Outline

1. Learning Objectives

2. Delays in Control
   - Back to the Drawing Desk: Quantum the Ping Pong Ball
   - Quantum the Time-triggered Ping Pong Ball
   - The Impact of Delays on Events
   - Cartesian Demon
   - Predictive Control
   - Design-by-Invariant
   - Controlling the Control Points
   - Short Invariants

3. Proof

4. Summary
   - Zeno’s Quantum Turtles
   - A Note on Assignments
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3 Proof

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Learning Objectives
Reactions & Delays

using loop invariants
design time-triggered control
design-by-invariant

modeling CPS
designing controls
time-triggered control
reaction delays
discrete sensing

semantics of time-triggered control
operational effect
finding control constraints
model-predictive control
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**Quantum’s Ping Pong Proof Invariants**

**Proposition (Quantum can play ping pong safely)**

\[
0 \leq x \wedge x \leq 5 \wedge v \leq 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \wedge f \geq 0 \Rightarrow \\
\left(((x' = v, v' = -g \& x \geq 0 \wedge x \leq 5) \cup (x' = v, v' = -g \& x \geq 5)) ; \\
\text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \wedge v \geq 0) v := -fv\right)^* (0 \leq x \leq 5)
\]

**Proof**

@invariant(0 \leq x \leq 5 \wedge (x = 5 \rightarrow v \leq 0))
Proposition (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]
\[ (((x' = v, v' = -g \land x \geq 0 \land x \leq 5) \cup (x' = v, v' = -g \land x \geq 5)); \]
\[ \text{if}(x = 0) v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) v := -fv) \]
\[ ](0 \leq x \leq 5)\]

Proof

@invariant(0 \leq x \leq 5 \land (x = 5 \rightarrow v \leq 0))

Just can’t implement . . .
Physical vs. Controller Events

1. Justifiable: Physical events (on ground $x = 0$)
2. Justifiable: Physical evolution domains (above ground $x \geq 0$)
3. Questionable: Controller evolution domain ($x \leq 5$)
4. Unlike physics, controllers won’t run all the time. Just often.
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]

\[ ([\{ x' = v, v' = -g \land x \geq 0 \};
\quad \text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) v := -fv]^{*})(0 \leq x \leq 5) \]

Proof? Ask René Descartes
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \wedge x \leq 5 \wedge v \leq 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \wedge f \geq 0 \rightarrow \]

\[
(\{x' = v, v' = -g \& x \geq 0\};
\quad \text{if}(x=0) v := -cv \quad \text{else if}(4 \leq x \leq 5 \wedge v \geq 0) v := -fv)^*
\]

(0 \leq x \leq 5)

Proof?

Ask René Descartes who says no!

Could miss if-then event
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]
\[ [(\{x' = v, v' = -g \land x \geq 0 \land t \leq 1\}; \]
\[ \text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) v := -fv) \] * \] \[ ) (0 \leq x \leq 5) \]

Proof?
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]

\[ [(x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1); \]

\[ \text{if}(x=0) \text{then } v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) \text{ then } v := -fv] \]

\( (0 \leq x \leq 5) \)
Conjecture (Quantum can play ping pong safely)

\[
0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \\
[ (t := 0; \{ x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1 \}; \\
  \text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) v := -fv)^* ](0 \leq x \leq 5)
\]

Proof? Ask René Descartes
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \wedge x \leq 5 \wedge v \leq 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \wedge f \geq 0 \rightarrow \]

\[(\text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \wedge v \geq 0) v := -fv; \]

\[ t := 0; \{x' = v, v' = -g, t' = 1 \& x \geq 0 \wedge t \leq 1\})^*]\(0 \leq x \leq 5)\)

Proof? Ask René Descartes

Control before physics
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]

\[ \left( \text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) v := -fv; t := 0; \{ x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1 \} \right)^* \] \( (0 \leq x \leq 5) \)

Proof? Ask René Descartes

Could act early or late
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \implies \]

\[ \left( \text{if}(x=0) \quad v := -cv \quad \text{else if}(4 \leq x \leq 5 \land v \geq 0) \quad v := -fv; \quad t := 0; \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1 \} \right)^* \text{ for } (0 \leq x \leq 5) \]

Proof?

Ask René Descartes who says no!

Could miss event off cycle
Delays vs. Events

1. Periodically/frequently monitoring for an event with a polling frequency / reaction time
2. Delays may make the controller miss events.
3. Discrepancy event-driven idea vs. real time-triggered implementation.
4. Slow controllers monitoring small regions of a fast moving system.
5. Issues indicate poor event abstraction
6. Controller need to be aware of its own delay
Cartesian Doubt: René Descartes’s Cartesian Demon 1641

Outwit the Cartesian Demon

Skeptical about the truth of all beliefs until justification has been found.
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g > 0 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]
\[
[\text{if}(x=0) v := -cv \text{ else if}(4 \leq x \leq 5 \land v \geq 0) v := -fv; \\
t := 0; \{x' = v, v' = -g, t' = 1 \& x \geq 0 \land t \leq 1\}]^* (0 \leq x \leq 5)
\]

Proof? Ask René Descartes who says no!

Could miss event off cycle
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \wedge x \leq 5 \wedge v \leq 0 \wedge g = 1 \wedge 1 \geq c \geq 0 \wedge f \geq 0 \rightarrow \\
\left[ (\text{if}(x=0) \ v := -cv \ 	ext{else if}(x>5\frac{1}{2}) \ v := -fv; \right. \\
\left. t := 0; \{x' = v, v' = -g, t' = 1 \wedge x \geq 0 \wedge t \leq 1\} \right)^* \right] (0 \leq x \leq 5) \]

Proof? Ask René Descartes

predict: \( x + v - \frac{g}{2} > 5 \)
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \]

\[ [(\text{if}(x=0) v := -cv \text{ else if}(x>5 \frac{1}{2} - v \land v \geq 0) v := -fv;\]
\[ t := 0; \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1\}]^\ast (0 \leq x \leq 5) \]

Proof? Ask René Descartes who says no!

Safe after 1 s but not until then

All depends on sampling
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow\]

\[
\left\{ \begin{array}{l}
\text{if}(x = 0) v := -cv \\
\text{else if}(x > 5.5) v := -fv; \\
t := 0; \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1\}\right.\]

\]

\] (0 \leq x \leq 5)

Proof? Ask René Descartes who says no!

Safe after 1 s but not until then

All depends on sampling
Design-by-Invariant

\[ 2g x = 2g H - v^2 \land x \geq 0 \land c = 1 \land g > 0 \]

bouncing ball invariant
Quantum Discovers Design-by-Invariant

Design-by-Invariant

\[ 2g \times = 2gH - v^2 \land x \geq 0 \land c = 1 \land g = 1 \]

simplify arithmetic
Design-by-Invariant

\[ 2x = 2H - \nu^2 \land x \geq 0 \]
Design-by-Invariant

\[ 2x = 2 \cdot H - v^2 \land x \geq 0 \]
Design-by-Invariant

$$2x = 2 \cdot 5 - v^2 \land x \geq 0$$
Quantum Discovers Design-by-Invariant

Design-by-Invariant

\[ 2x > 2 \cdot 5 - \sqrt{2} \land x \geq 0 \]

potential exceeds safe height
Design-by-Invariant

\[ 2x > 2 \cdot 5 - v^2 \land x \geq 0 \]

use invariant for control
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 \geq c \geq 0 \land f \geq 0 \rightarrow \\
\left( \begin{array}{l}
\text{if}\ (x=0) \ v := -cv \ \text{else if}\ ((x>5\frac{1}{2} - v \lor 2x>2.5 - v^2) \land v \geq 0) \ v := -fv; \\
\ t := 0; \ \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1\} \right)^* \right) (0 \leq x \leq 5) \\
\]

Proof? Ask René Descartes
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]

\[
\left( \begin{array}{l}
\text{if}(x=0) v := -cv \\
\text{else if}((x>5\frac{1}{2}) \lor 2x>2.5-v^2) \land v\geq 0) v := -fv; \\
t := 0; \{x' = v, v' = -g, t' = 1 \& x \geq 0 \land t \leq 1\}^* \end{array} \right)\]

Proof? Ask René Descartes
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]
\[ \left( \text{if}(x=0) v := -cv \text{ else if } (x>5\frac{1}{2} - v \lor 2x > 2 \cdot 5 - v^2) \land v \geq 0 \right) v := -fv; \]
\[ t := 0; \{ x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1 \} \right) \rightarrow (0 \leq x \leq 5) \]

Proof? Ask René Descartes who says no!

No control near ground
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]
\[ \left( \text{if}(x=0) v := -cv; \text{if}(x>5\frac{1}{2} - v \lor 2x > 2 \cdot 5 - v^2) \land v \geq 0) v := -fv; \right. \]
\[ t := 0; \{ x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1 \} \right) \]
\[ (0 \leq x \leq 5) \]

Proof? Ask René Descartes

Control despite ground
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]
\[ \left( \text{if}(x=0) \ v := -cv; \text{if}(x > 5\frac{1}{2} - v \lor 2x > 2 \cdot 5 - v^2) \land v \geq 0) \ v := -fv; \right. \]
\[ t := 0; \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1\} \times (0 \leq x \leq 5) \]

Proof? Ask René Descartes who says yes
Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]
\[ \left( \begin{array}{l}
  \text{if} (x=0) v := -cv; \\
  \text{if} ((x>5^{\frac{1}{2}} - v \lor 2x>2.5-v^2) \land v \geq 0) v := -fv; \\
  t := 0; \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1\} \right)^* \] (0 \leq x \leq 5)

Proof? Ask René Descartes who says yes but should have said no!
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]
\[ [(\text{if}(x=0) v := -cv; \text{if}((x>5\frac{1}{2})-v \lor 2x>2.5-v^2) \land v \geq 0) v := -fv; \]
\[ t := 0; \{x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1\}]^* \]
\[ (0 \leq x \leq 5) \]

Proof?

Ask René Descartes who says yes but should have said no!

Invariants are \textit{invariants}!

True ever \(\sim\) true initially
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \wedge x \leq 5 \wedge v \leq 0 \wedge g = 1 \wedge 1 = c \wedge f = 1 \rightarrow \]

\[
\left( \text{if}(x=0) v := -cv; \text{if}((x>5\frac{1}{2} - v \vee 2x>2.5 - v^2 \land v<1) \land v \geq 0) v := -fv; \right.
\]

\[
t := 0; \{ x' = v, v' = -g, t' = 1 \& x \geq 0 \land t \leq 1 \} \right) \ast (0 \leq x \leq 5)
\]

Proof? Ask René Descartes

\[ v(t) = v - gt = v - t < 0 \]

Slow turn around
Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow \]
\[ \left( \text{if}(x=0) \, v := -cv; \text{if}((x>5\frac{1}{2}) \land v \lor 2x>2\cdot5-v^2 \land v<1) \land v \geq 0) \, v := -fv; \right. \]
\[ \left. t := 0; \{ x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1 \} \right) \] 
\[ * \] 
\[ (0 \leq x \leq 5) \]

**Proof?** Ask René Descartes who says yes

![Graph showing the movement of the ping pong ball over time](image-url)
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3 Proof

4 Summary
   - Zeno’s Quantum Turtles
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Proposition (Quantum can play ping pong safely in real-time)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g=1 \land 1=c \geq 0 \land 1=f \geq 0 \rightarrow \]

\[ \left( \text{if } (x=0) v := -cv; \text{ if } ((x>5^{1/2}) - v \lor 2x>2.5-v^2 \land v<1) \land v \geq 0) v := -fv; \right. \]

\[ t := 0; (x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1) \right) ^\ast \]

\[ 0 \leq x \leq 5 \]
Quantum’s Time-triggered Ping Pong Proof Invariants

Proposition (Quantum can play ping pong safely in real-time)

\[ 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 > 0 \land 1 = c \geq 0 \land 1 = f \geq 0 \rightarrow \\
\left( \text{if}(x = 0) v := -cv; \text{if}(x > 5 \frac{1}{2} - v \lor 2x > 2 \cdot 5 - v^2 \land v < 1) \land v \geq 0) v := -fv; \\
t := 0; (x' = v, v' = -g, t' = 1 \land x \geq 0 \land t \leq 1) \right)^* (0 \leq x \leq 5) \]

Proof

@invariant(2x = 2H - v^2 \land x \geq 0 \land x \leq 5)
Quantum’s Time-triggered Ping Pong Proof Invariants

Proposition (Quantum can play ping pong safely in real-time)

\[ 2x = 2H - v^2 \land 0 \leq x \land x \leq 5 \land v \leq 0 \land g = 1 \geq 0 \land 1 = c \geq 0 \land 1 = f \geq 0 \rightarrow \]
\[ \left[ \text{if}(x=0) v := -cv; \text{if}(x > 5\frac{1}{2}) v := -fv; \right. \]
\[ t := 0; (x' = v, v' = -g, t' = 1 \land x > 0 \land t \leq 1) \right]^* \]
\[ (0 \leq x \leq 5) \]

Proof

@invariant(\[ 2x = 2H - v^2 \land x \geq 0 \land x \leq 5 \])
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Summary: Time-triggered Control

1. Common paradigm for designing real controllers
2. Periodical or pseudo-periodical control (jitter)
3. Expects delays, expects inertia
4. Implementation: discrete-time sensing
5. Predict events, not just if(eventnow(x))... 
6. Safe controllers know their own reaction delays
7. Burden of event detection brought to attention of CPS programmer
8. Time-triggered controls are implementable and more robust, but make design and verification more challenging!
9. Use knowledge gained from verified event-triggered model as a basis for designing a time-triggered controller
Example (Quantum the Bouncing Ball)

\[
(x' = v, v' = -g \& x \geq 0; \text{if}(x = 0) v := -cv)^*
\]
Example (Quantum the Bouncing Ball)

\[
(x' = v, \quad v' = -g \quad \& \quad x \geq 0; \quad \text{if}(x = 0) \quad v := -cv)^*
\]
Example (Quantum the Bouncing Ball)

\((x' = v, v' = -g \& x \geq 0; \text{if}(x = 0) v := -cv)\)
How Quantum Met Achilles and His Tortoise

Example (Quantum the Bouncing Ball experiences time)

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]
Example (Quantum the Bouncing Ball experiences time)

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i}
\]
Example (Quantum the Bouncing Ball experiences time)

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}}
\]
Example (Quantum the Bouncing Ball experiences time)

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 \]
Example (Quantum the Bouncing Ball experiences time)

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty \]
How Quantum Met Achilles and His Tortoise

Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty$$
Example (Quantum the Bouncing Ball experiences time)

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty
\]
How Quantum Met Achilles and His Tortoise

Example (Quantum the Bouncing Ball experiences time)

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty \]
Example (Quantum the Bouncing Ball experiences time)

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty
\]
How Quantum Met Achilles and His Tortoise

Zeno Paradox
Quantum’s model causes a time freeze

Example (Quantum the Bouncing Ball experiences time)

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty \]
What to do with assignments

\[
\begin{align*}
[x := e] p(x) &\iff p(e) \\
[x := x^2] x \neq 0 &\iff x^2 \neq 0 \\
\frac{x := e] p(x) &\iff p(e) \\
[x := x^2] [y := 2x] x > 0 &\iff [y := 2x^2] x^2 > 0 \\
\frac{x := e] p(x) &\iff p(e) \\
[x := x^2] x \neq x &\iff x^2 \neq x \\
\frac{x := e] p(x) &\iff p(e) \\
[x := 5y] [y := 2x] (x > 0) &\iff [y := 2(5y)] (5y > 0) \\
\frac{x := e] p(x) &\iff p(e) \\
[x := x^2] [x' = 2x] x > 0 &\iff [x' = 2x^2] x^2 > 0 \\
\frac{x := e] p(x) &\iff p(e) \\
[x := x^2] [(x := x+1)^*] x \geq 0 &\iff [(x := x^2 + 1)^*] x^2 \geq 0 \\
\end{align*}
\]

\( e \leadsto x^2, p(\cdot) \leadsto \cdot \neq 0 \)

\( e \leadsto x^2, p(\cdot) \leadsto [y := 2\cdot] (\cdot > 0) \)

\( e \leadsto x^2, p(\cdot) \leadsto \cdot \neq x \)

\( e \leadsto 5y, p(\cdot) \leadsto [y := 2\cdot] (\cdot > 0) \)

\( e \leadsto x^2, p(\cdot) \leadsto [\cdot' = 2\cdot] \cdot > 0 \)

\( e \leadsto x^2, p(\cdot) \leadsto [(x := \cdot + 1)^*] \cdot \)
What to do with assignments and what not to do!

\[
\begin{align*}
[x:=e]p(x) & \iff p(e) \\
[x:=x^2]x \not= 0 & \iff x^2 \not= 0 \\
[x:=e]p(x) & \iff p(e) \\
[x:=x^2][y:=2x]x > 0 & \iff [y:=2x^2]x^2 > 0 \\
[x:=e]p(x) & \iff p(e) \\
[x:=5y][y:=2x](x > 0) & \iff [y:=2(5y)](5y > 0) \\
[x:=e]p(x) & \iff p(e) \\
[x:=x^2][x' = 2x]x > 0 & \iff [x' = 2x^2]x^2 > 0 \\
[x:=e]p(x) & \iff p(e) \\
[x:=x^2][(x:=x+1)^*]x \geq 0 & \iff [(x:=x^2+1)^*]x^2 \geq 0
\end{align*}
\]

\[e \leadsto x^2, p(\cdot) \leadsto \cdot \not= 0 \]

\[e \leadsto x^2, p(\cdot) \leadsto [y:=2\cdot](\cdot > 0) \]

\[e \leadsto x^2, p(\cdot) \leadsto \cdot \not= x \]

\[e \leadsto 5y, p(\cdot) \leadsto [y:=2\cdot](\cdot > 0) \]

\[e \leadsto x^2, p(\cdot) \leadsto [\cdot' = 2\cdot] \cdot > 0 \]

\[e \leadsto x^2, p(\cdot) \leadsto [(x:=\cdot + 1)^*] \cdot \]
What else to do with assignments

\[\begin{align*}
\text{[:=]} & \quad [x := e]p(x) \leftrightarrow p(e) \\
\Gamma, x = e & \vdash P, \Delta \\
\Gamma & \vdash [x := e]P, \Delta
\end{align*}\]
What else to do with assignments and what not to do!

\[ [:=] \quad [x := e]p(x) \leftrightarrow p(e) \]

\[
\Gamma, x = e \vdash P, \Delta \\
\frac{}{\Gamma \vdash [x := e]P, \Delta} \quad \text{if } x \notin \Gamma, \Delta
\]
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