11: Differential Equations & Proofs

15-424: Foundations of Cyber-Physical Systems

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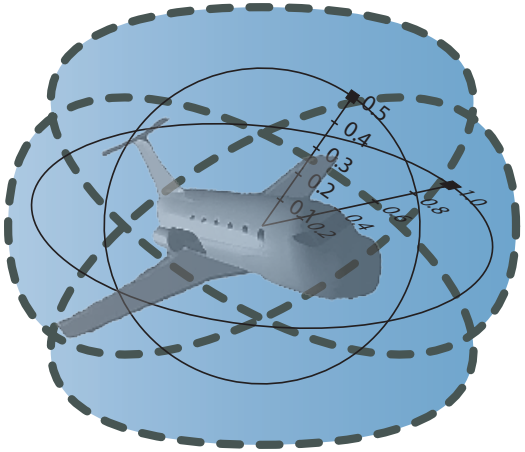

Outline

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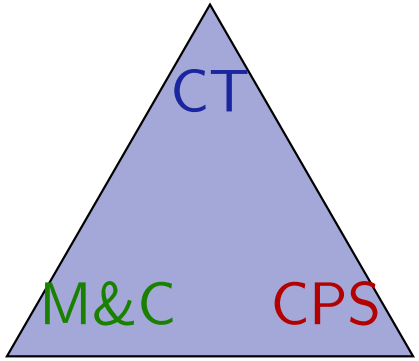
Learning Objectives

Differential Equations & Proofs

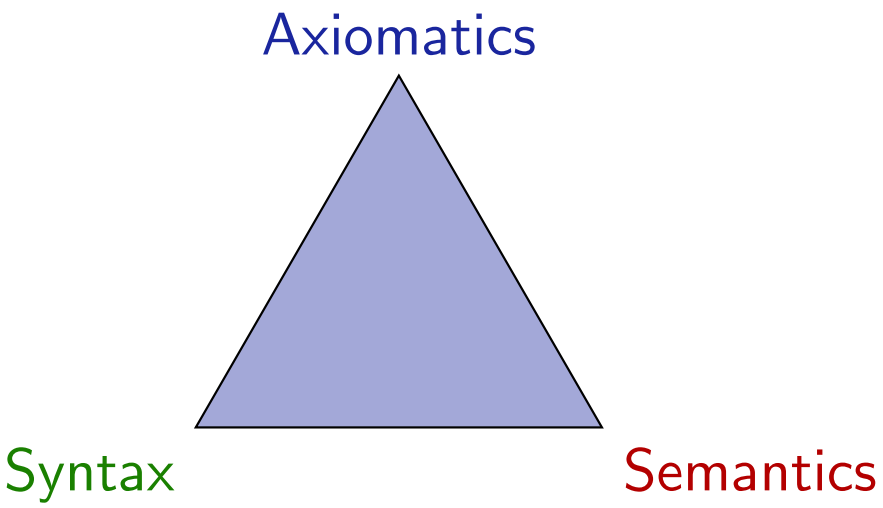
discrete vs. continuous analogy
rigorous reasoning about ODEs
beyond differential invariant terms
differential invariant formulas
cut principles for differential equations
axiomatization of ODEs
differential facet of logical trinity



understanding continuous dynamics
relate discrete+continuous
design-by-invariant

operational CPS effects
state changes along ODE

Differential Facet of Logical Trinity



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of A relate to semantics of $A \wedge B$, syntactically? If A is true, is $A \wedge B$ true, too? Conversely?

Outline

Differentials

Syntax

$$e ::= x \mid x' \mid c \mid e + k \mid e \cdot k \mid (e)'$$

Semantics

$$[\![(e)']\!]\omega = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Axioms

$$(e+k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \text{for constants/numbers } c()$$

$$(x)' = x' \qquad \text{for variables } x \in \mathcal{V}$$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0), \varphi(r)) : \varphi \models x' = f(x) \wedge Q \text{ for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(\zeta)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(\zeta)$$

Differential Substitution Lemmas

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r > 0$*, then for all* $0 \leq z \leq r$*:*

$$[\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \wedge Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e + k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \textit{for constants/numbers } c()$$

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Differential Substitution Lemmas $\rightsquigarrow$ Proofs

Lemma (Differential lemma) (Differential value vs. Time-derivative)

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Lemma (Differential assignment) (Effect on Differentials)

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$+'$	$(e + k)' = (e)' + (k)'$
$\cdot'$	$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
c'	$(c())' = 0$
x'	$(x)' = x'$

Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

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Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

Proof.

$[\![(e+k)']\!]\omega =$

$\square$

Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

Proof.

$$[\![(e+k)']\!]\omega = \sum_x \omega(x') \frac{\partial [\![e+k]\!]}{\partial x}(\omega)$$

□

Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

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Proof.

$$[\![(e+k)']\!]\omega = \sum_x \omega(x')\frac{\partial [\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x')\frac{\partial([\![e]\!]+[\![k]\!])}{\partial x}(\omega)$$

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$$= \sum_x \omega(x')\Big(\frac{\partial[\![e]\!]}{\partial x}(\omega) + \frac{\partial[\![k]\!]}{\partial x}(\omega)\Big)$$

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\end{aligned}$$

□

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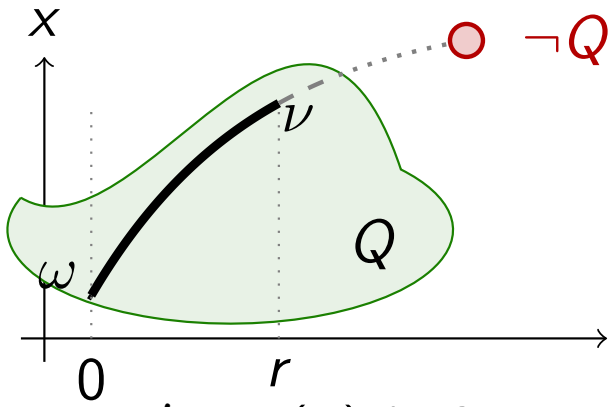
Differential Weakening



$x' = f(x) \,\&\, Q$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0), \varphi(r)) : \varphi \models x' = f(x) \wedge Q \text{ for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

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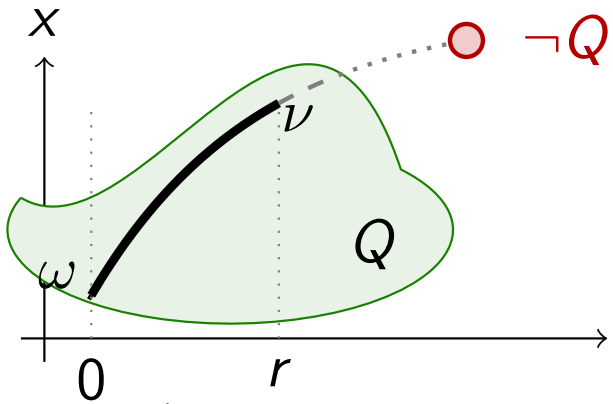


DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

$x' = f(x) \,\&\, Q$

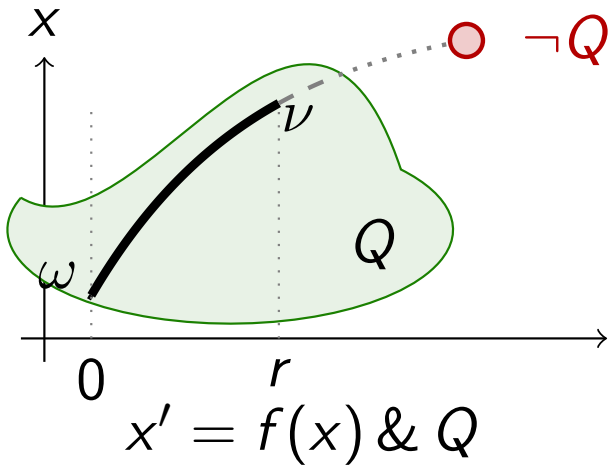
ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0), \varphi(r)) : \varphi \models x' = f(x) \wedge Q \text{ for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(\zeta)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(\zeta)$$

Differential equations cannot leave their domains.

Differential Weakening



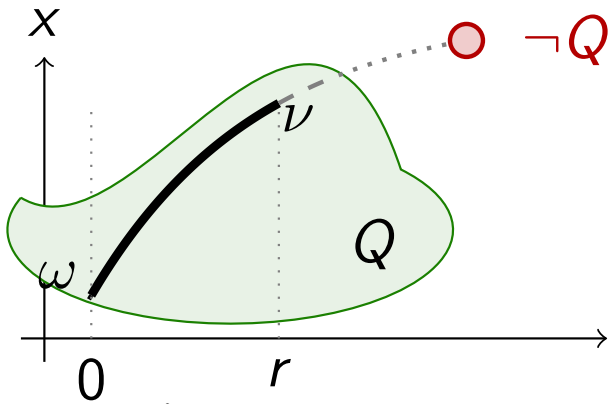
DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

Example (Bouncing ball)

$$\text{DW}\,\frac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening



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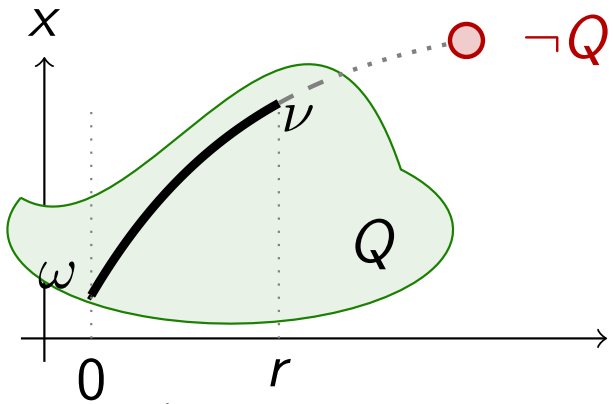
Differential Weakening



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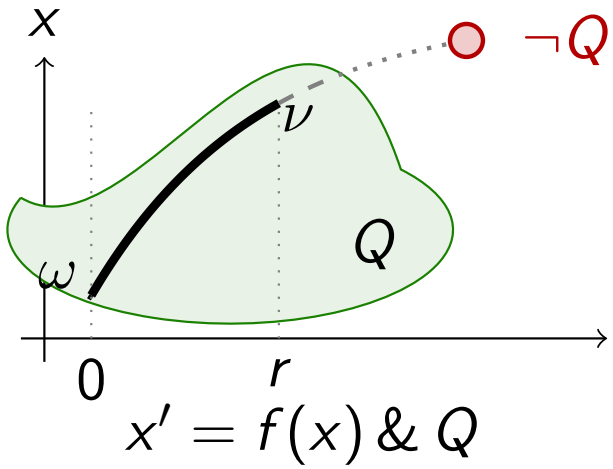

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Differential Weakening

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$$\text{DW}\ \frac{}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]p(x), \Delta}$$

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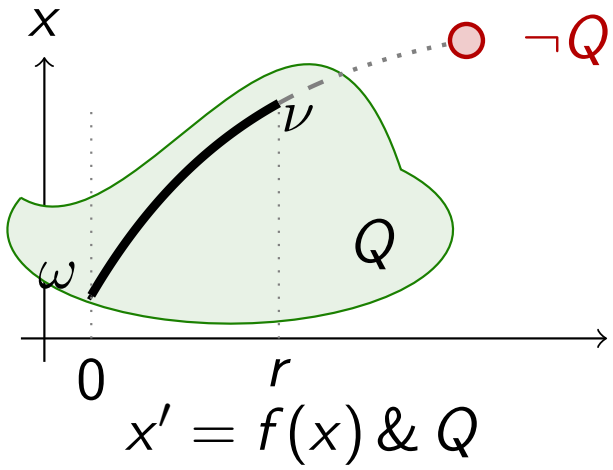

Example (Bouncing ball)

$$\text{DW}\ \frac{\text{G}\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash x \geq 0 \rightarrow 0 \leq x}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \rightarrow 0 \leq x)}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]0 \leq x}$$

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Differential Weakening

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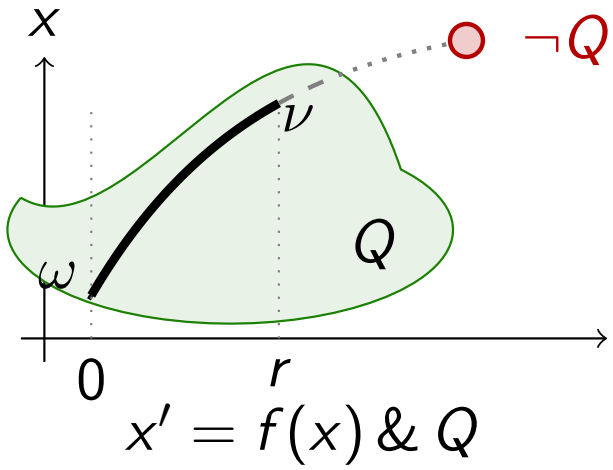
$$\text{DW}\quad \frac{\Gamma \vdash \forall x\,(q(x) \to p(x)), \Delta}{\Gamma \vdash [x' = f(x)\,\&\,q(x)]p(x), \Delta}$$



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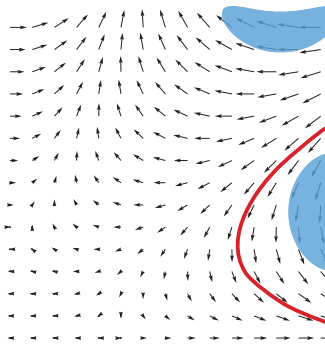
No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Invariant Terms for Differential Equations

Differential Invariant

$$\text{dI}\quad \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$



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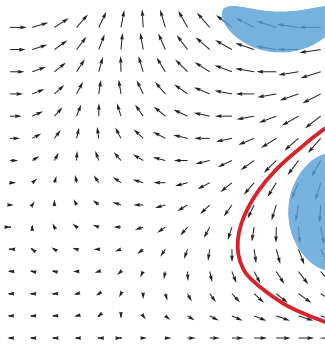

Differential Invariant

$$\text{dI}\quad \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

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dI is a derived rule:

$$\text{DI}\;\dfrac{\text{DE}\;\dfrac{\text{DW}\;\dfrac{\text{G},\rightarrow\text{R}\;\dfrac{Q \vdash [x' := f(x)](e)' = 0}{\vdash [x' = f(x) \,\&\, Q](Q \rightarrow [x' := f(x)](e)' = 0)}}{\vdash [x' = f(x) \,\&\, Q][x' := f(x)](e)' = 0}}{\vdash [x' = f(x) \,\&\, Q](e)' = 0}}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

$$\text{G}\quad \frac{P}{[\alpha]P}$$

Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \ \textit{for}\ r > 0 \ \Rightarrow\ \forall 0 \leq z \leq r \ \ [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

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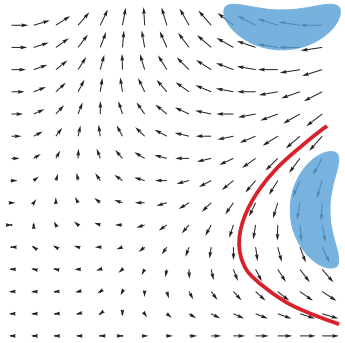

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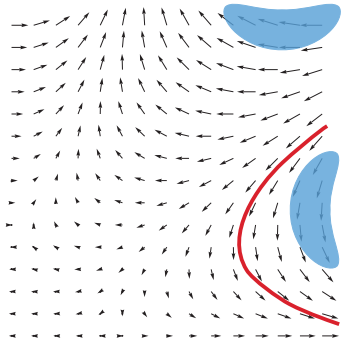

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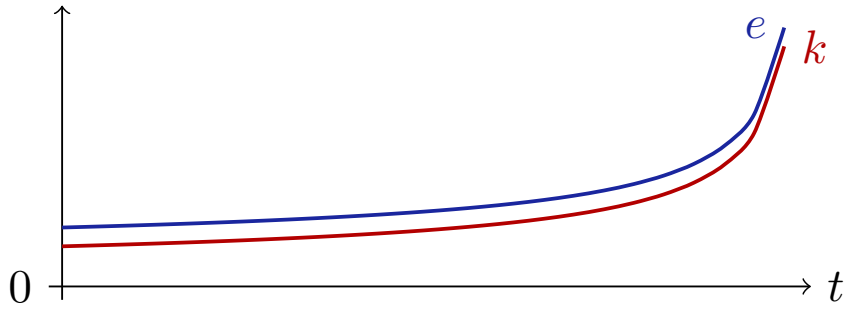

Proof (= rate of change from = initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) = [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

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Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \geq (k)'}{e \geq k \vdash [x' = f(x)]e \geq k}$$

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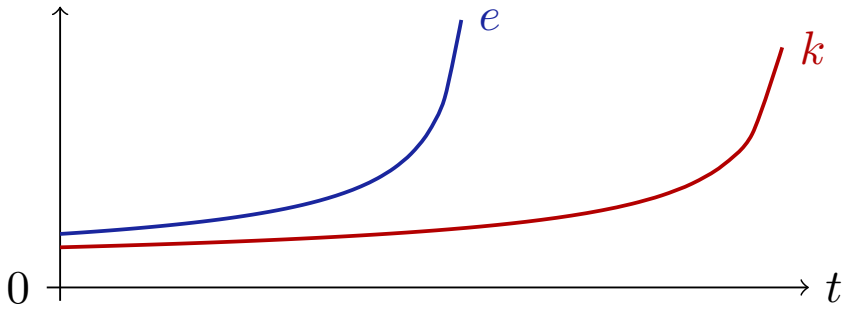

Proof ($\geq$ rate of change from $\geq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \geq [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

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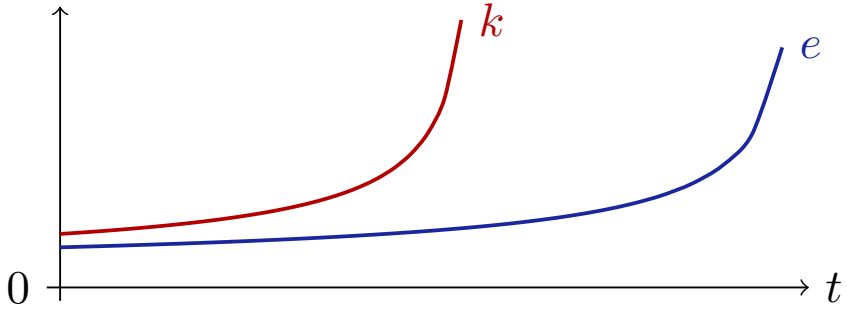
$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \leq (k)'}{e \leq k \vdash [x' = f(x)]e \leq k}$$

$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' \leq (k)'}{e \leq k \vdash [x' = f(x)]e \leq k}$$



Proof ($\leq$ rate of change from $\leq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \leq [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

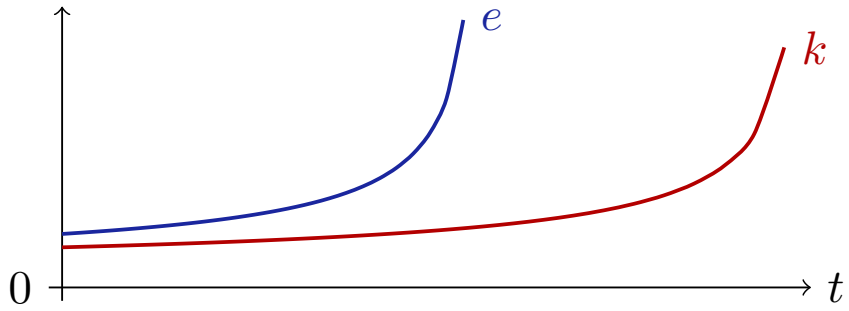
Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' > (k)'}{e > k \vdash [x' = f(x)]e > k}$$

$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' > (k)'}{e > k \vdash [x' = f(x)]e > k}$$



Proof (> rate of change from > initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) > [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

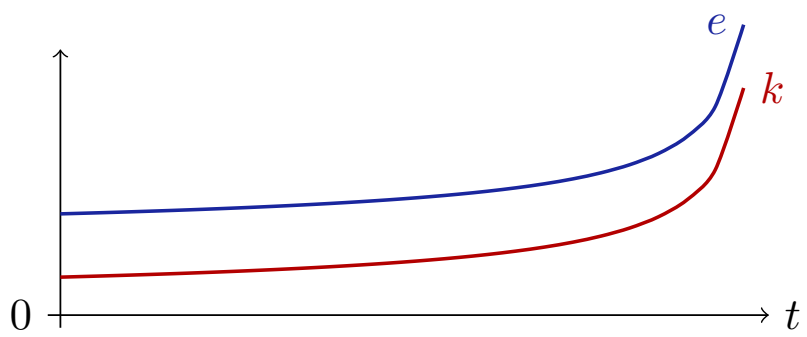
Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \geq (k)'}{e > k \vdash [x' = f(x)]e > k}$$

$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' \geq (k)'}{e > k \vdash [x' = f(x)]e > k}$$



Proof ($\geq$ rate of change from $>$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \geq [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

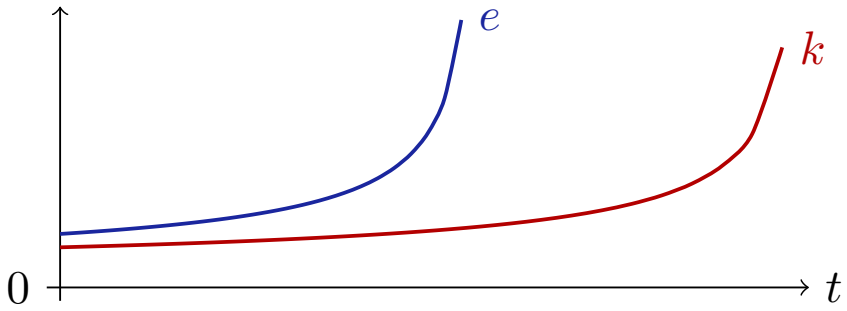
Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$

$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \neq [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

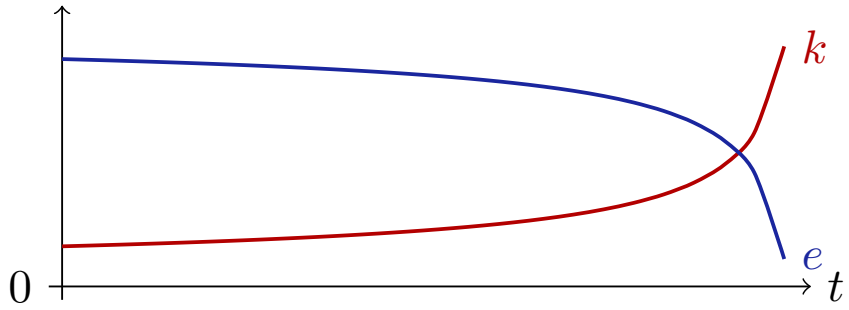
Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$

$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \neq [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

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Differential Invariant

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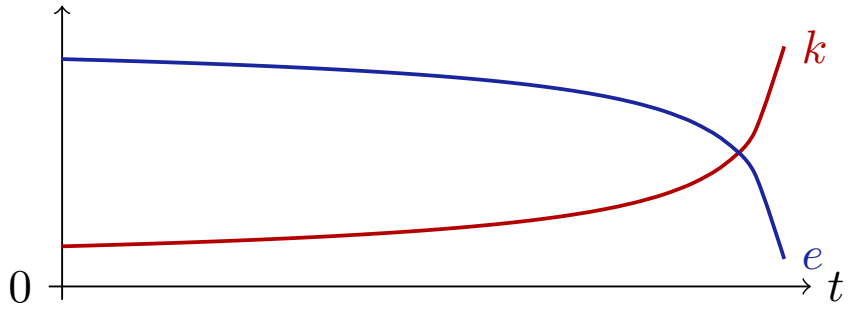
$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \neq [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

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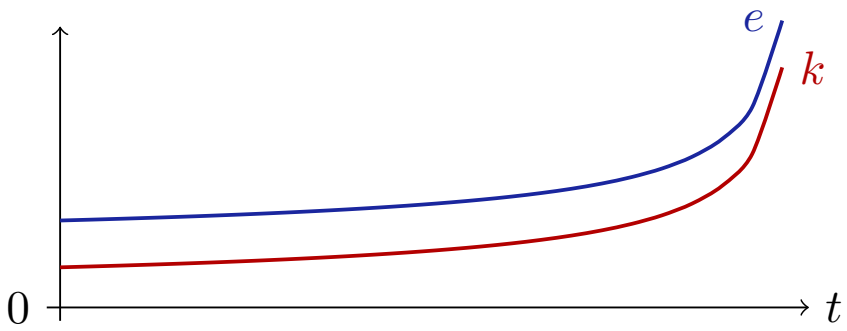
$$\text{DI} \quad \frac{\vdash [x' = f(x)](e)' = (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$

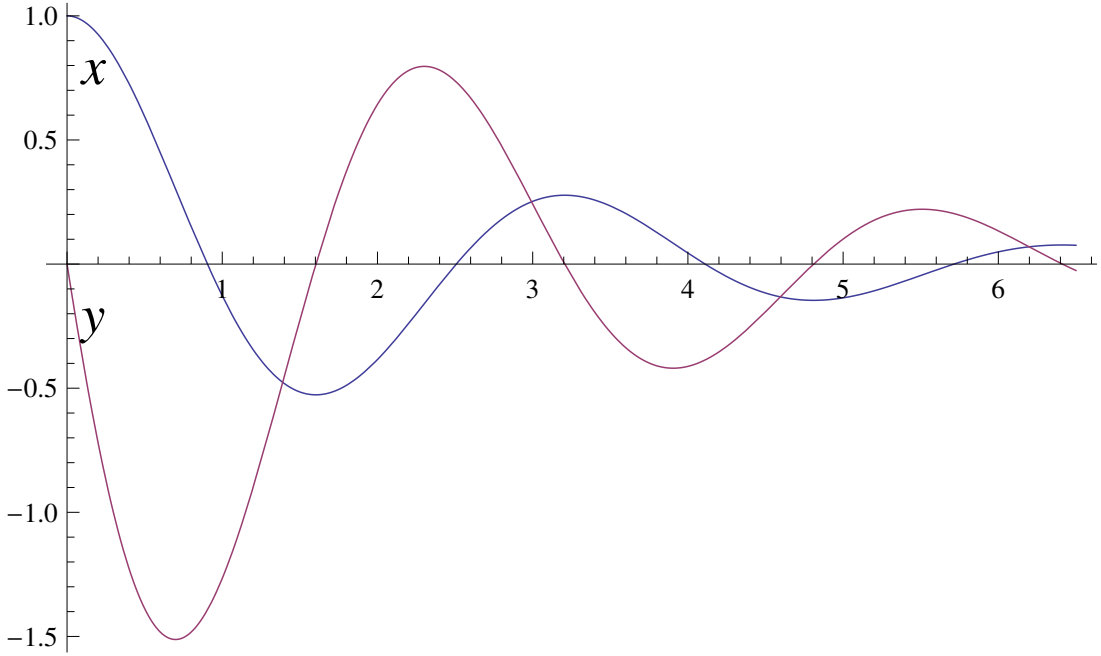


Proof (= rate of change from ≠ initial value. Mean-value theorem).

$$\frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) = [\![(k)']\!]\varphi(z) = \frac{\mathrm{d}[\![k]\!]\varphi(t)}{\mathrm{d}t}(z) \qquad \square$$

Example: Differential Invariant Inequalities

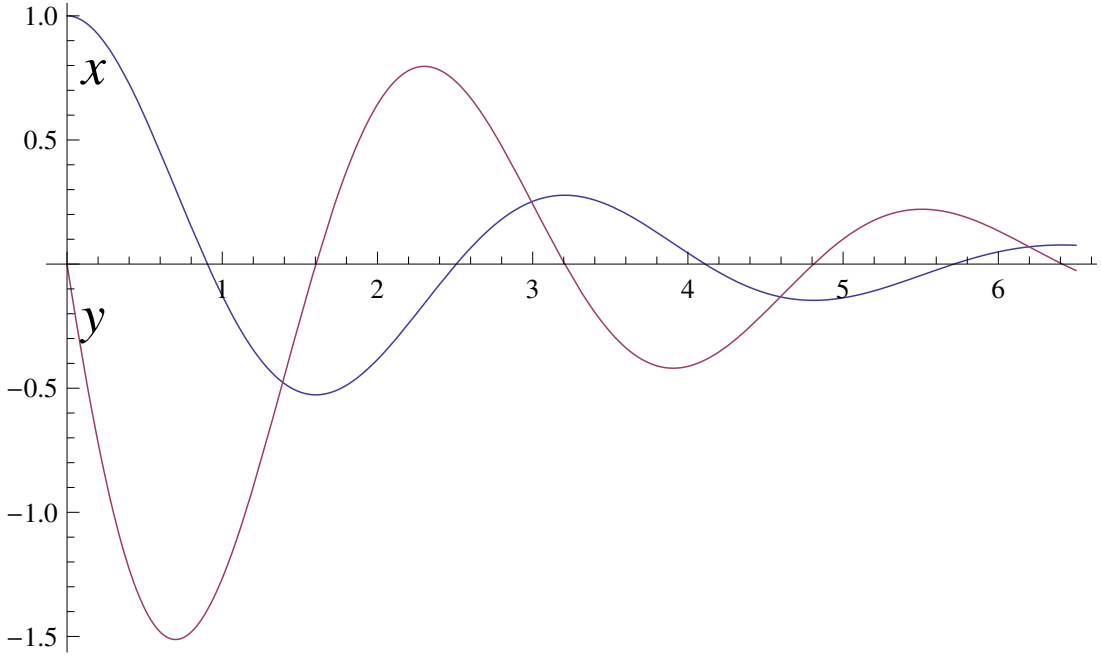
$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \wedge d\geq 0)]\, \omega^2x^2+y^2\leq c^2$$



Example: Differential Invariant Inequalities

$$\omega \geq 0 \wedge d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 x x' + 2y y' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega \geq 0 \wedge d \geq 0)]\, \omega^2 x^2 + y^2 \leq c^2$$

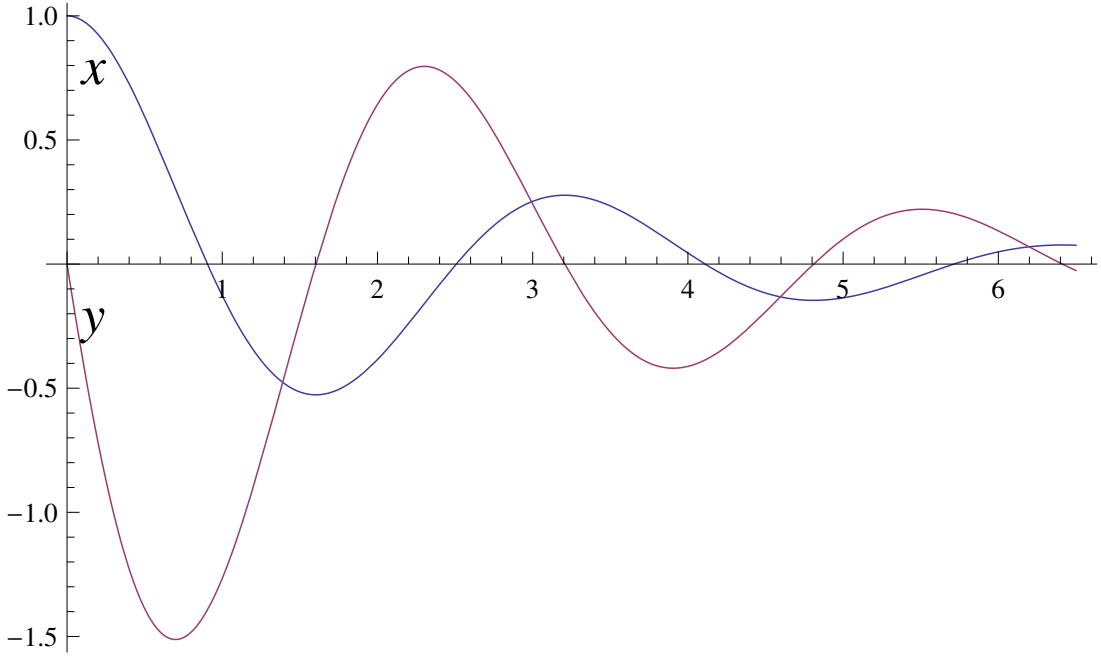


Example: Differential Invariant Inequalities

$$\omega \geq 0 \land d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega \geq 0 \land d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 x x' + 2yy' \leq 0$$

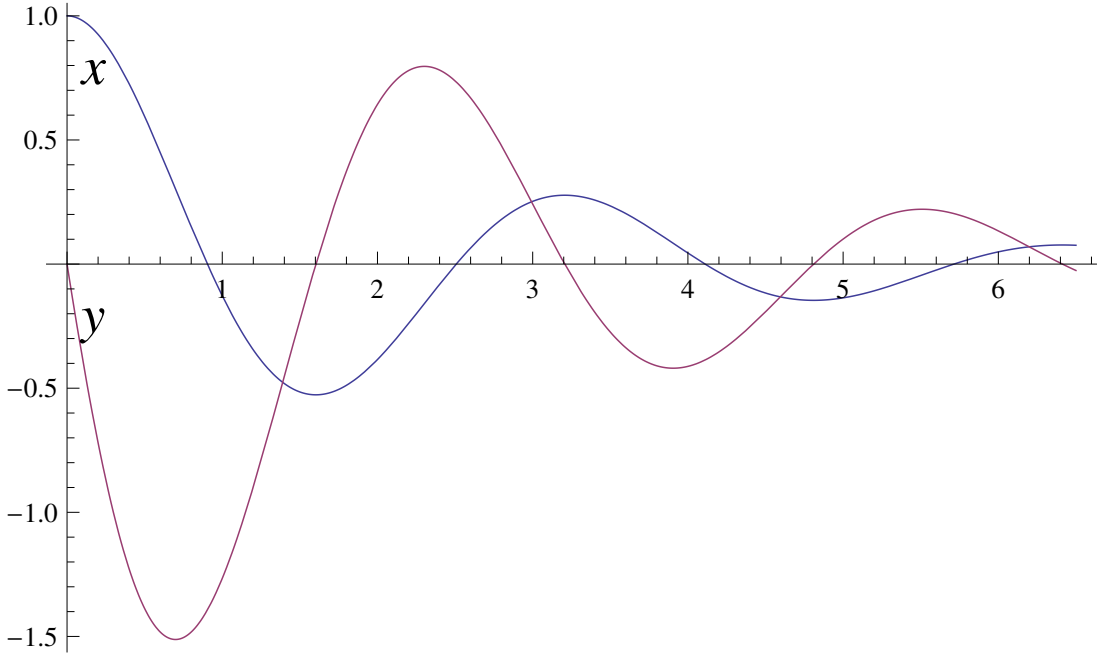
$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega \geq 0 \land d \geq 0)]\, \omega^2 x^2 + y^2 \leq c^2$$



Example: Differential Invariant Inequalities

$$\frac{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0 \quad *}{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]2\omega^2 xx' + 2yy' \leq 0}$$

$$\frac{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]2\omega^2 xx' + 2yy' \leq 0}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \land d\geq 0)]\, \omega^2x^2+y^2\leq c^2}$$



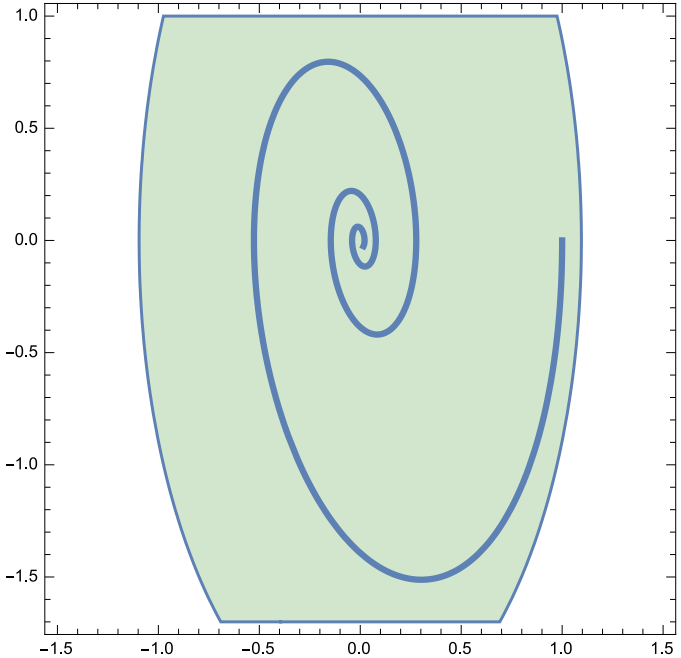
Example: Differential Invariant Inequalities: Oscillator

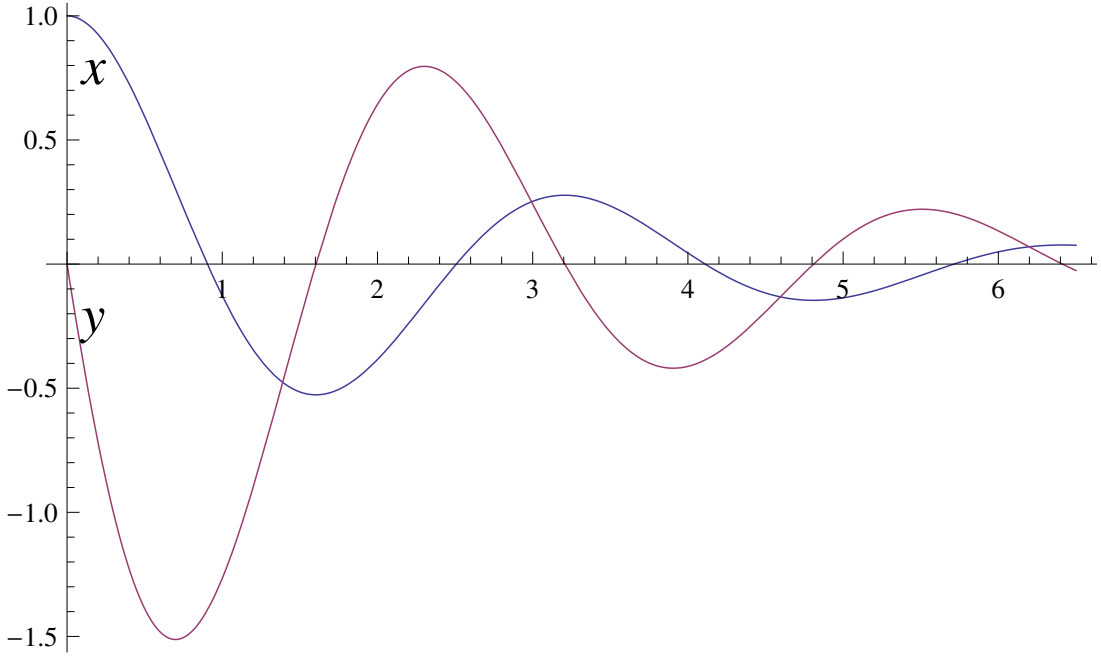
$$*$$

$$\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0$$

$$\omega{\geq}\,0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}{-}\omega^2x - 2d\omega y]2\omega^2xx' + 2yy' \leq 0$$

$$\omega^2x^2{+}y^2{\leq}c^2 \vdash [x' = y, y' = -\omega^2x - 2d\omega y \,\&\, (\omega{\geq}0 \land d{\geq}0)]\, \omega^2x^2{+}y^2{\leq}c^2$$

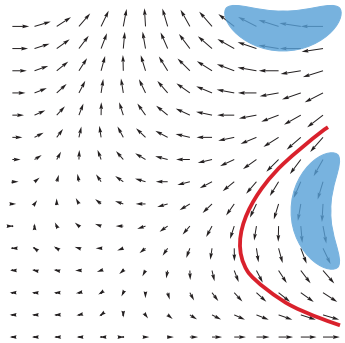




Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\ \frac{}{P \wedge Q \vdash [x' = f(x)](P \wedge Q)}$$

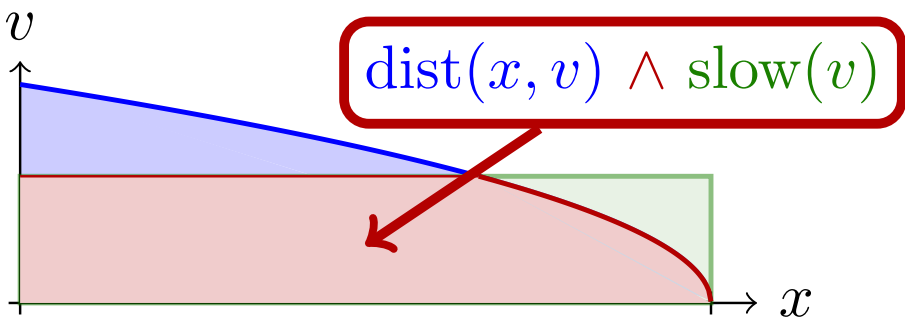


Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((P)' \land (Q)')}{P \land Q \vdash [x' = f(x)](P \land Q)}$$

$$\text{DI}\quad \frac{\vdash [x' = f(x)]((P)' \land (Q)')}{P \land Q \vdash [x' = f(x)](P \land Q)}$$

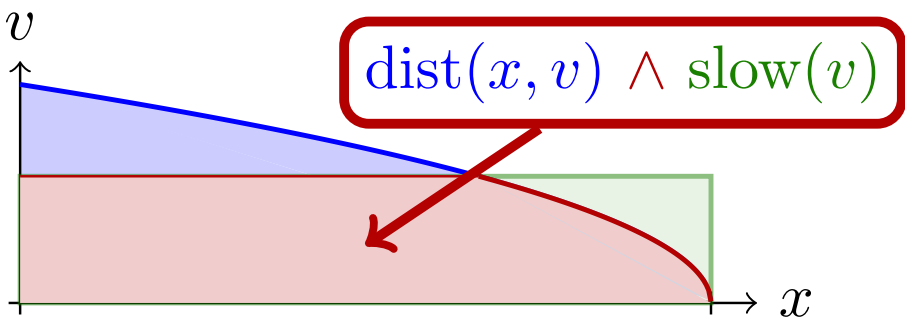


Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((P)' \wedge (Q)')}{P \wedge Q \vdash [x' = f(x)](P \wedge Q)}$$

$$\text{DI}\quad \frac{\vdash [x' = f(x)]((P)' \wedge (Q)')}{P \wedge Q \vdash [x' = f(x)](P \wedge Q)}$$



Proof (separately).

$$[]\wedge,\text{WL}\ \frac{\text{DI}\ \dfrac{\vdash [x' = f(x)](P)'}{P \vdash [x' = f(x)]P} \qquad \text{DI}\ \dfrac{\vdash [x' = f(x)](Q)'}{Q \vdash [x' = f(x)]Q}}{P \wedge Q \vdash [x' = f(x)](P \wedge Q)}$$

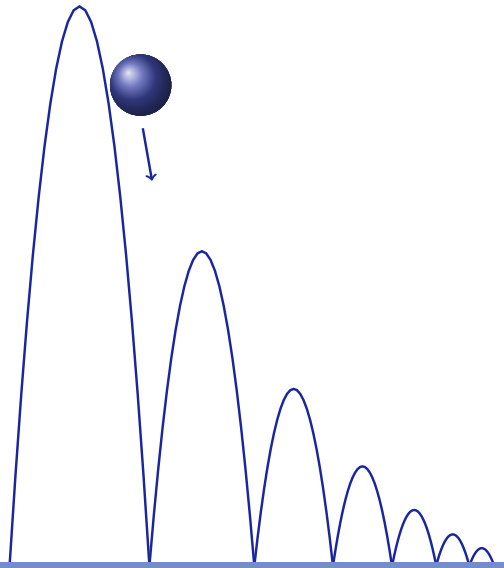
□

$[]\wedge\quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$

Quantum's Back for a Differential Invariant Proof

$$2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \land x{\geq}0)$$

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$[]\wedge \quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$

$$[]\wedge \frac{\dfrac{}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \wedge x{\geq}0)}$$

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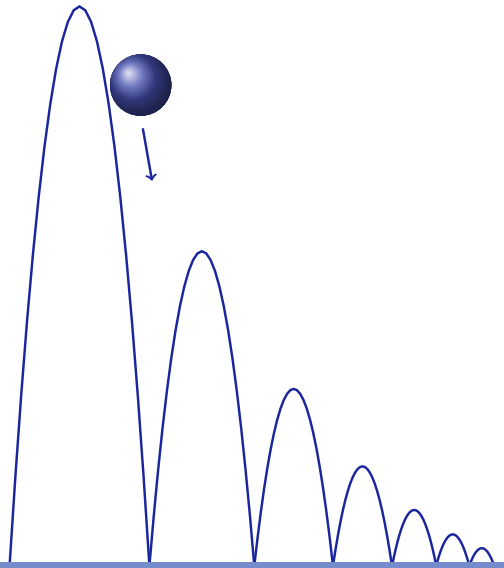

Quantum's Back for a Differential Invariant Proof

$$
[]\wedge\dfrac{\mathrm{dI}\dfrac{\dfrac{}{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x\geq 0]2gx{=}2gH{-}v^2} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x\geq 0]x\geq 0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}
$$

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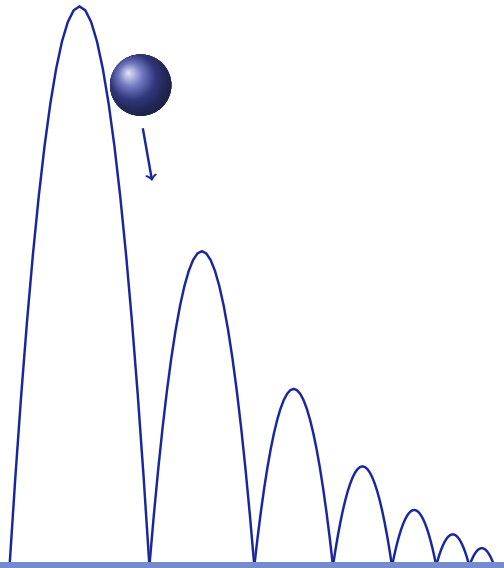

Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{x{\geq}0 \vdash 2gv = -2v(-g)}}{x{\geq}0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}\,[']}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2}\,\mathsf{dI} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \land x{\geq}0)}\,[]\land
$$

No solutions but still a proof.
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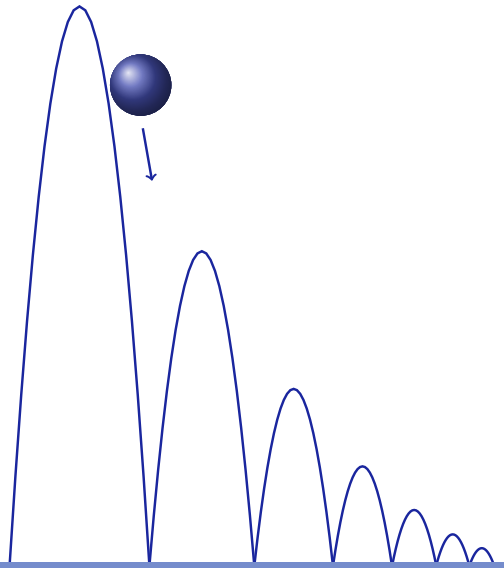
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}\,[':=]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-g}\,\&\,x\geq 0]2gx{=}2gH{-}v^2}\,\mathsf{dI} \qquad \dfrac{}{\vdash [x''{=}{-g}\,\&\,x\geq 0]x\geq 0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g\,\&\,x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}\,[]\wedge
$$

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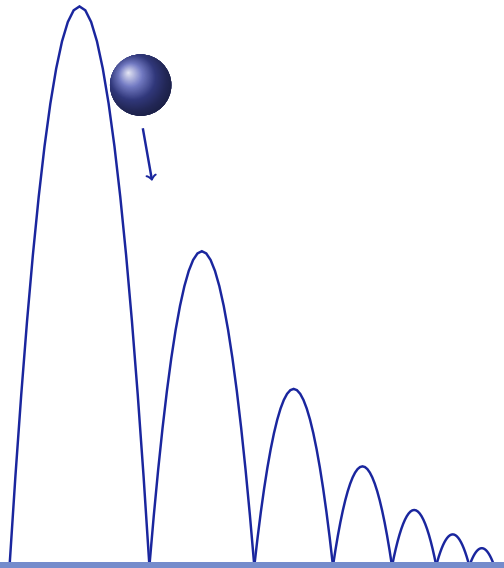

Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq 0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}\,[{}'{:=}]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x\geq 0]2gx{=}2gH{-}v^2}\,\mathrm{dI} \quad \dfrac{\dfrac{}{x\geq 0 \vdash x\geq 0}}{\vdash [x''{=}{-}g \,\&\, x\geq 0]x\geq 0}\,\mathrm{dW}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}\,[]\wedge
$$

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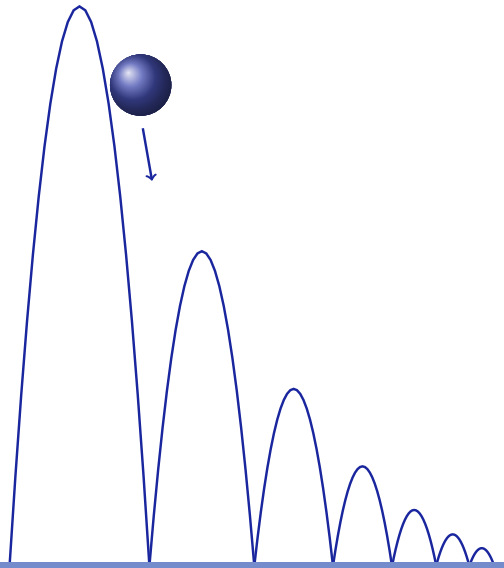

Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq 0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}\,[':=]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x\geq 0]2gx{=}2gH{-}v^2}\,\mathrm{dI} \quad \dfrac{\dfrac{*}{x\geq 0 \vdash x\geq 0}\,\mathrm{id}}{\vdash [x''{=}{-}g \,\&\, x\geq 0]x\geq 0}\,\mathrm{dW}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}\,[]\wedge
$$

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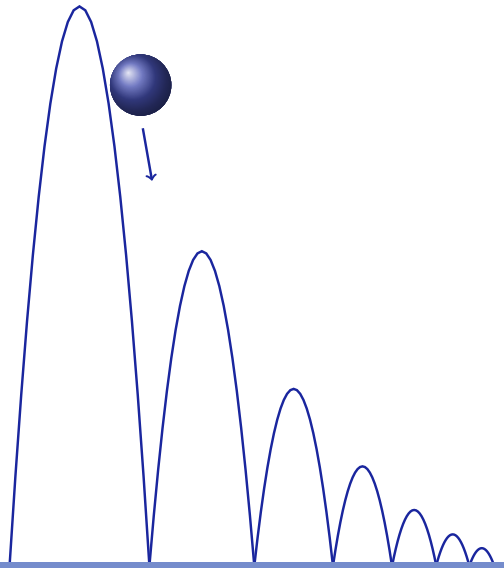
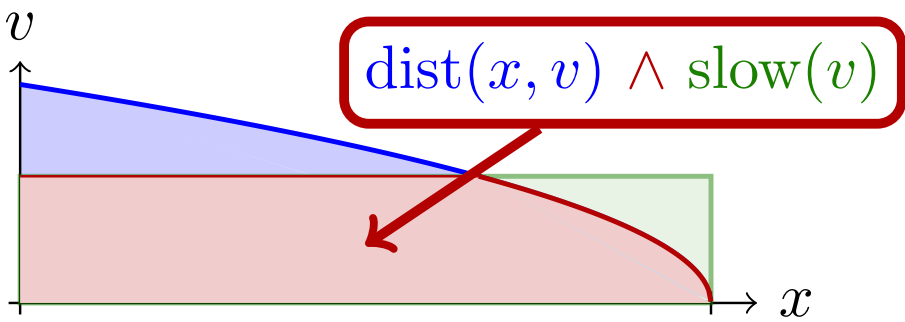


Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((P)' \land (Q)')}{P \land Q \vdash [x' = f(x)](P \land Q)}$$

$$\text{DI}\quad \frac{\vdash [x' = f(x)]((P)' \land (Q)')}{P \land Q \vdash [x' = f(x)](P \land Q)}$$

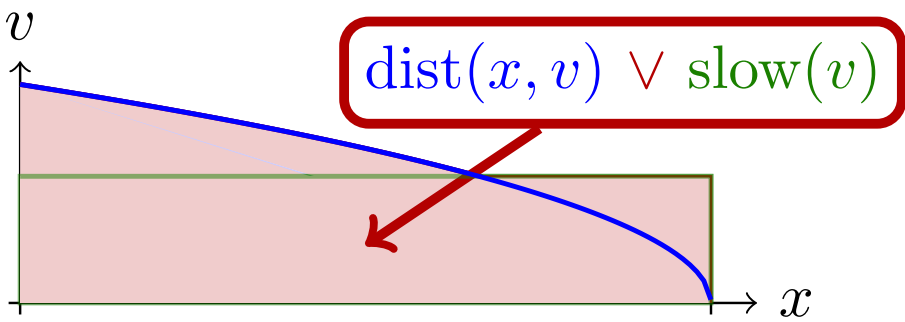


Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((P)' \vee (Q)')}{P \vee Q \vdash [x' = f(x)](P \vee Q)}$$

$$\text{DI}\quad \frac{\vdash [x' = f(x)]((P)' \vee (Q)')}{P \vee Q \vdash [x' = f(x)](P \vee Q)}$$



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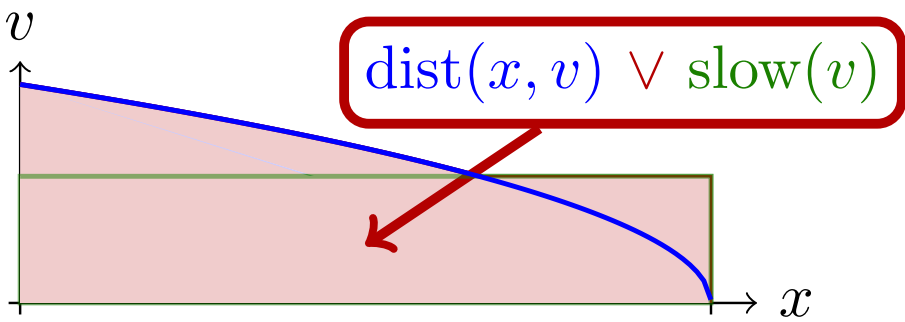

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$$\text{dI}\quad \frac{\vdash [x' := f(x)]((P)' \land (Q)')}{P \lor Q \vdash [x' = f(x)](P \lor Q)}$$

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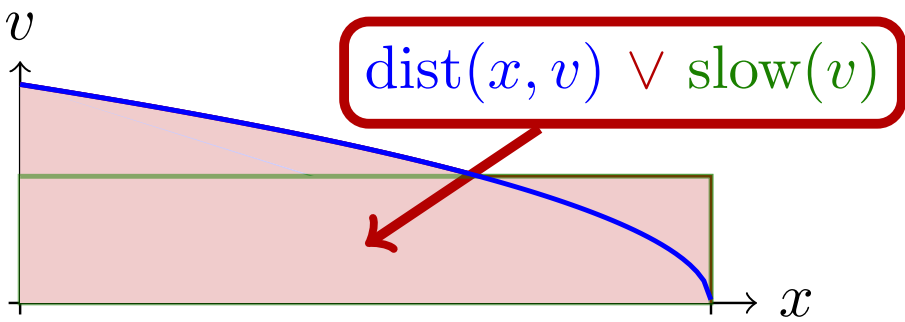

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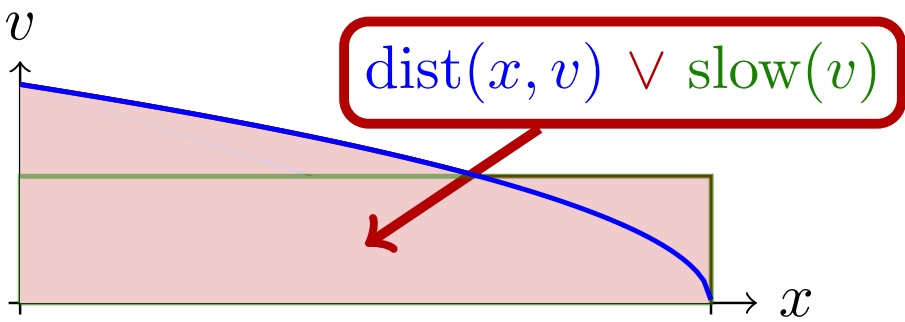
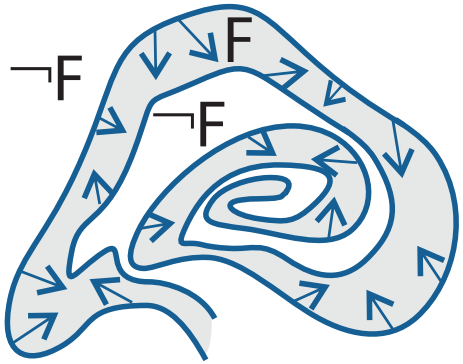


Proof (separately).

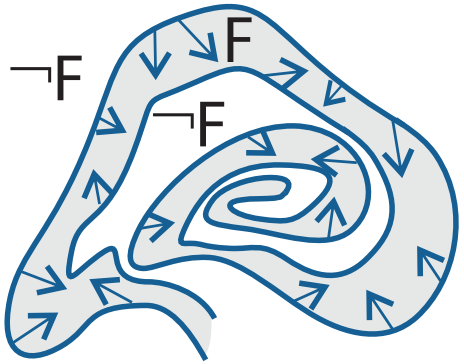
$$\vee\text{L}\ \frac{\text{MR}\ \dfrac{\dfrac{*}{P \vdash P \lor Q} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](P)'}{P \vdash [x'=f(x)]P}}{P \vdash [x' = f(x)](P \lor Q)} \qquad \text{MR}\ \dfrac{\dfrac{*}{Q \vdash P \lor Q} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](Q)'}{Q \vdash [x'=f(x)]Q}}{Q \vdash [x' = f(x)](P \lor Q)}}{P \lor Q \vdash [x' = f(x)](P \lor Q)}$$

□

Assuming Differential Invariance



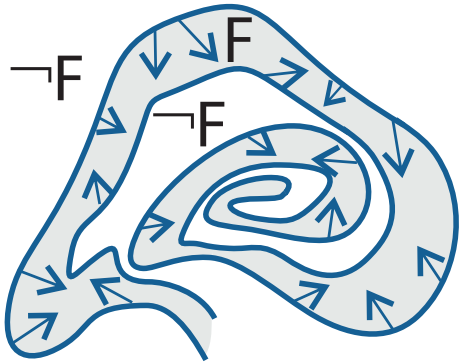
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{loop} \quad \frac{F \vdash [\alpha]F}{F \vdash [\alpha^*]F}$$

Assuming Differential Invariance



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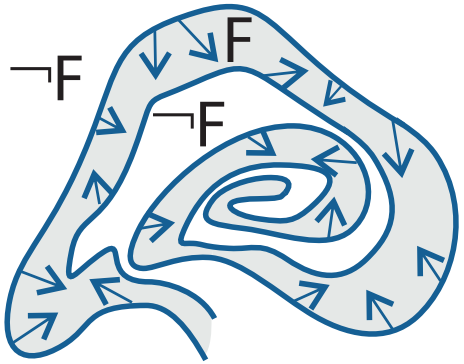
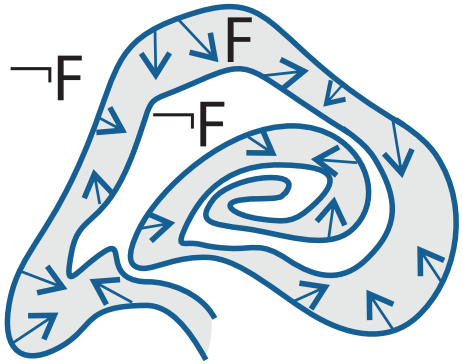


$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

$$\frac{}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance



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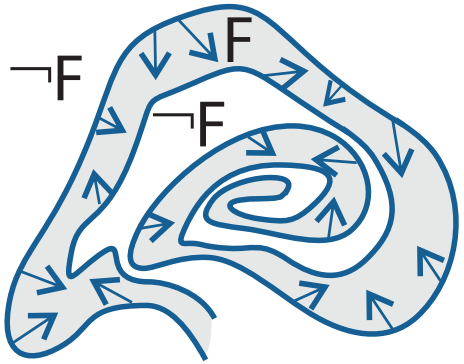

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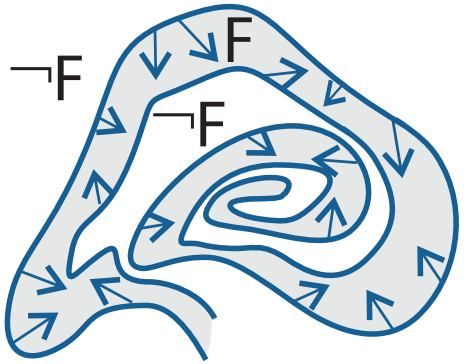
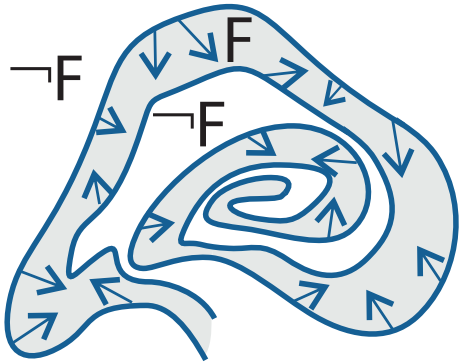
$$\frac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance



$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



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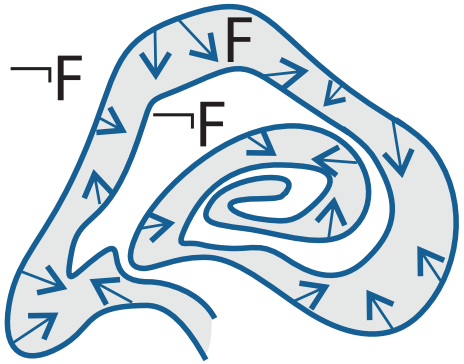
Assuming Differential Invariance



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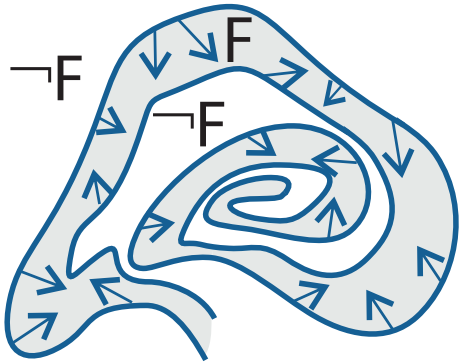

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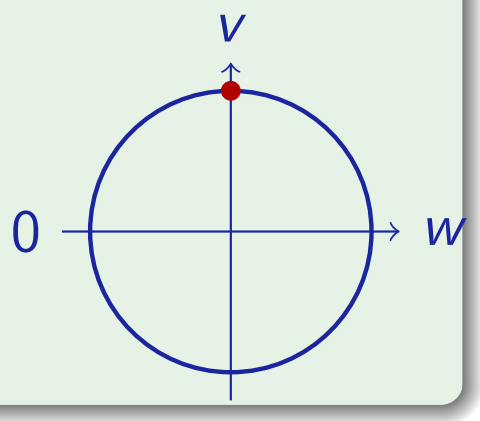
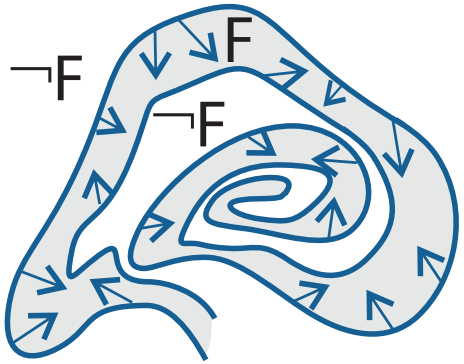


Assuming Differential Invariance



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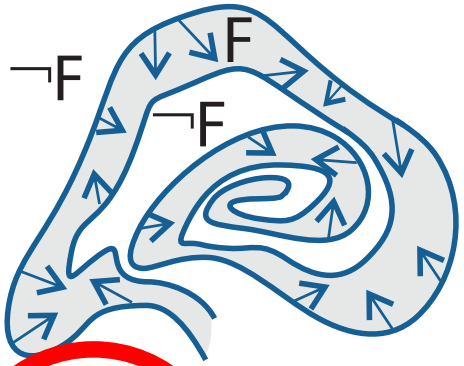
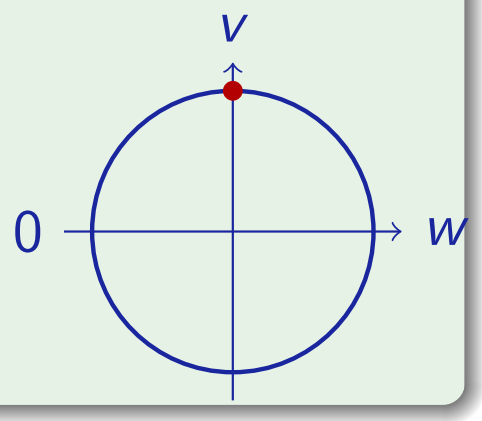


$$\frac{F \wedge Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions are unsound!)

$$\frac{\dfrac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}\,(\text{unsound})}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

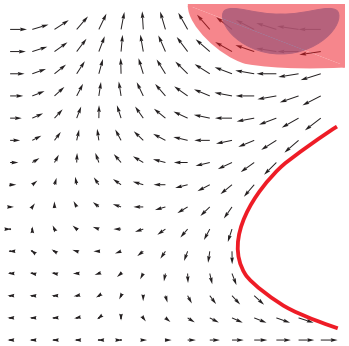


Outline

Differential Cuts

Differential Cut

$$\frac{}{F \vdash [x' = f(x)]F}$$



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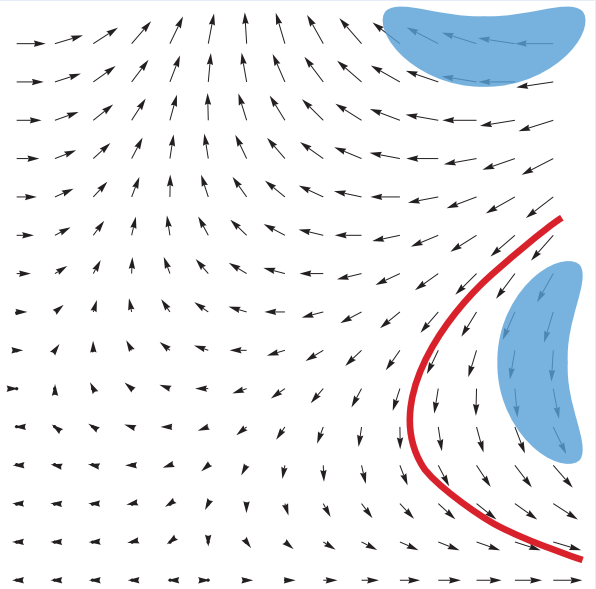

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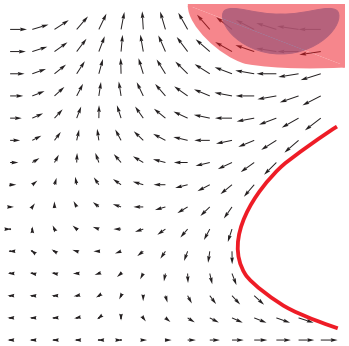
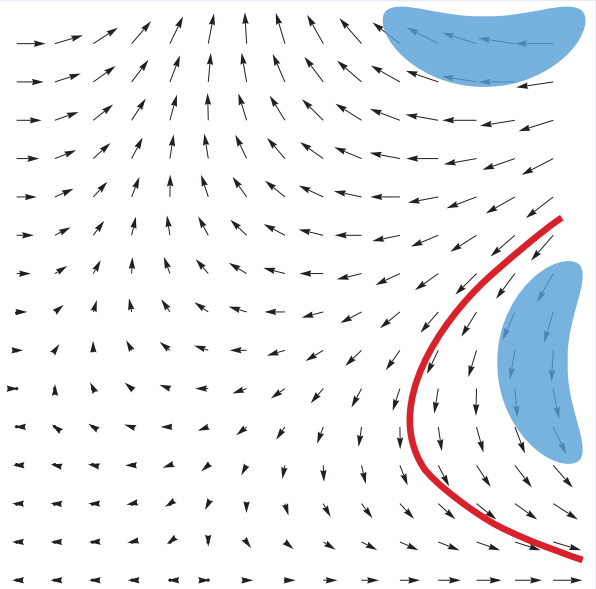


Differential Cut

$$\frac{F \vdash [x' = f(x)]C}{F \vdash [x' = f(x)]F}$$

Differential Cut



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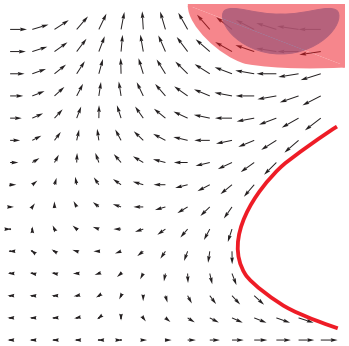

Differential Cut

$$\frac{F \vdash [x' = f(x)]C \qquad F \vdash [x' = f(x) \,\&\, C]F}{F \vdash [x' = f(x)]F}$$

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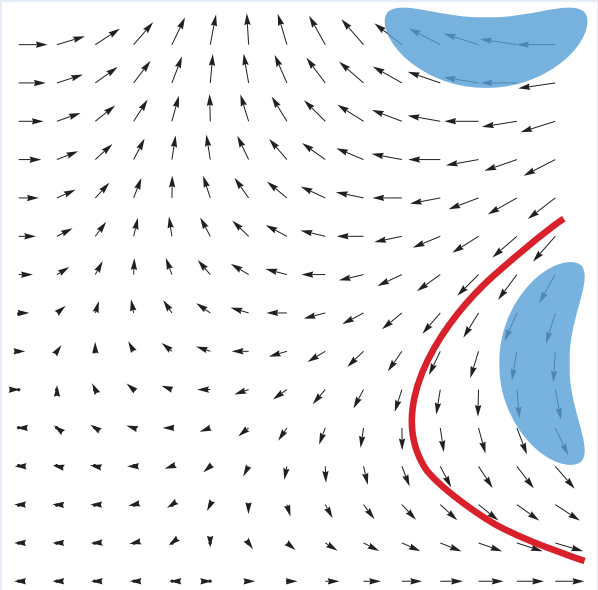

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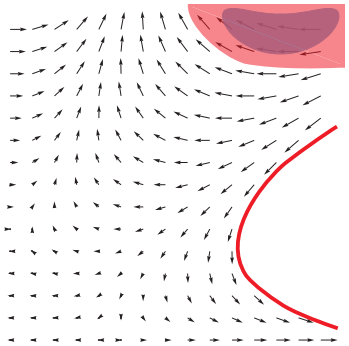
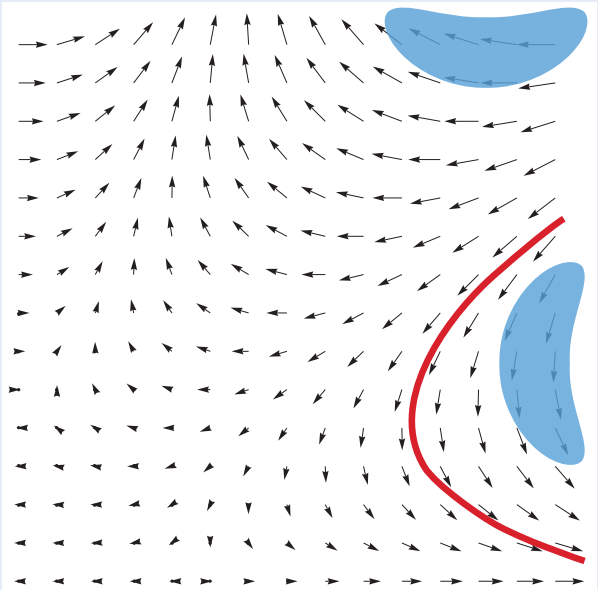


Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



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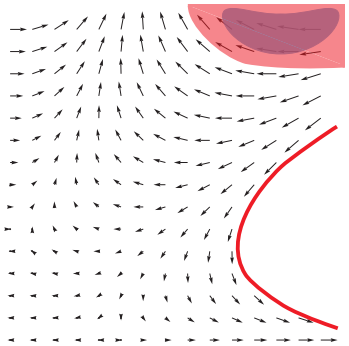

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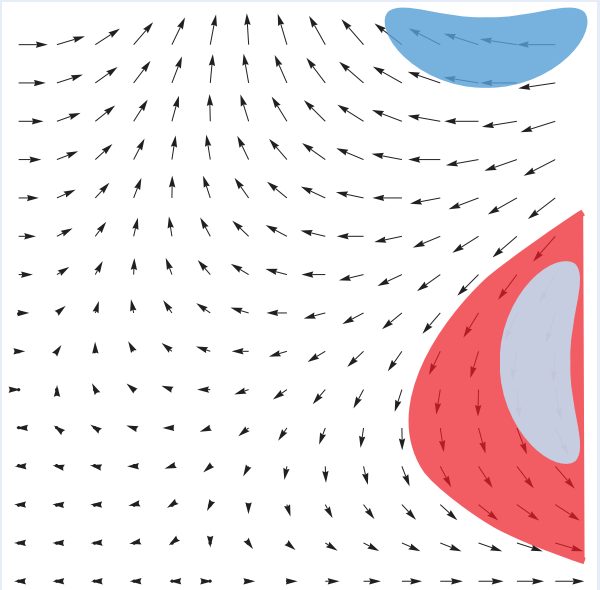

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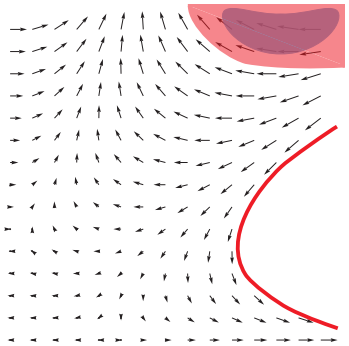

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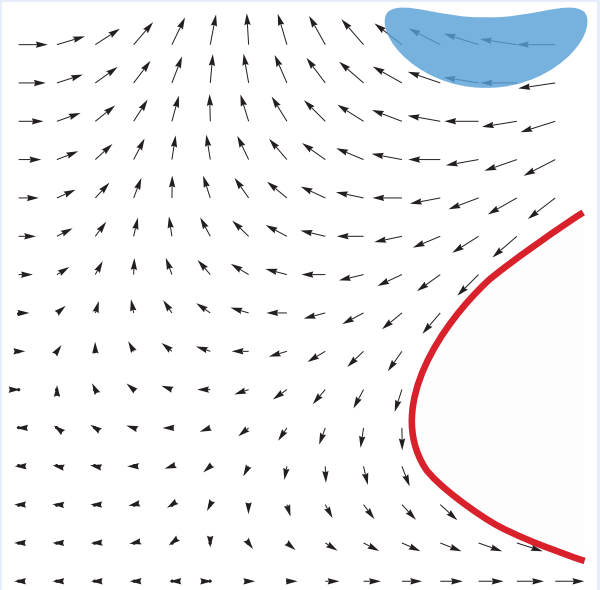

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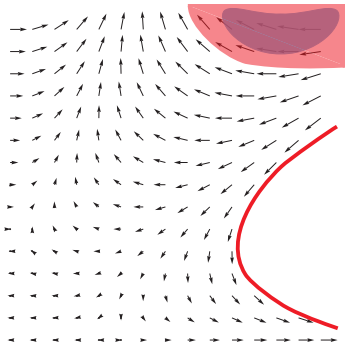

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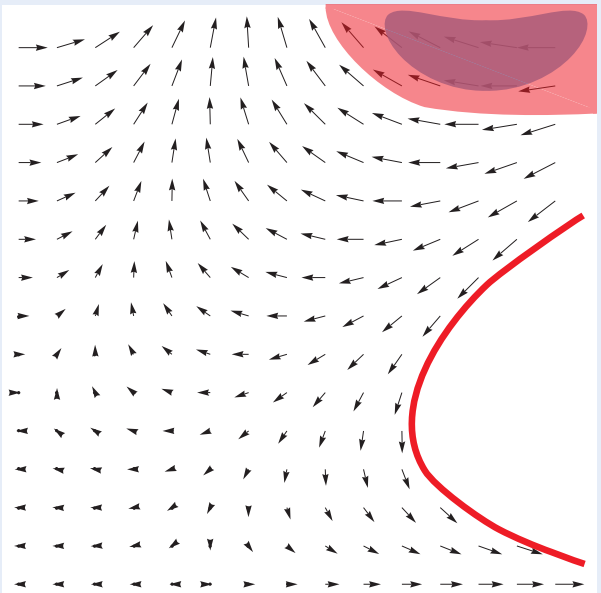

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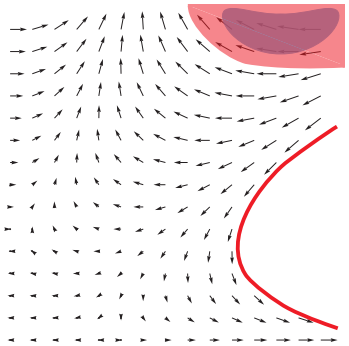

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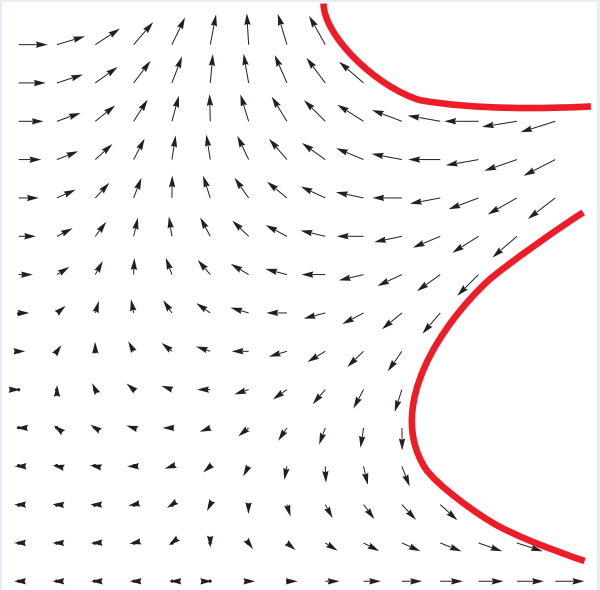

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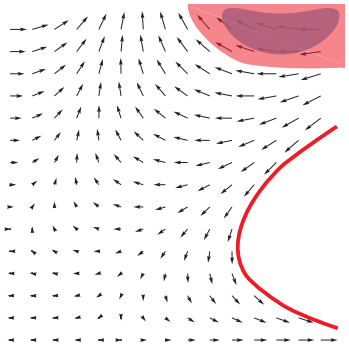

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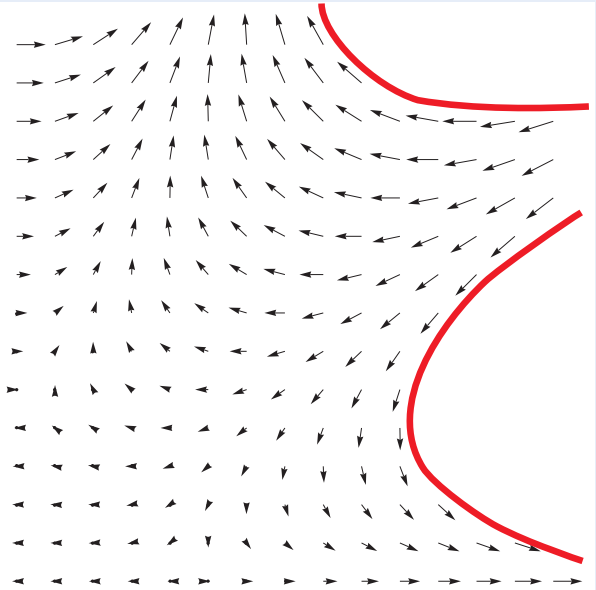

Proof (Soundness).

Let $\varphi \models x' = f(x) \wedge Q$ starting in $\omega \in [\![F]\!]$.
$\omega \in [\![[x' = f(x) \,\&\, Q]C]\!]$ by left premise.
Thus, $\varphi \models x' = f(x) \wedge Q \wedge C$.
Thus, $\varphi(r) \in [\![F]\!]$ by second premise. $\square$

Ex: Differential Cuts

$$\text{DC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

Ex: Differential Cuts

$$\text{DC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\text{DC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{\text{['}:=\text{]}\ \dfrac{}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\text{DC}\ \frac{}{x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{\text{[}':=\text{]}\ \dfrac{\mathbb{R}\ \dfrac{}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$
\begin{array}{ll}
\text{DC} & x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1 \\
\hline
\mathbb{R} & \qquad * \\
\hline
\mathbb{R} & \qquad \vdash 5y^4 y^2 \geq 0 \\
\hline
[':=] & \qquad \vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0 \\
\hline
\text{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
$$

Ex: Differential Cuts

$$\text{dI}\ \frac{}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \ \triangleright}$$

$$\text{DC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{\text{[}'{:=}\text{]}\ \frac{\mathbb{R}\ \frac{*}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x'{:=}(x-2)^4 + y^5][y'{:=}y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$
\begin{array}{ll}
[':=] & y^5 \geq 0 \vdash [x':=(x-2)^4+y^5][y':=y^2]2x^2x' \geq 0 \\
\hline
\text{dI} & x^3 \geq -1 \vdash [x' = (x-2)^4+y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright \\
\hline
\text{DC} & x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\hline
\mathbb{R} & \vdash 5y^4y^2 \geq 0 \\
\hline
[':=] & \vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0 \\
\hline
\text{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]y^5 \geq 0
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$$

Ex: Differential Cuts

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[':=] & y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0 \\
\hline
\text{dI} & x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright \\
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Outline

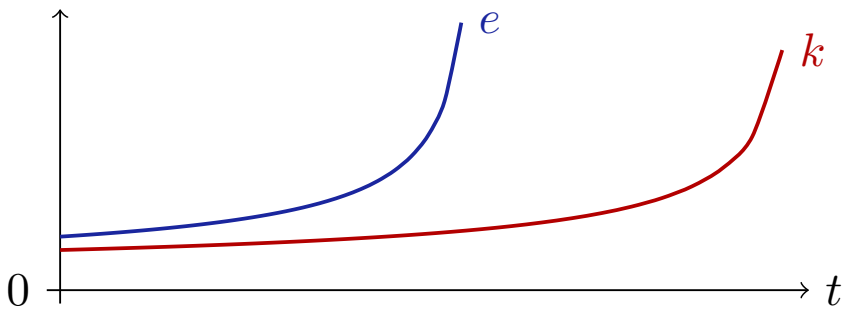
Soundness Proof: Differential Invariants

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; [\![(e)']\!]\varphi(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{DI}\;\; \frac{\vdash [x' = f(x) \,\&\, Q](e)' \geq 0}{e \geq 0 \vdash [x' = f(x) \,\&\, Q]e \geq 0}$$



Proof ($\geq$ rate of change from $\geq$ initial value. Case $r = 0$ is easier.)

$h(t) \stackrel{\text{def}}{=} [\![e]\!]\varphi(t)$ is differentiable on $[0, r]$ if $r > 0$.

$\frac{\mathrm{d}h(t)}{\mathrm{d}t}(z) = \frac{\mathrm{d}[\![e]\!]\varphi(t)}{\mathrm{d}t}(z) = [\![(e)']\!]\varphi(z) \geq 0$ by lemma + premise for all z.

$h(r) - \underbrace{h(0)}_{\geq 0} = \underbrace{(r - 0)}_{>0} \underbrace{\frac{\mathrm{d}h(t)}{\mathrm{d}t}(\xi)}_{\geq 0} \geq 0$ by mean-value theorem for some ξ. $\square$

Outline

Differential Invariants for Differential Equations

Differential Weakening

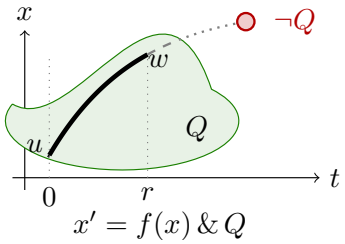
$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$

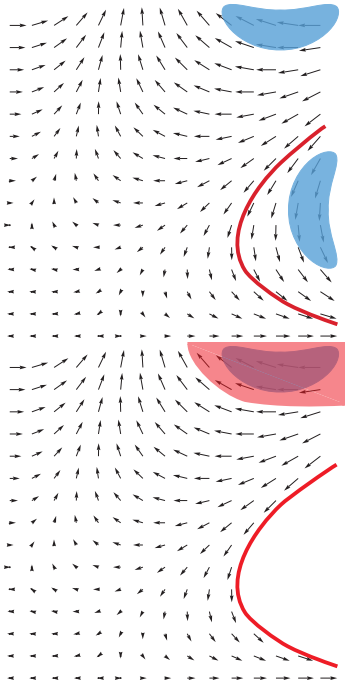
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





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