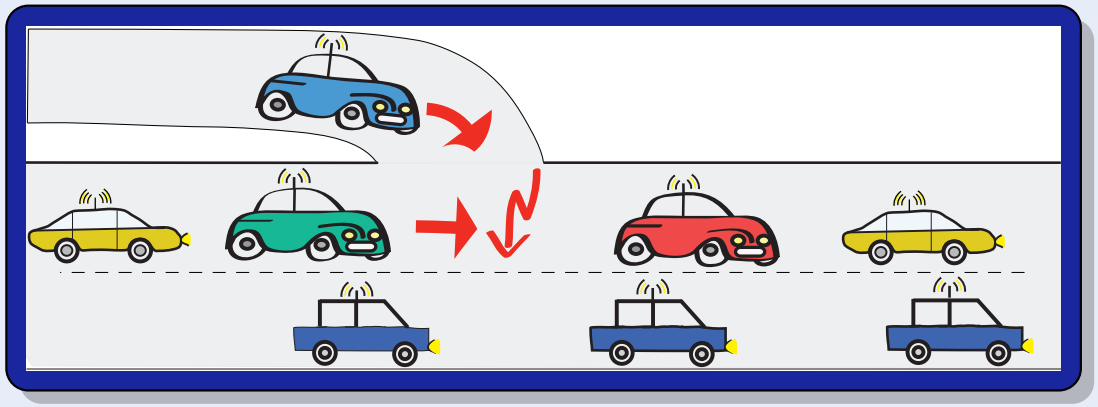
- Structural dynamics (communication/coupling)
  $\ell(i) := carInFrontOf(i)$



Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)



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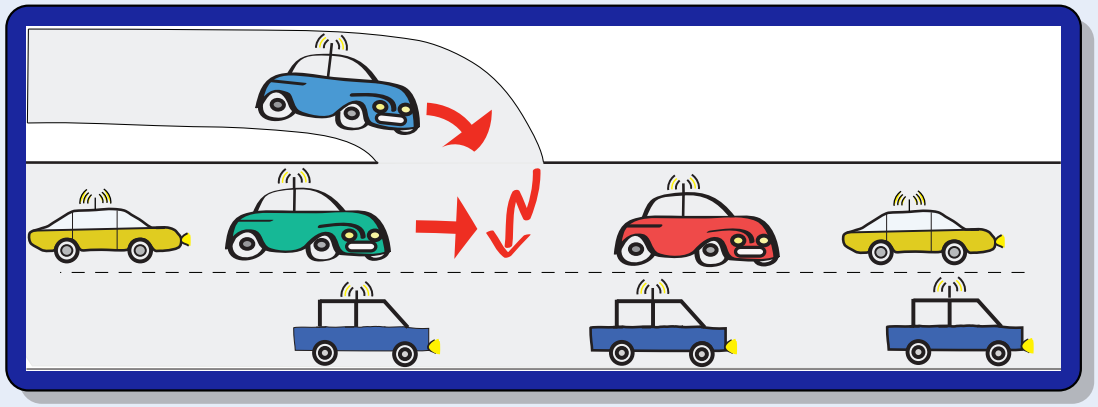
- Structural dynamics (communication/coupling)
  $\ell(i) := \mathit{carInFrontOf}(i)$
- Dimensional dynamics (appearance)

Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)



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  $n := \texttt{new}\ \mathit{Car}$

Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)

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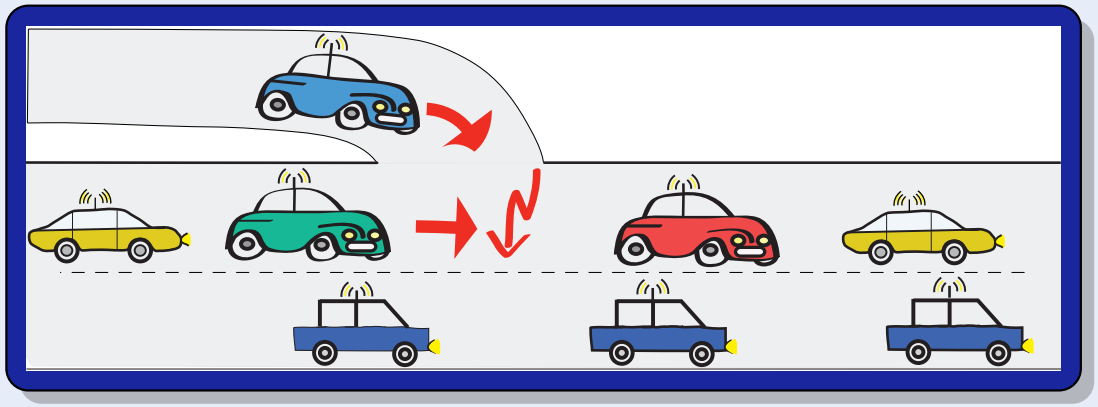
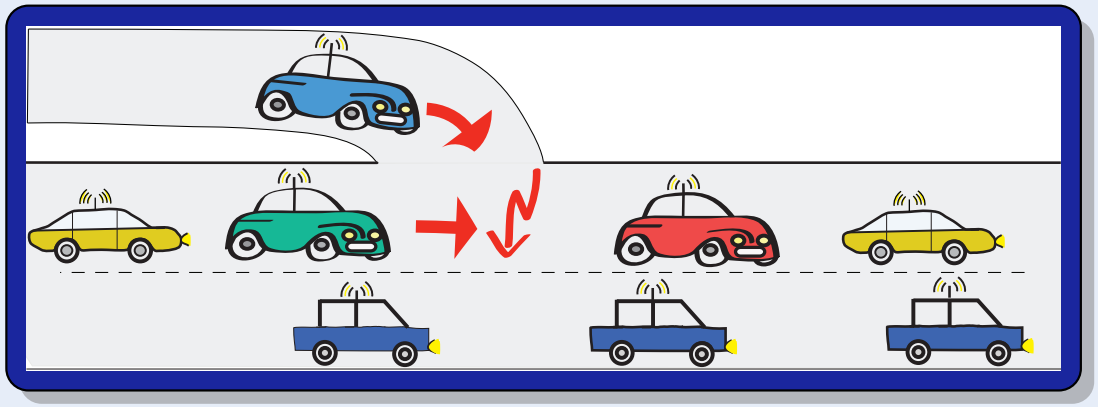


$\Rightarrow$ Communication

$$d(i,\ell(i)) := d(i,\ell(i)) + 10$$

Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)



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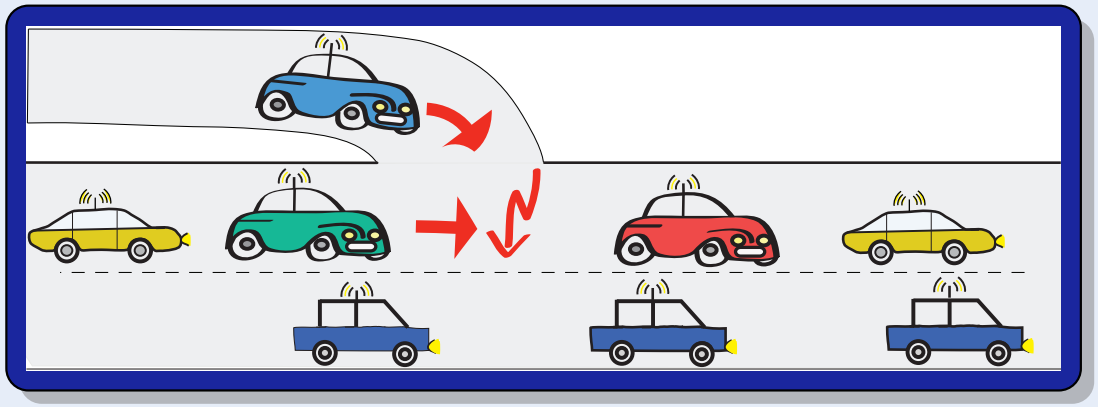
Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)



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  $\ell(i) := \mathit{carInFrontOf}(i)$
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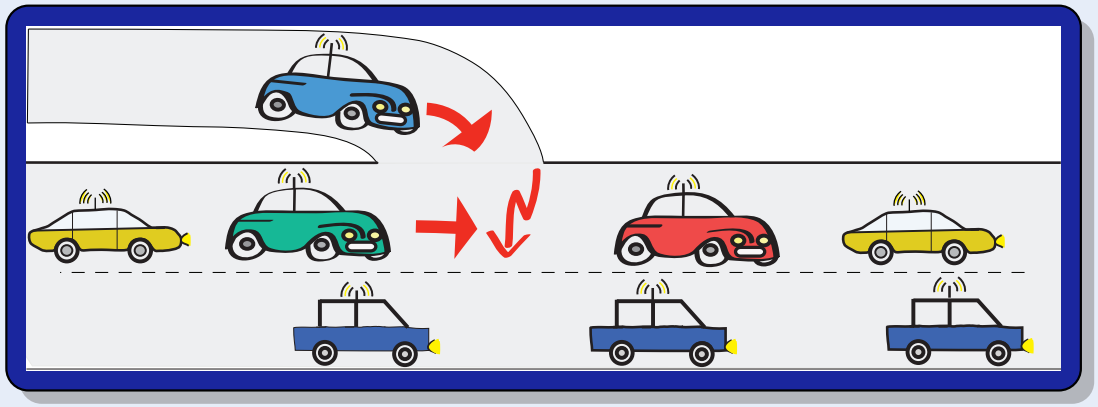
  $n := \texttt{new}\ \mathit{Car}$

$\Rightarrow$ Communication
$\forall i\, d(i, \ell(i)) := d(i, \ell(i)) + 10$

$\Rightarrow$ Discrete structural dynamics
$\ell(i) := \ell(\ell(i))$

Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)



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  $$n := \texttt{new}\ \mathit{Car}$$

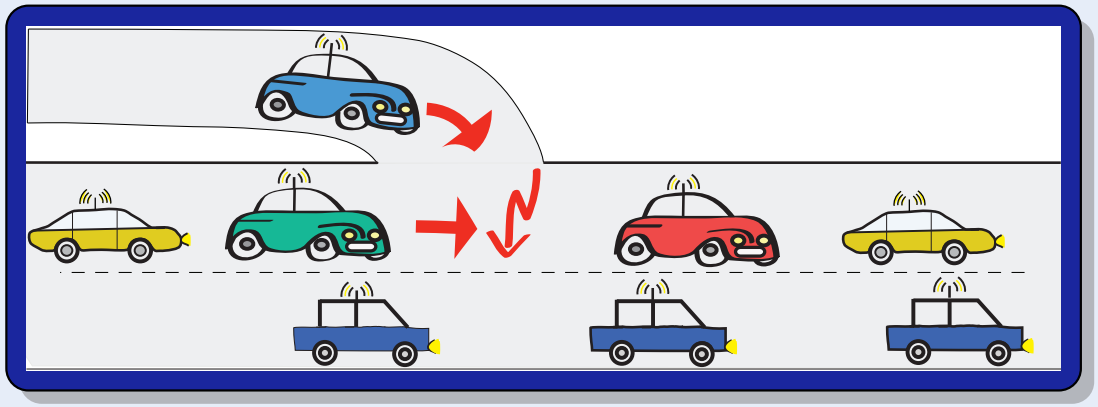
$\Rightarrow$ Communication
$$\forall i\, d(i,\ell(i)) := d(i,\ell(i)) + 10$$

$\Rightarrow$ Discrete structural dynamics
$$\ell(i) := \ell(\ell(i))$$

$\Rightarrow$ Continuous structural dynamics
$$x(i)'' = a(i) + c(i,\ell(i))a(\ell(i))$$

Q: How to model distributed hybrid systems A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)



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7 Formal Details
- Soundness Proof
- Completeness Proof

8 Differential Algebraic Dynamic Logic DAL (Excerpt)
- Differential Invariants

9 Differential Temporal Dynamic Logic dTL (Excerpt)

10 Deduction Modulo Real Algebraic and Computer Algebraic Constraints

11 European Train Control System

12 Collision Avoidance Maneuvers in Air Traffic Control

13 Hybrid Automata Embedding

14 Distributed Hybrid Systems

15 Car Control Verification

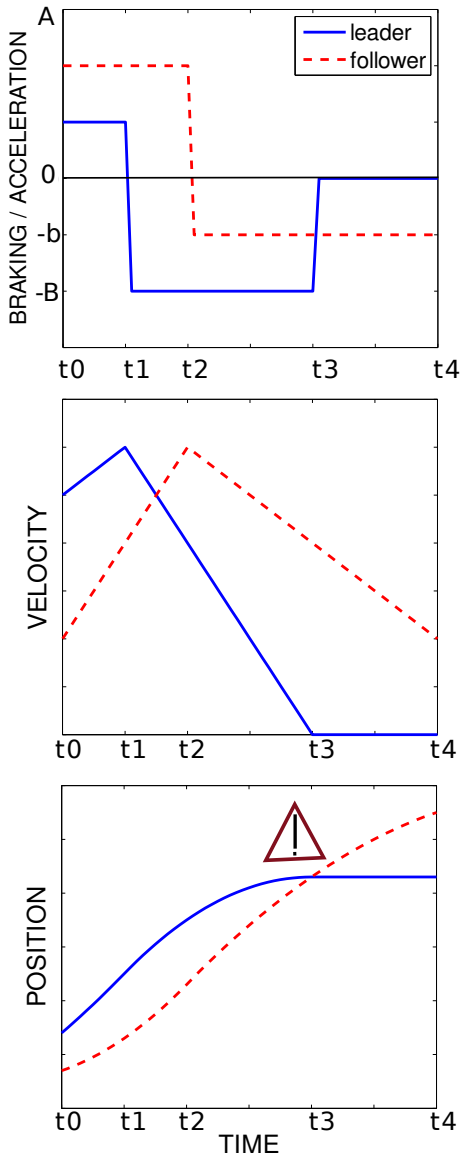
16 Stochastic Hybrid Systems

Challenge: Local lane dynamics

- A car controller for a differential equation respects separation of local lane.

Challenge: Local lane dynamics

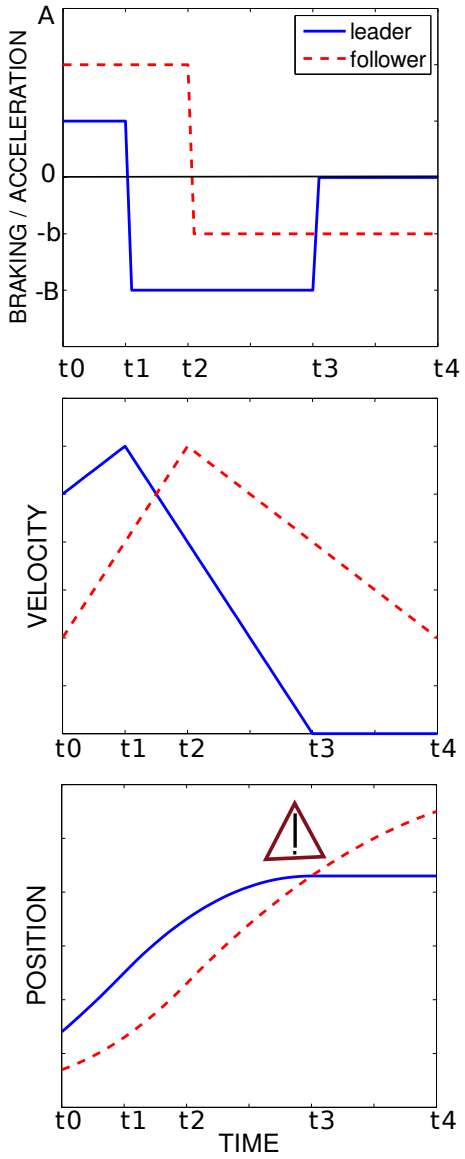
- A car controller for a differential equation respects separation of local lane.
- Follower car maintains safe distance to leader:



Challenge: Local lane dynamics

- A car controller for a differential equation respects separation of local lane.
- Follower car maintains safe distance to leader:

$$f \ll \ell \rightarrow [(a_i := ctrl;\ x_i'' = a_i)^*]\, f \ll \ell$$



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- A car controller for a differential equation respects separation of local lane.
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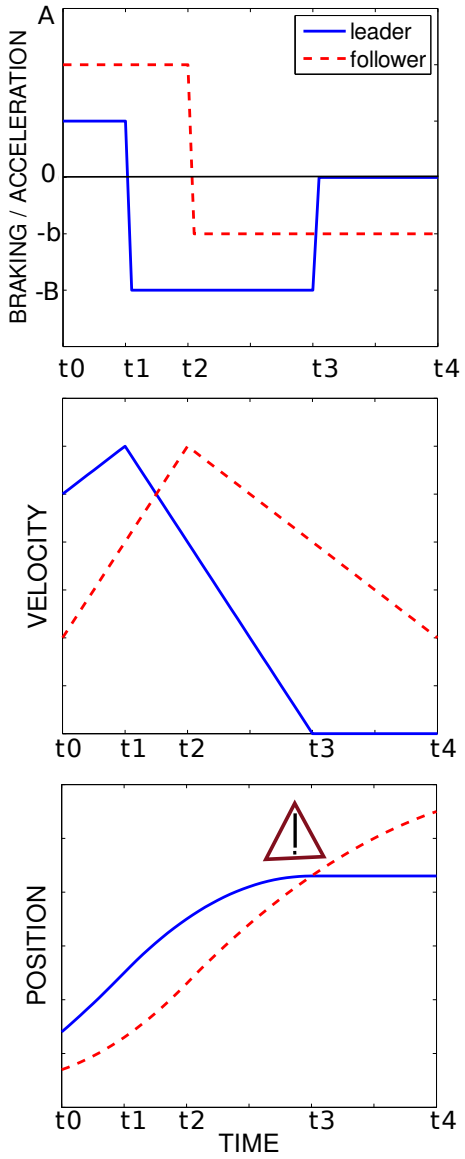
$$f \ll \ell \rightarrow [(a_i := ctrl;\ x_i'' = a_i)^*]\, f \ll \ell$$

$$f \ll \ell \equiv (x_f \leq x_\ell) \wedge (f \neq \ell) \rightarrow (x_\ell > x_f + \frac{v_f^2}{2b} - \frac{v_\ell^2}{2B} \wedge x_\ell > x_f \wedge v_f \geq 0 \wedge v_\ell \geq 0)$$



Challenge: Global lane dynamics

- All controllers for arbitrarily many differential equations respect separation globally on lane.

Challenge: Global lane dynamics

- All controllers for arbitrarily many differential equations respect separation globally on lane.
- Each car safe behind all others

Challenge: Global lane dynamics

- All controllers for arbitrarily many differential equations respect separation globally on lane.
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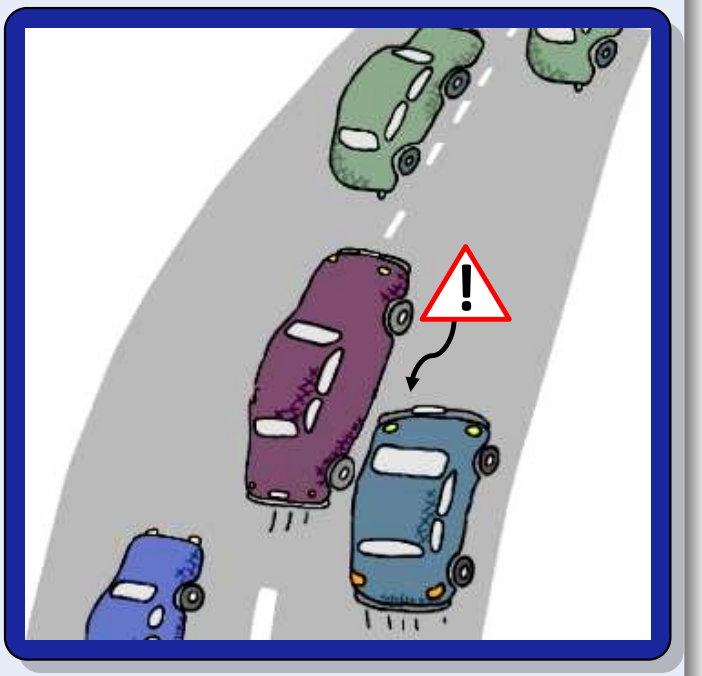
$$[(\forall i\, a(i) := ctrl;\ \forall i\, x(i)'' = a(i))^*]\forall i,j\, i \ll j$$

Challenge: Local highway dynamics

- All controllers for arbitrarily many differential equations respect separation locally on highway.

Challenge: Local highway dynamics

- All controllers for arbitrarily many differential equations respect separation locally on highway.
- For each lane: all controllers for the differential equations respect separation even if cars appear or disappear.



Challenge: Local highway dynamics

- All controllers for arbitrarily many differential equations respect separation locally on highway.
- For each lane: all controllers for the differential equations respect separation even if cars appear or disappear.
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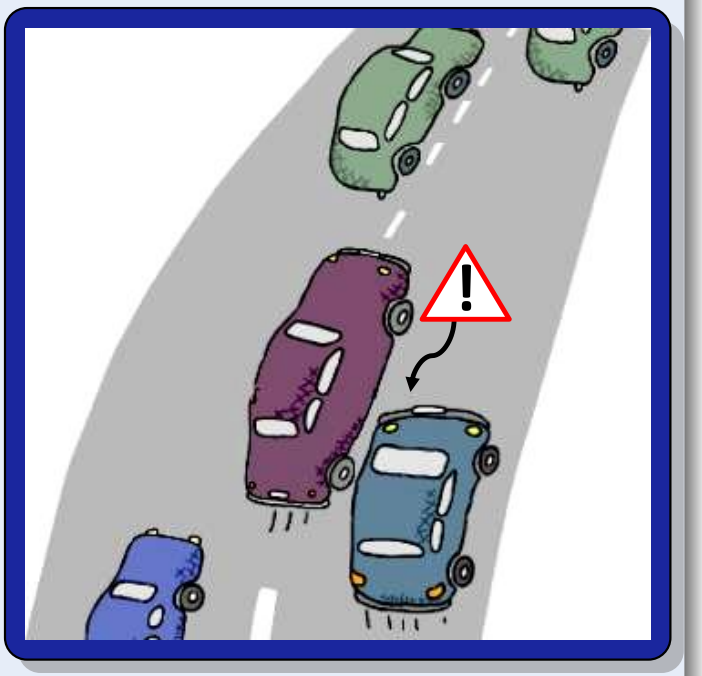

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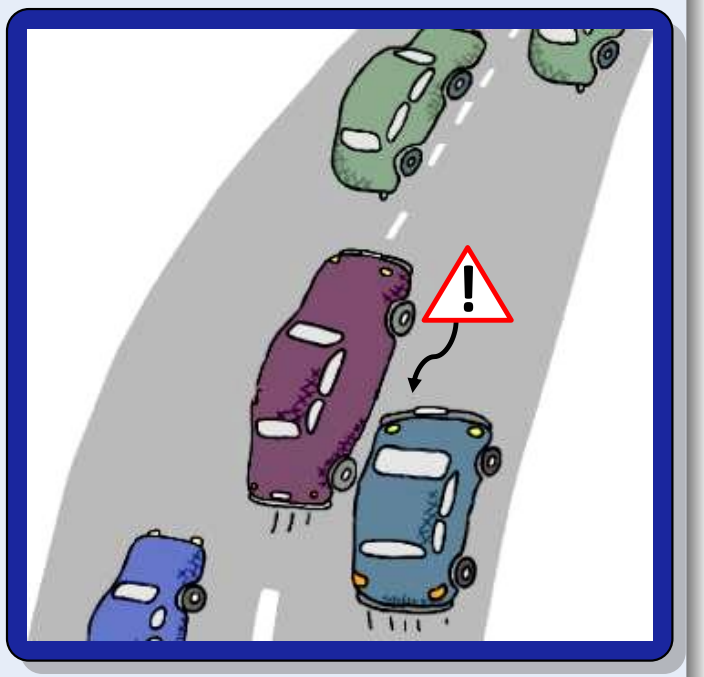
- For each lane: all controllers for the differential equations respect separation even if cars appear or disappear.
- Each car safe behind all others, even if new cars appear or disappear.



$$[(n := \texttt{new}\ C;\ \forall i\ a(i) := ctrl;\ \forall i\ x(i)'' = a(i))^*]\forall i,j\ i \ll j$$

Challenge: Global highway dynamics

- All controllers for arbitrarily many differential equations respect separation globally on highway.

Challenge: Global highway dynamics

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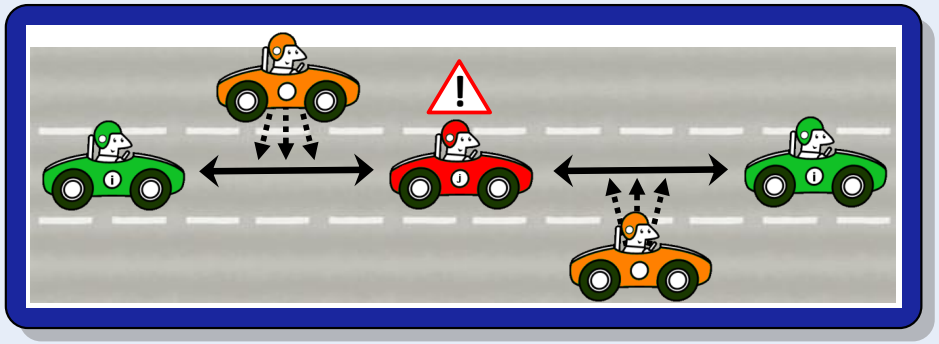
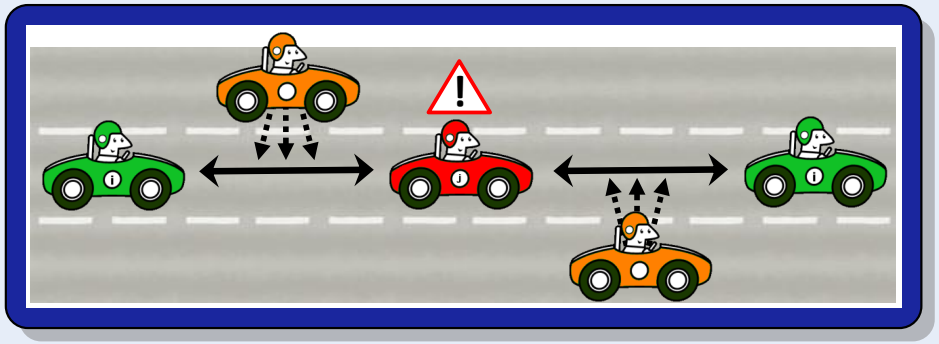
- All controllers for the differential equations respect separation even if cars switch lanes.

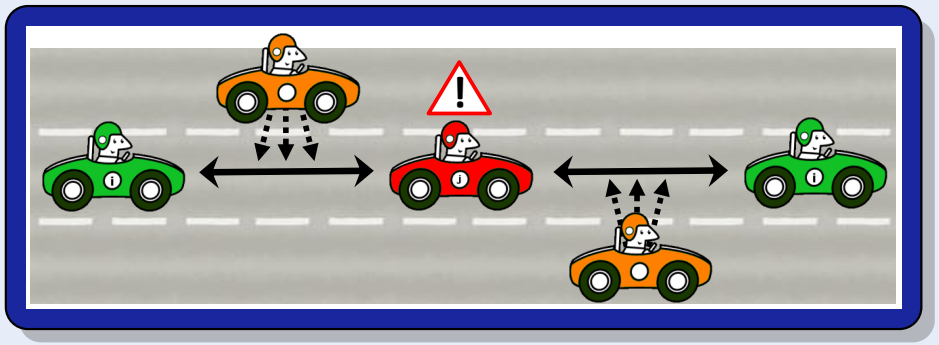


Challenge: Global highway dynamics

- All controllers for arbitrarily many differential equations respect separation globally on highway.
- All controllers for the differential equations respect separation even if cars switch lanes.
- On all lanes, all car safe behind all others on their lanes, even if cars switch lanes.



Challenge: Global highway dynamics



- All controllers for arbitrarily many differential equations respect separation globally on highway.
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7 Formal Details
- Soundness Proof
- Completeness Proof

8 Differential Algebraic Dynamic Logic DAL (Excerpt)
- Differential Invariants

9 Differential Temporal Dynamic Logic dTL (Excerpt)

10 Deduction Modulo Real Algebraic and Computer Algebraic Constraints

11 European Train Control System

12 Collision Avoidance Maneuvers in Air Traffic Control

13 Hybrid Automata Embedding

14 Distributed Hybrid Systems

15 Car Control Verification

16 Stochastic Hybrid Systems

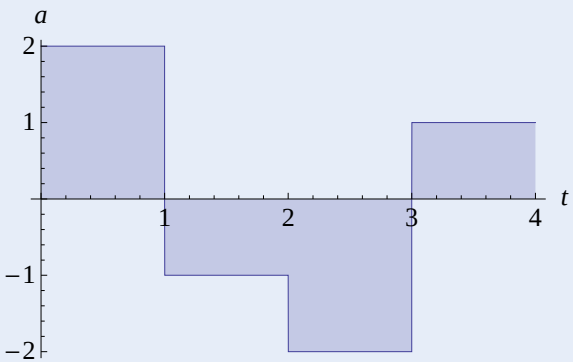
Q: I want to verify trains

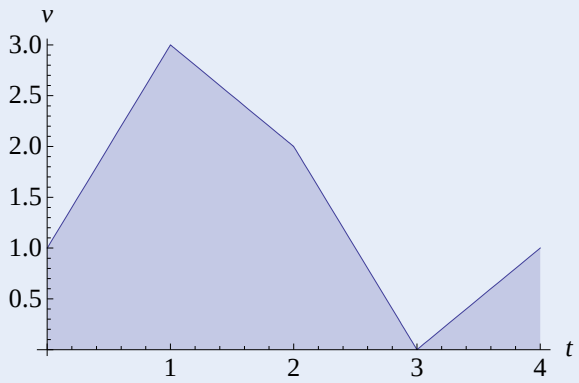
Challenge

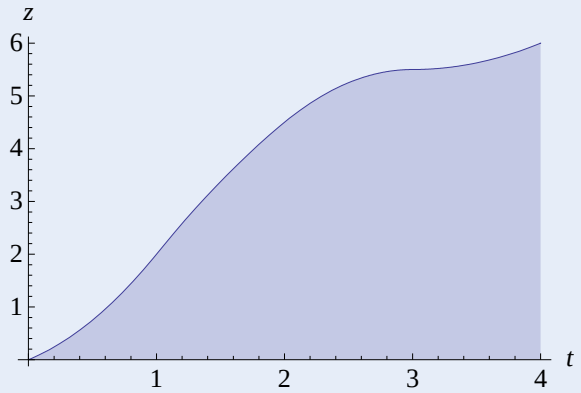
Q: I want to verify trains A: Hybrid systems

Challenge (Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)



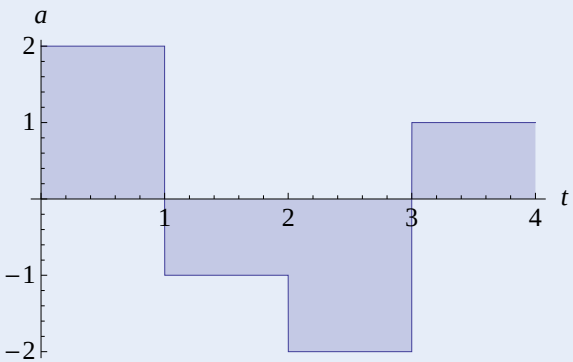


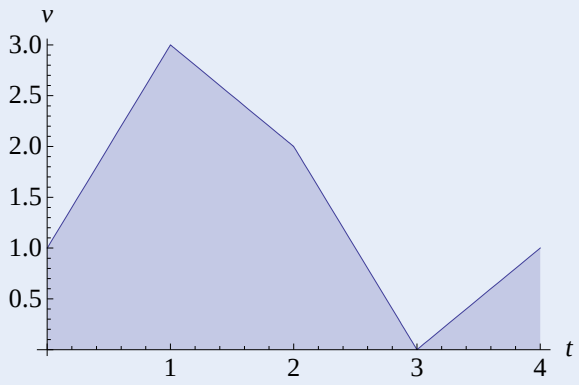


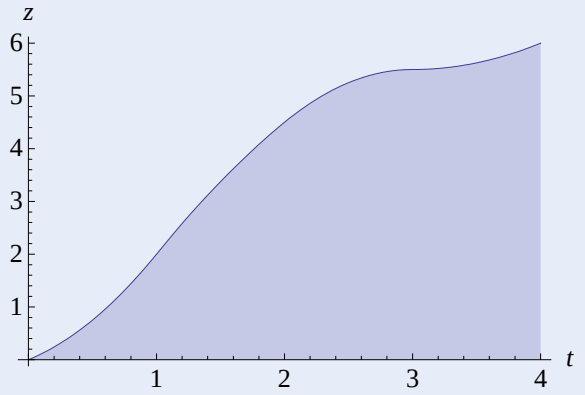
Q: I want to verify trains A: Hybrid systems Q: But there's uncertainties!

Challenge (Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)

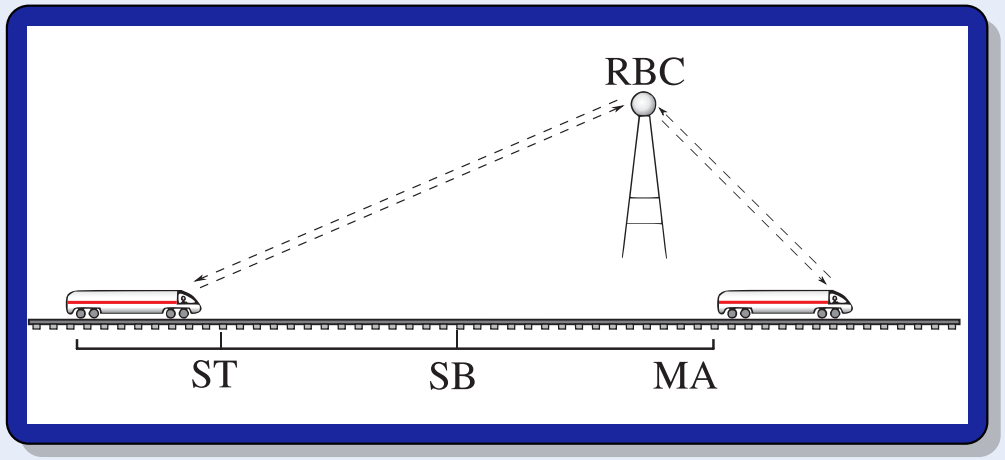






Q: I want to verify uncertain trains

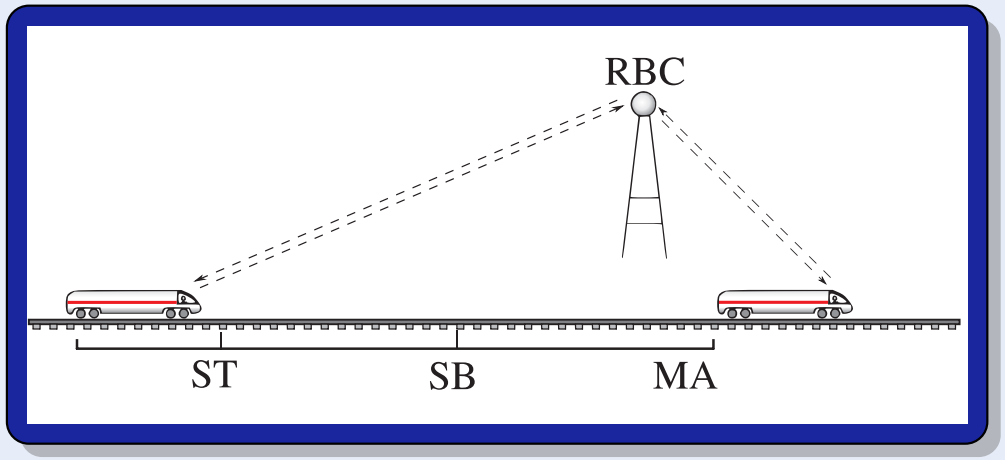
Challenge

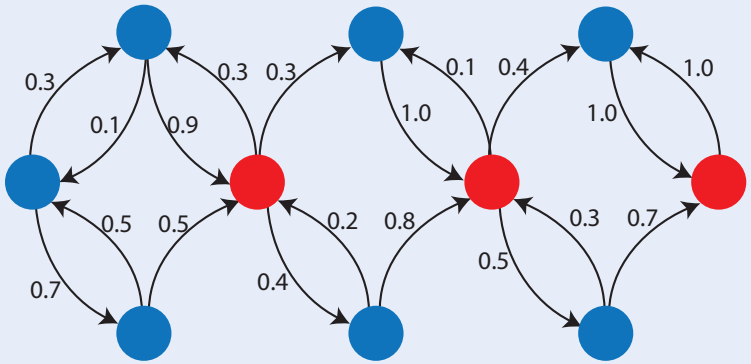


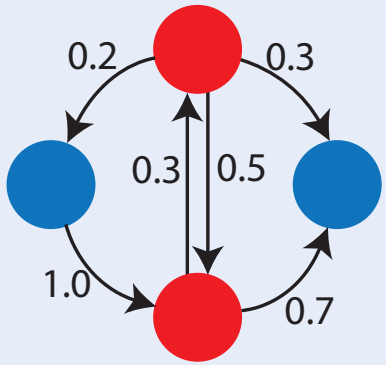
Q: I want to verify uncertain trains A: Markov chains

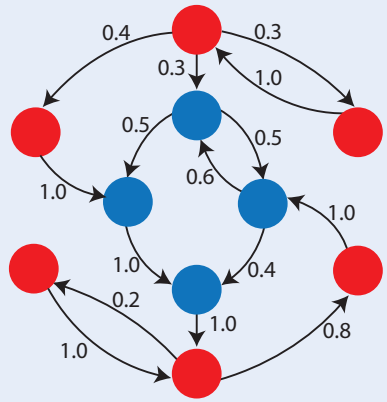
Challenge (Probabilistic Systems)

- Directed graph
  (Countable state space)
- Weighted edges
  (Transition probabilities)





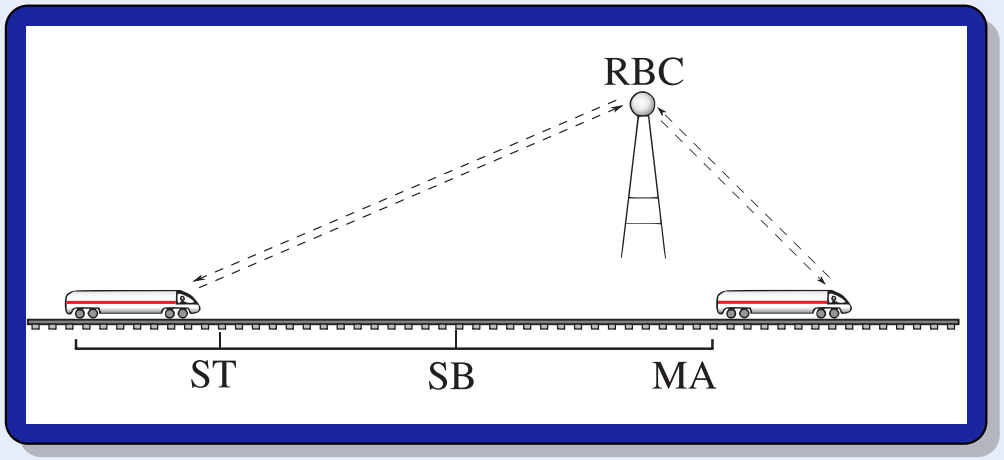


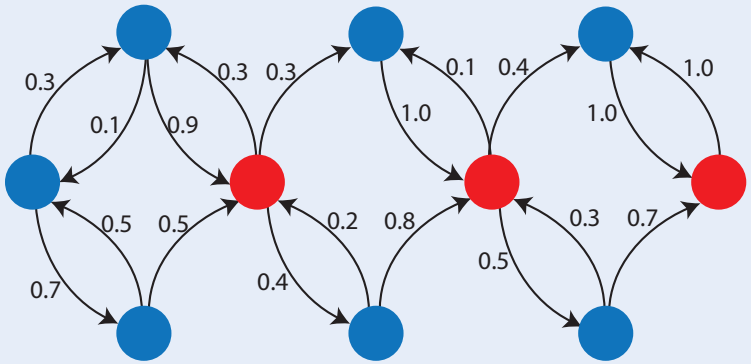


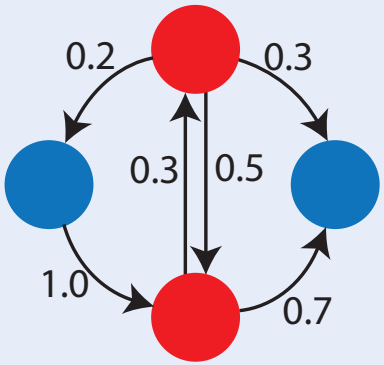
Q: I want to verify uncertain trains A: Markov chains Q: But trains move!

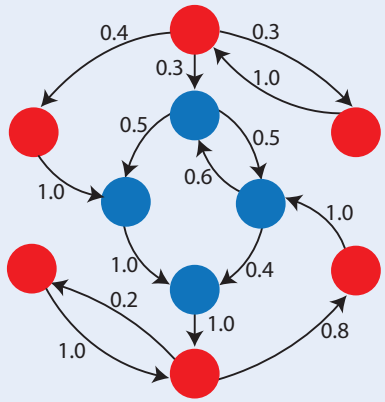
Challenge (Probabilistic Systems)

- Directed graph
  (Countable state space)
- Weighted edges
  (Transition probabilities)



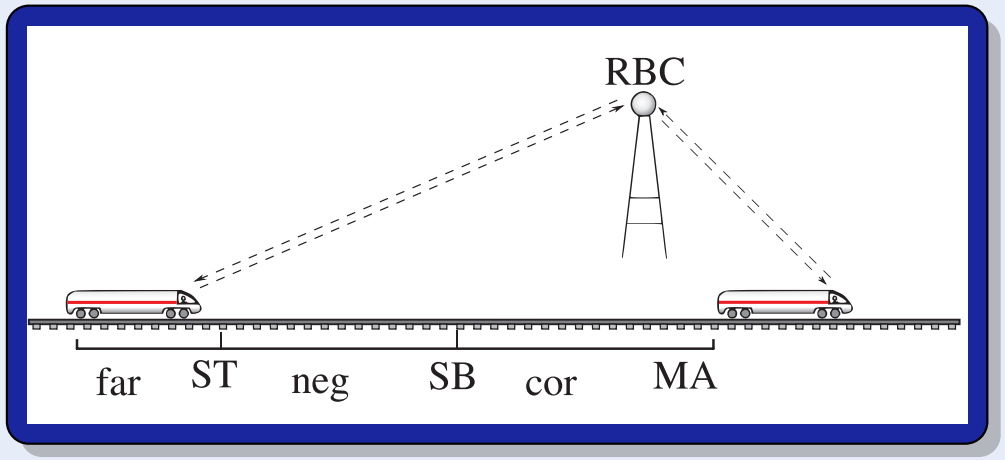






Q: I want to verify uncertain systems

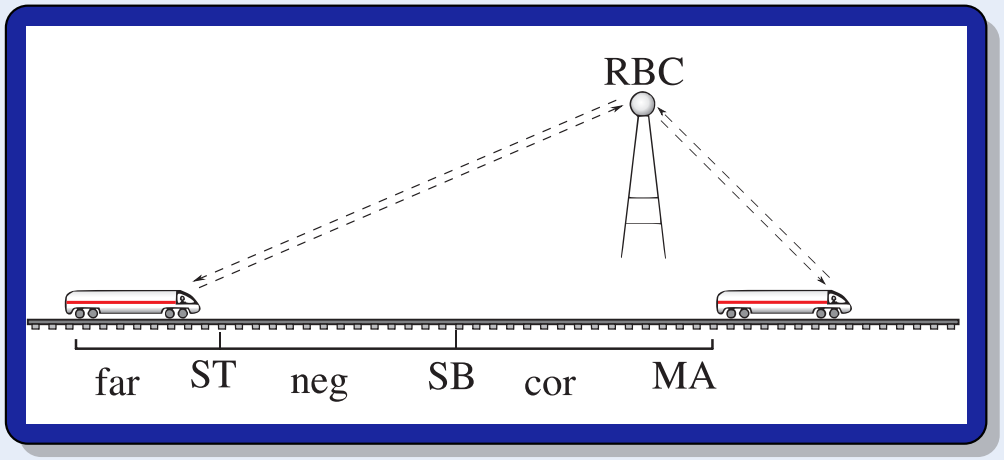
Challenge



Q: I want to verify uncertain systems A: Stochastic hybrid systems

Challenge (Stochastic Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Stochastic dynamics (uncertainty)

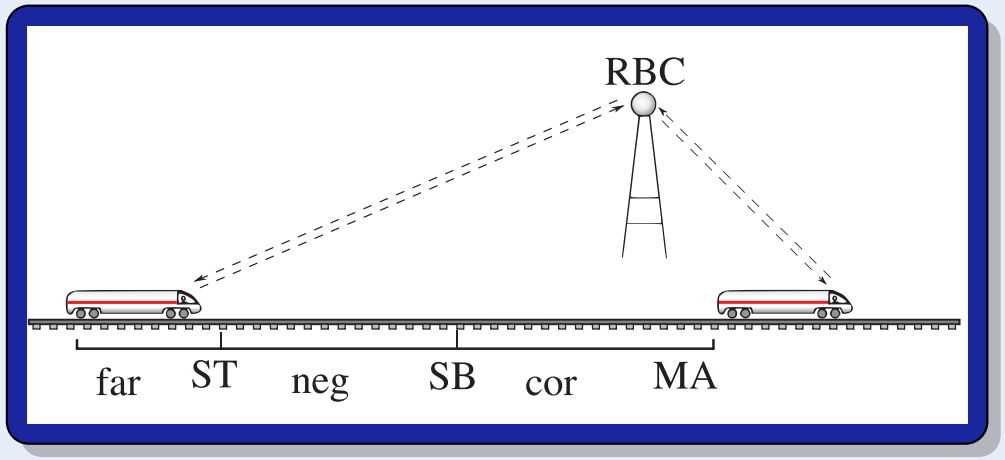


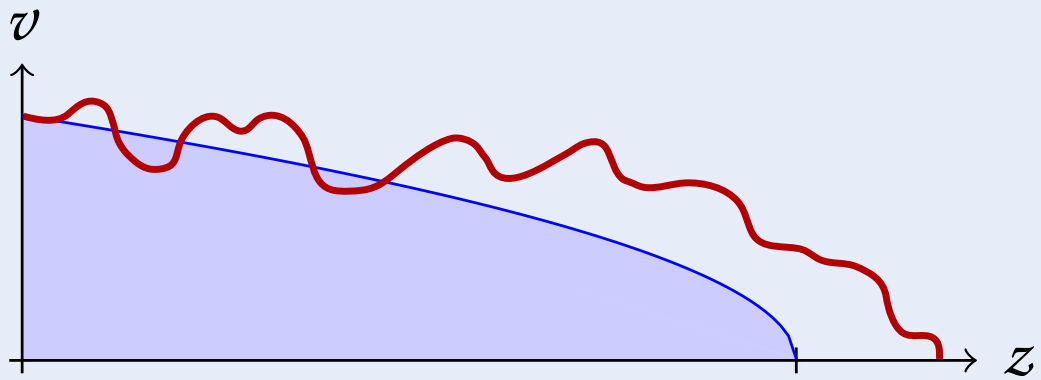
Q: I want to verify uncertain systems A: Stochastic hybrid systems

Challenge (Stochastic Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Stochastic dynamics (uncertainty)

- Discrete stochastic (lossy communication)
- Continuous stochastic (wind, track)

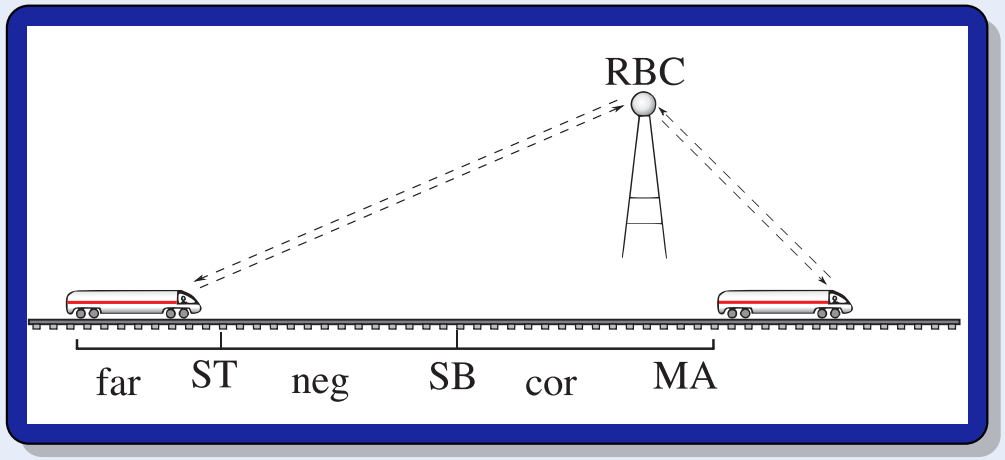


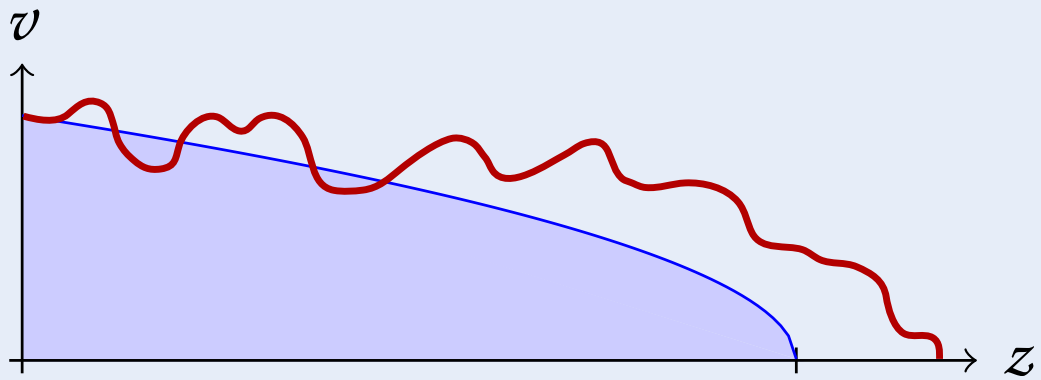


Q: I want to verify uncertain systems A: Stochastic hybrid systems Q: How?

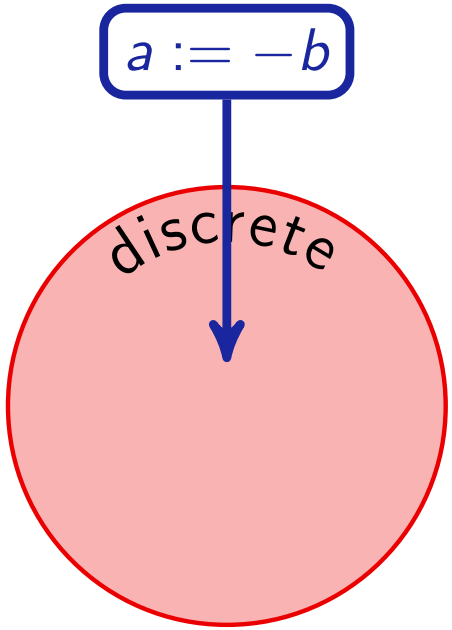
Challenge (Stochastic Hybrid Systems)

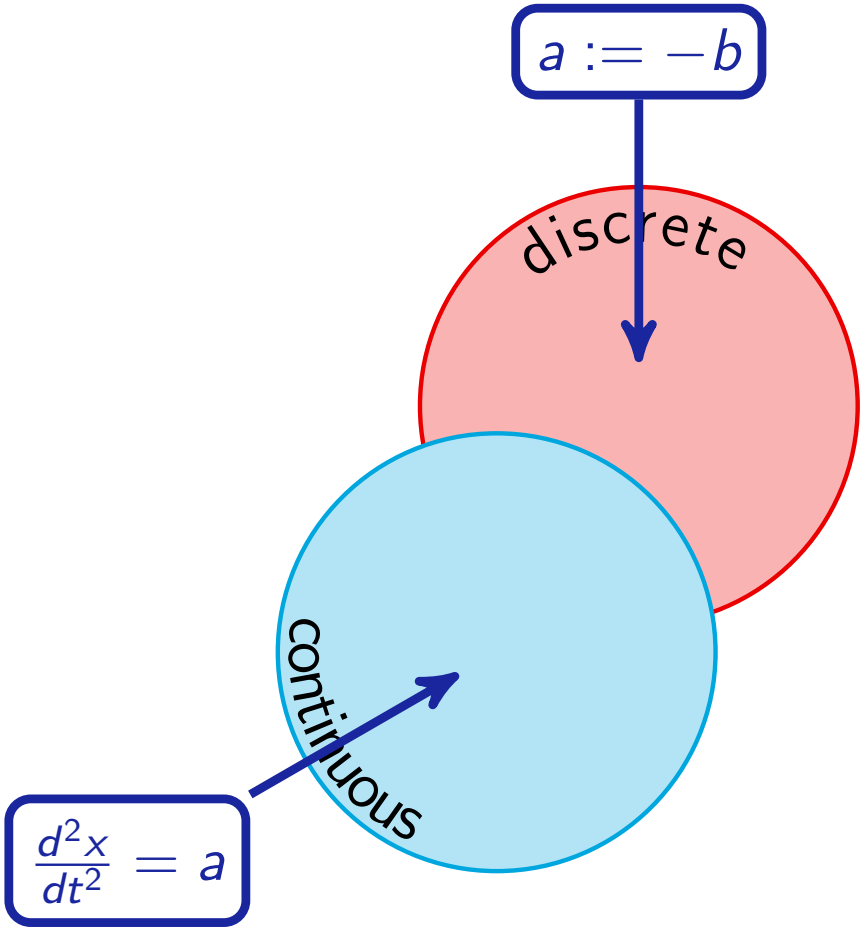
- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Stochastic dynamics (uncertainty)
- Discrete stochastic (lossy communication)
- Continuous stochastic (wind, track)



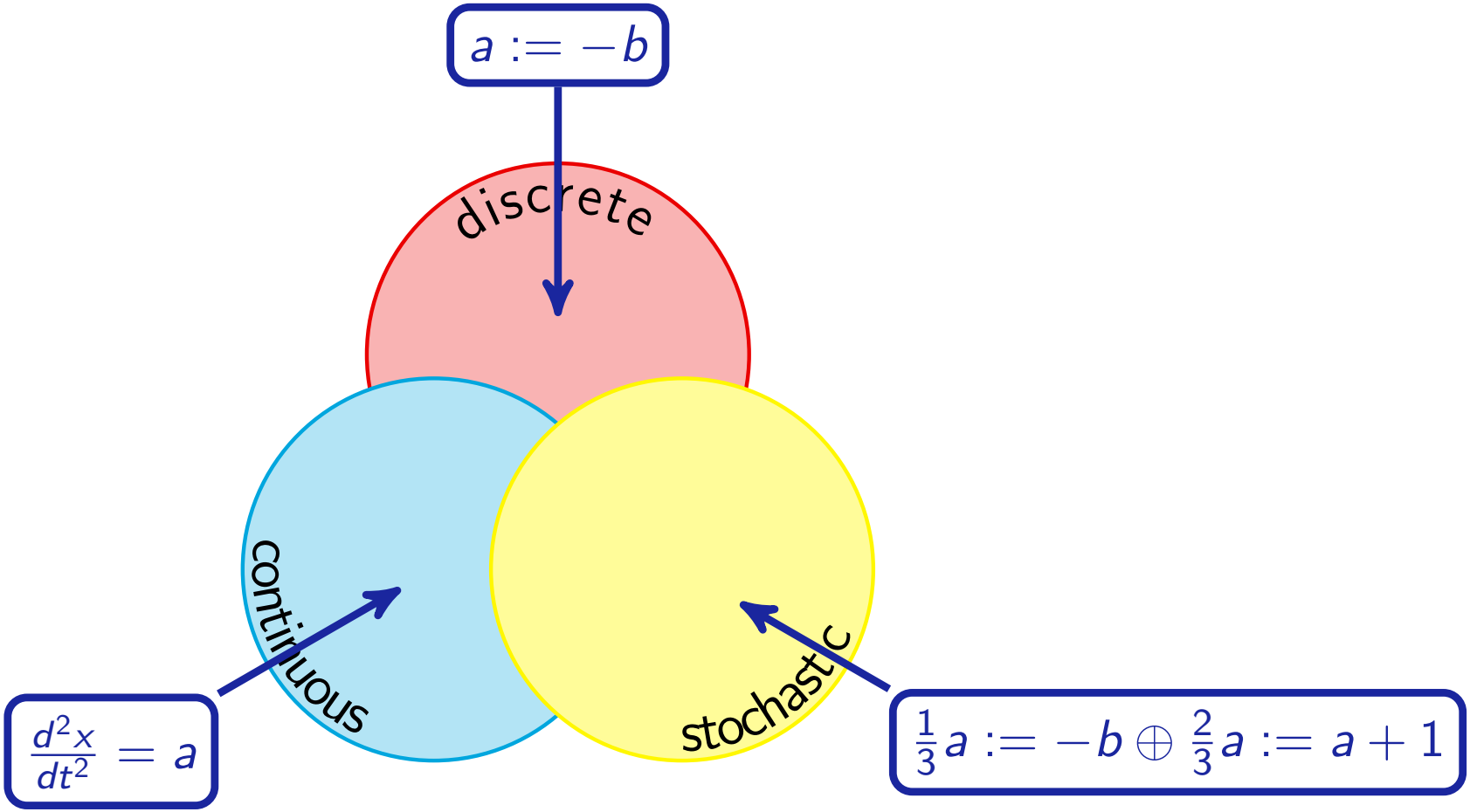


Model for Stochastic Hybrid Systems

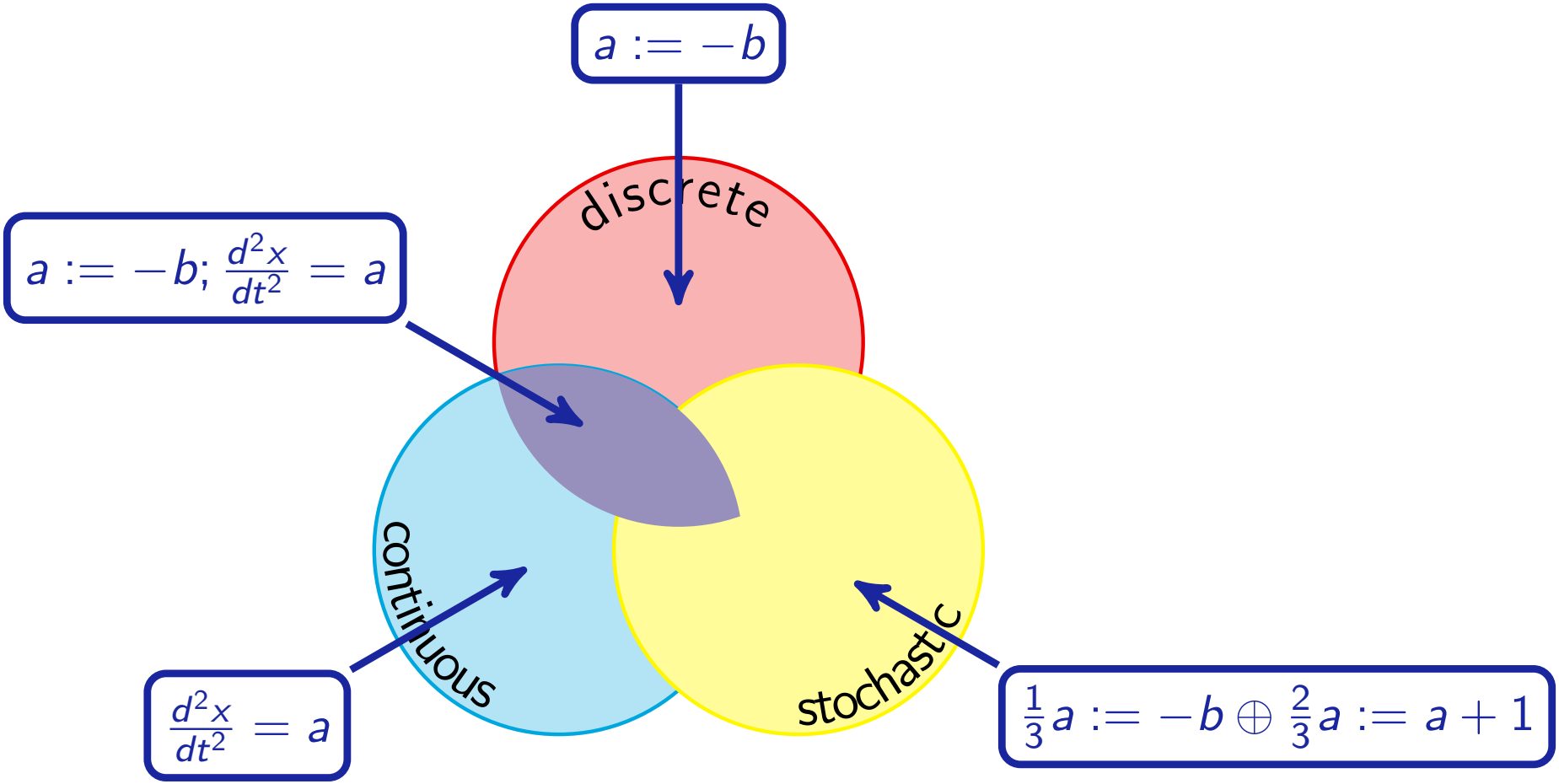



$a := -b$
discrete
continuous
$\frac{d^2x}{dt^2} = a$

Model for Stochastic Hybrid Systems



Model for Stochastic Hybrid Systems



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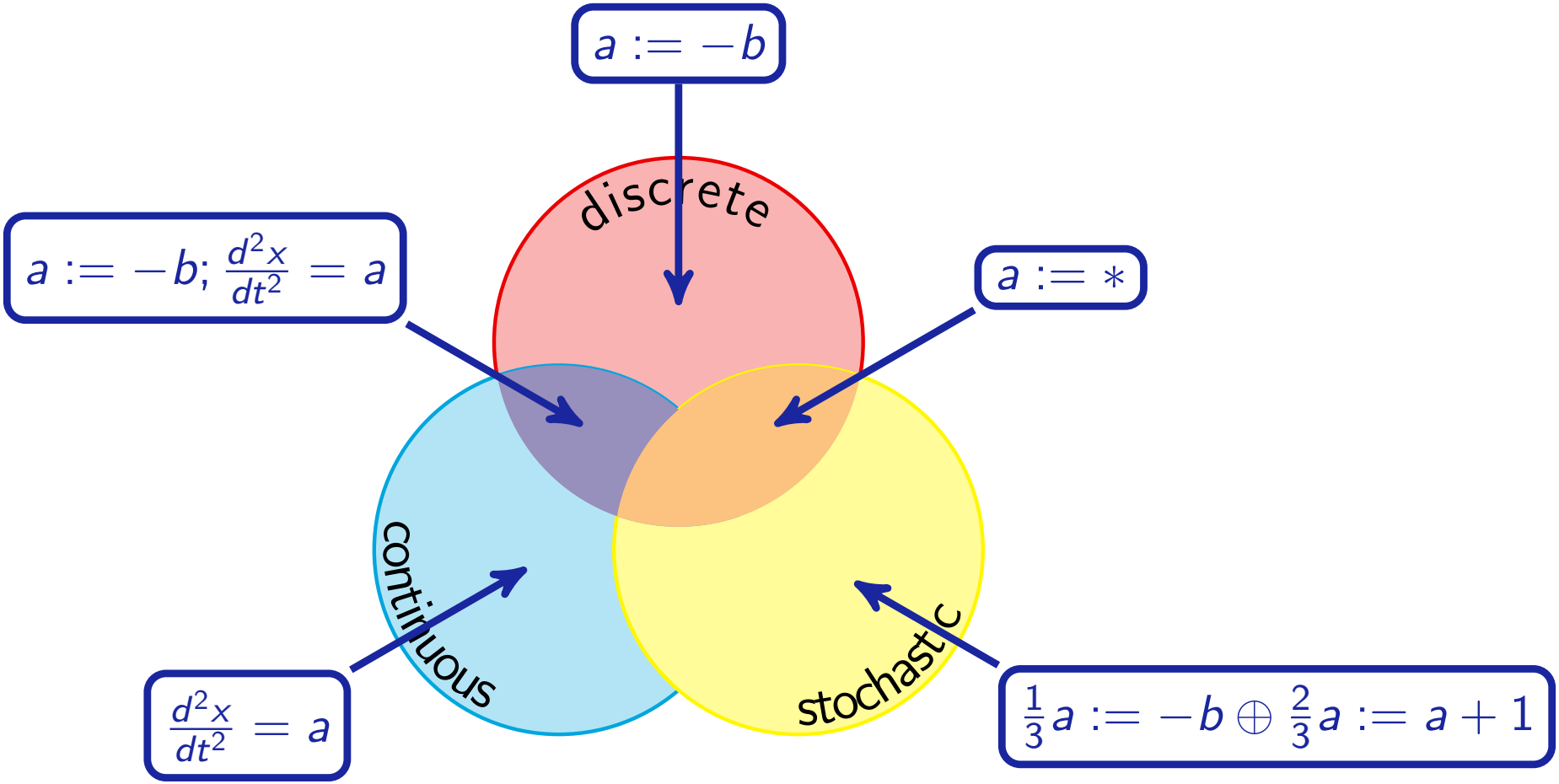

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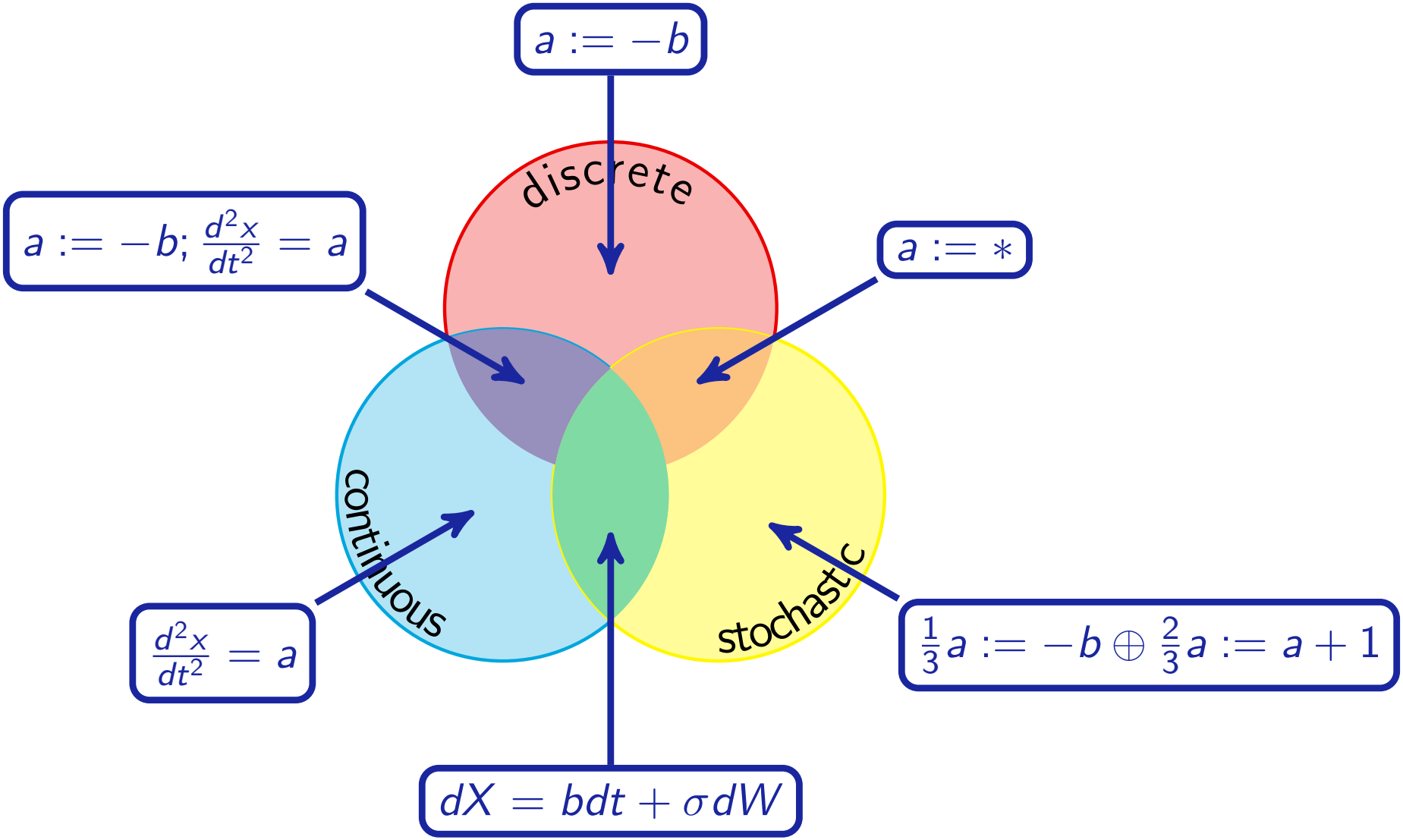

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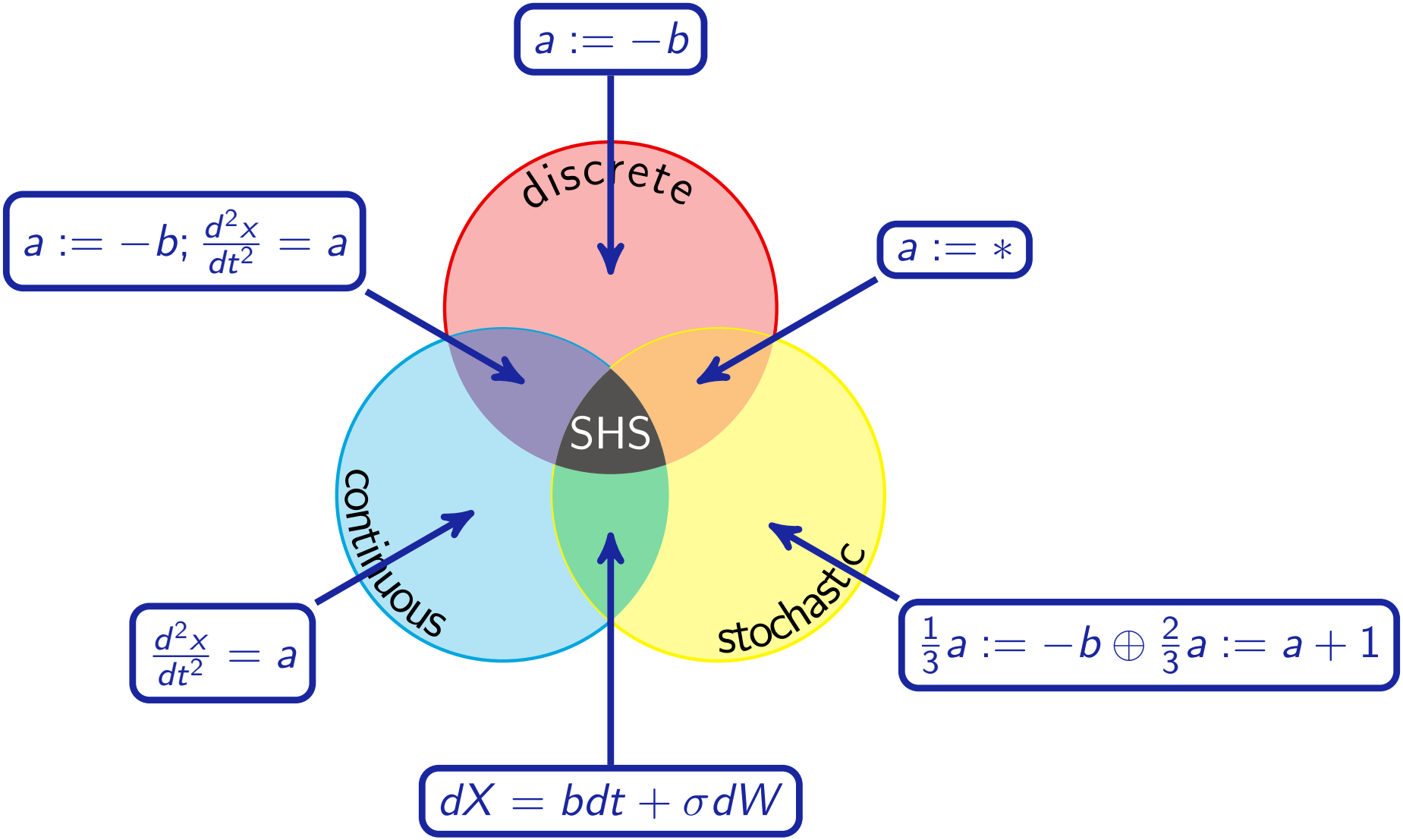
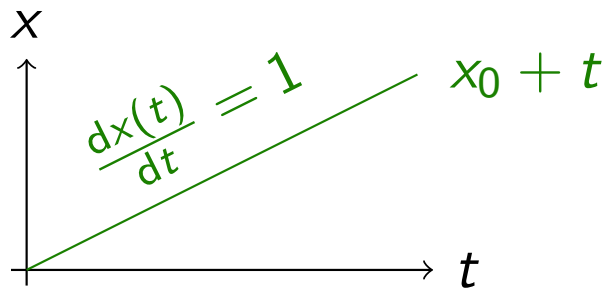


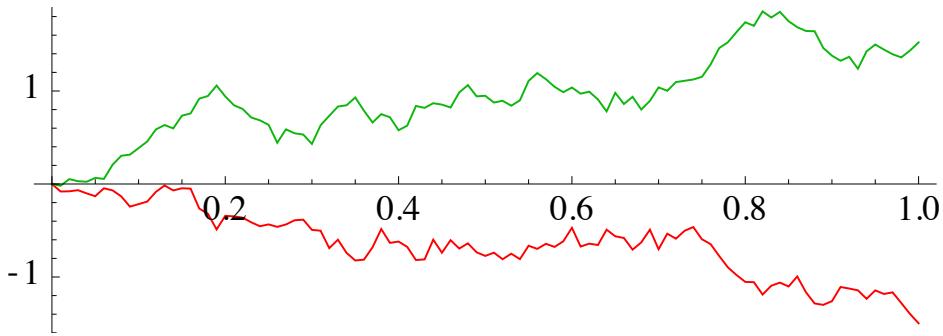
Definition (Ordinary differential equation (ODE))

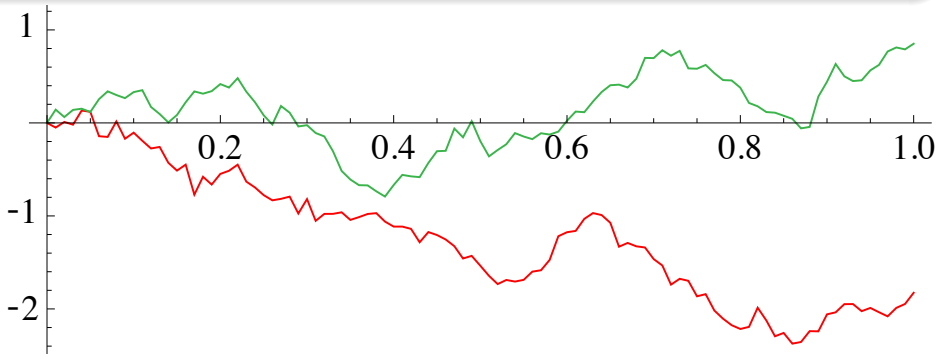
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = b(x(t)) \quad x(0) = x_0$$



Definition (Itō stochastic differential equation (SDE))

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \quad X_0 = Z$$





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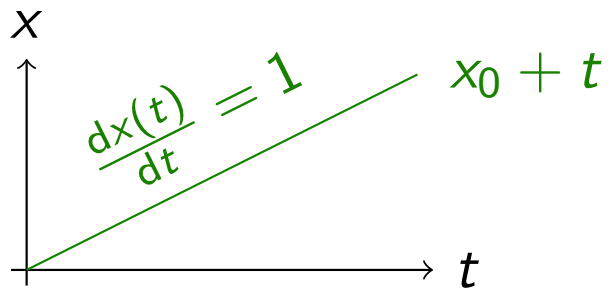
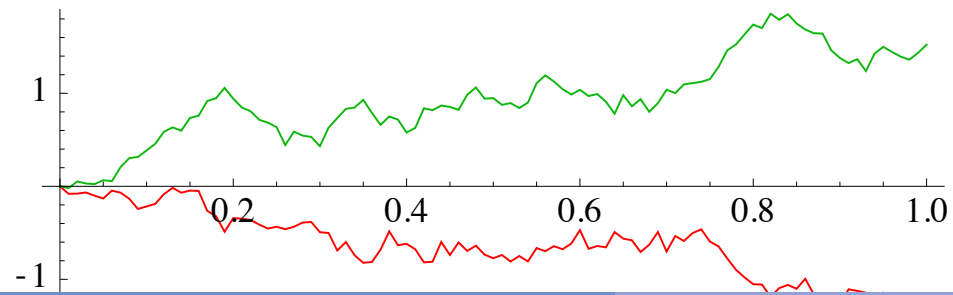
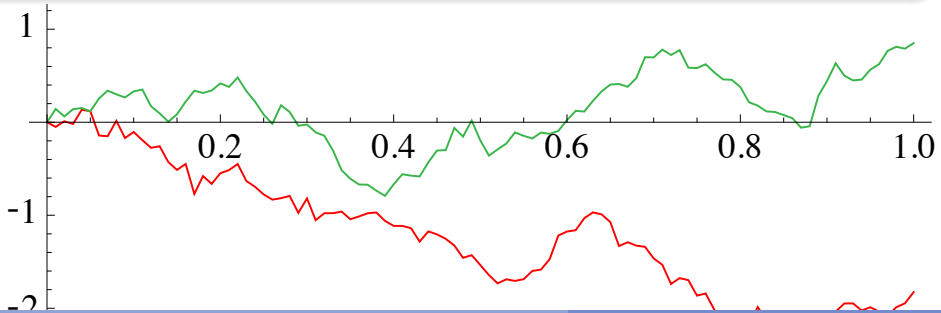


Definition (Itō stochastic differential equation (SDE))

$$X_s = Z + \int_0^s dX_t = Z + \int_0^s b(X_t)dt + \int_0^s \sigma(X_t)dW_t$$





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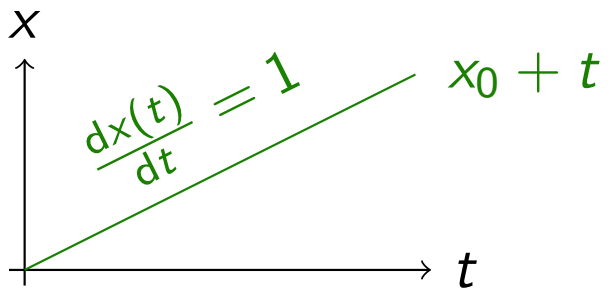

Calculus

Definition (Itō stochastic differential equation (SDE))

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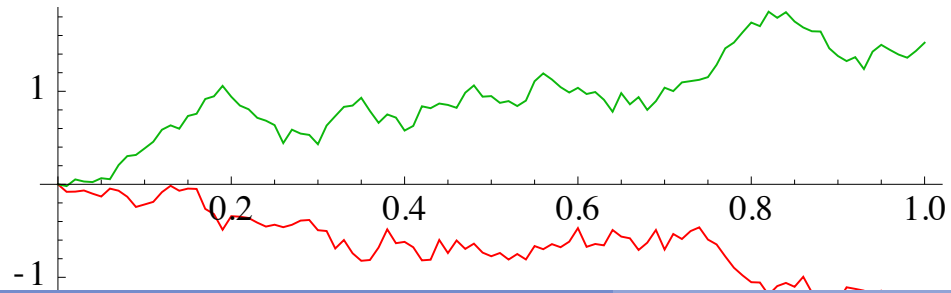

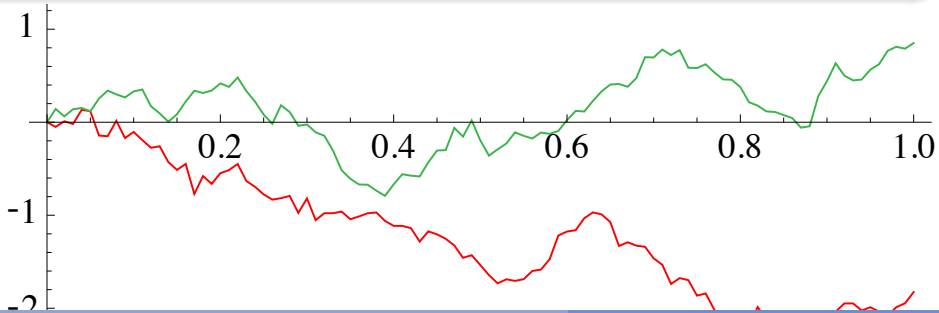

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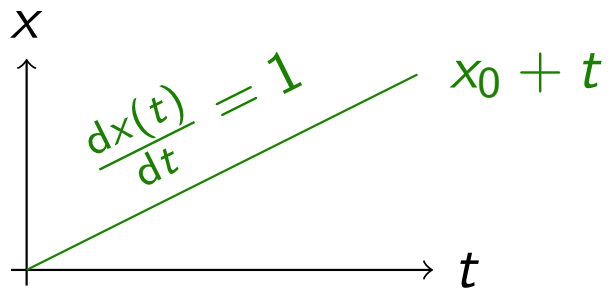

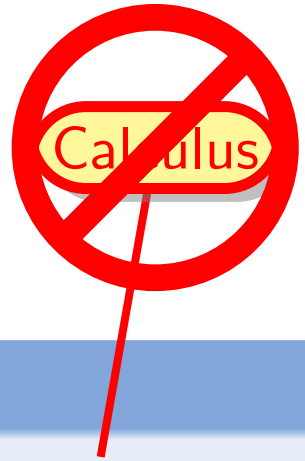

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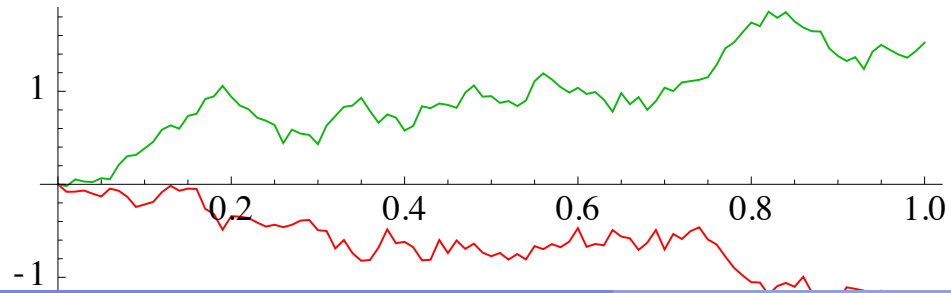

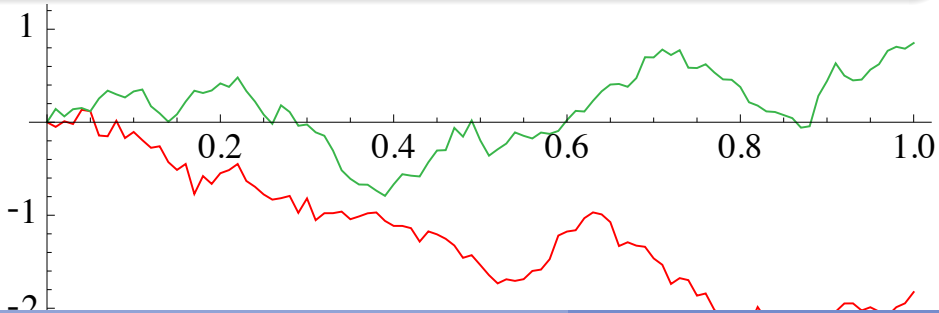

Stochastic Hybrid Programs: Syntax

Definition (Stochastic hybrid program α)

$x := \theta$	(assignment)	jump & test
$x := *$	(random assignment)	
$?H$	(conditional execution)	
$dx = bdt + \sigma dW \,\&\, H$	(SDE)	
$\alpha; \beta$	(seq. composition)	algebra
$\lambda\alpha \oplus \nu\beta$	(convex combination)	
α^*	(nondet. repetition)	