15-424: Foundations of Cyber-Physical Systems

Lecture Notes on
Game Proofs & Separations

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Lecture 19

1 Introduction

This lecture continues the study of hybrid games and their logic, differential game logic [Pla15]. Lecture 20 on Hybrid Systems & Games introduced hybrid games, Lecture 21 on Winning Strategies & Regions studied the winning region semantics, and Lecture 22 on Winning & Proving Hybrid Games identified the winning region semantics for loops in hybrid games as well as a study of the axioms of hybrid games.

These lecture notes are based on [Pla15], where more information can be found on logic and hybrid games.

2 Recap: Semantics of Hybrid Games

Recall the semantics of hybrid games and two results from Lecture 22 on Winning & Proving Hybrid Games.

Definition 1 (Semantics of hybrid games)**.** The *semantics of a hybrid game* α is a function $\varsigma_\alpha(\cdot)$ that, for each interpretation I and each set of Angel's winning states $X \subseteq \mathcal{S}$, gives the *winning region*, i.e. the set of states $\varsigma_\alpha(X)$ from which Angel has a winning strategy to achieve X (whatever strategy Demon chooses). It is defined inductively as follows[a]

1. $\varsigma_{x:=\theta}(X) = \{\omega \in \mathcal{S} : \omega_x^{[\![\theta]\!]\omega} \in X\}$
2. $\varsigma_{x'=f(x)\,\&\,Q}(X) = \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X$ for some $r \in \mathbb{R}_{\geq 0}$ and (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\varphi(\zeta) \in [\![Q]\!]$ and $\frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![\theta]\!]\varphi(\zeta)$ for all $0 \leq \zeta \leq r\}$
3. $\varsigma_{?Q}(X) = [\![Q]\!] \cap X$
4. $\varsigma_{\alpha\cup\beta}(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X)$
5. $\varsigma_{\alpha;\beta}(X) = \varsigma_\alpha(\varsigma_\beta(X))$
6. $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$
7. $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement$

The *winning region* of Demon, i.e. the set of states $\delta_\alpha(X)$ from which Demon has a winning strategy to achieve X (whatever strategy Angel chooses) is defined inductively as follows

1. $\delta_{x:=\theta}(X) = \{\omega \in \mathcal{S} : \omega_x^{[\![\theta]\!]\omega} \in X\}$
2. $\delta_{x'=f(x)\,\&\,Q}(X) = \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X$ for all $r \in \mathbb{R}_{\geq 0}$ and (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\varphi(\zeta) \in [\![Q]\!]$ and $\frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![\theta]\!]\varphi(\zeta)$ for all $0 \leq \zeta \leq r\}$
3. $\delta_{?Q}(X) = ([\![Q]\!])^\complement \cup X$
4. $\delta_{\alpha\cup\beta}(X) = \delta_\alpha(X) \cap \delta_\beta(X)$
5. $\delta_{\alpha;\beta}(X) = \delta_\alpha(\delta_\beta(X))$
6. $\delta_{\alpha^*}(X) = \bigcup\{Z \subseteq \mathcal{S} : Z \subseteq X \cap \delta_\alpha(Z)\}$
7. $\delta_{\alpha^d}(X) = (\delta_\alpha(X^\complement))^\complement$

[a] The semantics of a hybrid game is not merely a reachability relation between states as for hybrid systems [Pla12], because the adversarial dynamic interactions and nested choices of the players have to be taken into account.

Lemma 2 (Monotonicity [Pla15])**.** *The semantics is* monotone*, i.e.* $\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$*.*

Theorem 3 (Consistency & determinacy [Pla15])**.** *Hybrid games are consistent and determined, i.e.* $\vDash \neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$.

3 Hybrid Game Proofs

An axiomatization for differential game logic has been found in previous work [Pla15], where we refer to for more details.

Note 4 (Differential game logic axiomatization [Pla15])**.**

$$[\cdot]\ [\alpha]\phi \leftrightarrow \neg\langle\alpha\rangle\neg\phi$$

$$\langle:=\rangle\ \langle x := \theta\rangle\phi(x) \leftrightarrow \phi(\theta)$$

$$\langle'\rangle\ \langle x' = f(x)\rangle\phi \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle\phi \qquad (y'(t) = f(y))$$

$$\langle?\rangle\ \langle ?Q\rangle\phi \leftrightarrow (Q \land \phi)$$

$$\langle\cup\rangle\ \langle\alpha \cup \beta\rangle\phi \leftrightarrow \langle\alpha\rangle\phi \lor \langle\beta\rangle\phi$$

$$\langle;\rangle\ \langle\alpha;\beta\rangle\phi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\phi$$

$$\langle^*\rangle\ \phi \lor \langle\alpha\rangle\langle\alpha^*\rangle\phi \rightarrow \langle\alpha^*\rangle\phi$$

$$\langle^d\rangle\ \langle\alpha^d\rangle\phi \leftrightarrow \neg\langle\alpha\rangle\neg\phi$$

$$\text{M}\ \frac{\phi \rightarrow \psi}{\langle\alpha\rangle\phi \rightarrow \langle\alpha\rangle\psi}$$

$$\text{FP}\ \frac{\phi \lor \langle\alpha\rangle\psi \rightarrow \psi}{\langle\alpha^*\rangle\phi \rightarrow \psi}$$

$$\text{ind}\ \frac{\phi \rightarrow [\alpha]\phi}{\phi \rightarrow [\alpha^*]\phi}$$

The proof rules FP and ind are equivalent in the sense that one can be derived from the other in the dG$\mathcal{L}$ calculus [Pla15].

Example 4. The dual filibuster game formula from Lecture 20 proves easily in the dG$\mathcal{L}$

calculus by going back and forth between players [Pla15]:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{x=0\rightarrow 0=0\lor 1=0}\ \mathbb{R}}{x=0\rightarrow\langle x:=0\rangle x=0\lor\langle x:=1\rangle x=0}\ \langle:=\rangle}{x=0\rightarrow\langle x:=0\cup x:=1\rangle x=0}\ \langle\cup\rangle}{x=0\rightarrow\neg\langle x:=0\cap x:=1\rangle\neg x=0}\ \langle^d\rangle}{x=0\rightarrow[x:=0\cap x:=1]x=0}\ [\cdot]}{x=0\rightarrow[(x:=0\cap x:=1)^*]x=0}\ \text{ind}}{x=0\rightarrow\langle(x:=0\cup x:=1)^{\times}\rangle x=0}\ \langle^d\rangle
$$

4 Soundness

Theorem 5 (Soundness [Pla15])**.** *The* dGL *proof calculus in Fig. 4 is sound, i.e. all provable formulas are valid.*

Proof. The full proof can be found in [Pla15]. We just consider a few cases to exemplify the fundamentally more general semantics of hybrid games arguments compared to hybrid systems arguments. To prove soundness of an equivalence axiom $\phi\leftrightarrow\psi$, show $[\![\phi]\!]=[\![\psi]\!]$ for all interpretations I with any set of states $\mathcal{S}$.

- $\langle\cup\rangle$ $[\![\langle\alpha\cup\beta\rangle\phi]\!]=\varsigma_{\alpha\cup\beta}([\![\phi]\!])=\varsigma_\alpha([\![\phi]\!])\cup\varsigma_\beta([\![\phi]\!])=[\![\langle\alpha\rangle\phi]\!]\cup[\![\langle\beta\rangle\phi]\!]=[\![\langle\alpha\rangle\phi\lor\langle\beta\rangle\phi]\!]$
- $\langle;\rangle$ $[\![\langle\alpha;\beta\rangle\phi]\!]=\varsigma_{\alpha;\beta}([\![\phi]\!])=\varsigma_\alpha(\varsigma_\beta([\![\phi]\!]))=\varsigma_\alpha([\![\langle\beta\rangle\phi]\!])=[\![\langle\alpha\rangle\langle\beta\rangle\phi]\!]$.
- $\langle?\rangle$ $[\![\langle?Q\rangle\phi]\!]=\varsigma_{?Q}([\![\phi]\!])=[\![Q]\!]\cap[\![\phi]\!]=[\![Q\land\phi]\!]$
- $[\cdot]$ is sound by Theorem 3.
- M Assume the premise $\phi\rightarrow\psi$ is valid in interpretation I, i.e. $[\![\phi]\!]\subseteq[\![\psi]\!]$. Then the conclusion $\langle\alpha\rangle\phi\rightarrow\langle\alpha\rangle\psi$ is valid in I, i.e. $[\![\langle\alpha\rangle\phi]\!]=\varsigma_\alpha([\![\phi]\!])\subseteq\varsigma_\alpha([\![\psi]\!])=[\![\langle\alpha\rangle\psi]\!]$ by monotonicity (Lemma 2). □

5 Separating Axioms

The axioms of differential game logic in Fig. 4 are sound for hybrid systems as well, because every hybrid system is a (single player) hybrid game. With a few exceptions, they look surprisingly close to the axioms for hybrid systems from Lecture 5. In order to understand the fundamental difference between hybrid systems and hybrid games, it is instructive to also investigate separating axioms, i.e. axioms of hybrid systems that are not sound for hybrid games. Some of these are summarized in Fig. 1, referring to [Pla15] for details.

$\not{K}$ $[\alpha](P \to Q) \to ([\alpha]P \to [\alpha]Q)$	$M_{[\cdot]}$ $\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$
$\not{\overleftarrow{M}}$ $\langle\alpha\rangle(P \lor Q) \to \langle\alpha\rangle P \lor \langle\alpha\rangle Q$	M $\langle\alpha\rangle P \lor \langle\alpha\rangle Q \to \langle\alpha\rangle(P \lor Q)$
$\not{I}$ $[\alpha^*](P \to [\alpha]P) \to (P \to [\alpha^*]P)$	$\forall$I $\mathrm{Cl}_\forall (P \to [\alpha]P) \to (P \to [\alpha^*]P)$
$\not{B}$ $\langle\alpha\rangle\exists x\, P \to \exists x\, \langle\alpha\rangle P$ $(x \notin \alpha)$	$\overleftarrow{B}$ $\exists x\, \langle\alpha\rangle P \to \langle\alpha\rangle\exists x\, P$
$\not{V}$ $p \to [\alpha]p$ $(\mathrm{FV}(p) \cap \mathrm{BV}(\alpha) = \emptyset)$	VK $p \to ([\alpha]true \to [\alpha]p)$
$\not{G}$ $\dfrac{P}{[\alpha]P}$	$M_{[\cdot]}$ $\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$
$\not{R}$ $\dfrac{P_1 \land P_2 \to Q}{[\alpha]P_1 \land [\alpha]P_2 \to [\alpha]Q}$	$M_{[\cdot]}$ $\dfrac{P_1 \land P_2 \to Q}{[\alpha](P_1 \land P_2) \to [\alpha]Q}$
$\not{FA}$ $\langle\alpha^*\rangle P \to P \lor \langle\alpha^*\rangle(\neg P \land \langle\alpha\rangle P)$	

Figure 1: Separating axioms: The axioms and rules on the left are sound for hybrid systems but not for hybrid games. The related axioms or rules on the right are sound for hybrid games.

6 Repetitive Diamonds – Convergence vs. Iteration

More fundamental differences between hybrid systems and hybrid games also exist in terms of convergence rules, even if these have played a less prominent role in this course so far. These differences are discussed in detail elsewhere [Pla15]. In a nutshell, Harel's convergence rule [HMP77] is not a separating axiom, because it is sound for d$\mathcal{G}\mathcal{L}$, just unnecessary, and, furthermore, not even particularly useful for hybrid games [Pla15]. The hybrid version of Harel's convergence rule [Pla08] for d$\mathcal{L}$ reads as follows (it assumes that v does not occur in α):

$$\text{con}\ \frac{p(v+1) \land v+1 > 0 \vdash \langle\alpha\rangle p(v)}{\Gamma, \exists v\, p(v) \vdash \langle\alpha^*\rangle \exists v{\leq}0\, p(v), \Delta}$$

The d$\mathcal{L}$ proof rule con expresses that the variant $p(v)$ holds for some real number $v \leq 0$ after repeating α sufficiently often if $p(v)$ holds for some real number at all in the beginning (antecedent) and, by premise, $p(v)$ can decrease after some execution of α by 1 (or another positive real constant) if $v > 0$. This rule can be used to show positive progress (by 1) with respect to $p(v)$ by executing α. Just like the induction rule ind is often used with a separate premiss for the initial and postcondition check (loop from Lecture 7 on Loops & Invariants), rule con is often used in the following derived form that we simply call con:

$$\text{con}\ \frac{\Gamma \vdash \exists v\, p(v), \Delta \quad \forall v{>}0\, (p(v) \to \langle\alpha\rangle p(v-1)) \quad \exists v{\leq}0\, p(v) \vdash Q}{\Gamma \vdash \langle\alpha^*\rangle Q, \Delta}$$

The following sequent proof shows how convergence rule con can be used to prove a simple dL liveness property of a hybrid program:

$$
\dfrac{
\overset{\mathbb{R}}{\dfrac{*}{x \geq 0 \vdash \exists n\, x < n+1}}
\quad
\overset{\langle:=\rangle}{\dfrac{\overset{\mathbb{R}}{\dfrac{*}{x < n+2 \land n+1 > 0 \vdash x-1 < n+1}}}{x < n+2 \land n+1 > 0 \vdash \langle x := x-1\rangle x < n+1}}
\quad
\overset{\mathbb{R}}{\dfrac{*}{\exists n{\leq}0\, x < n+1 \vdash x < 1}}
}{
\dfrac{x \geq 0 \vdash \langle (x := x-1)^*\rangle x < 1 \quad (\text{con})}{x \geq 0 \rightarrow \langle (x := x-1)^*\rangle x < 1 \quad (\rightarrow\text{R})}
}
$$

Let's compare how dGL proves diamond properties of repetitions based on the iteration axiom $\langle * \rangle$.

Example 6 (Non-game system). The same simple non-game dGL formula

$$x \geq 0 \rightarrow \langle (x := x-1)^*\rangle 0 \leq x < 1$$

as above is provable without con, as shown in Fig. 2, where $\langle \alpha^* \rangle 0 \leq x < 1$ is short for $\langle (x := x-1)^*\rangle (0 \leq x < 1)$. Note that, as in many subsequent proofs, the extra

$$
\begin{array}{ll}
 & * \\ \hline
\mathbb{R} & \forall x\, (0 \leq x < 1 \lor p(x-1) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x)) \\ \hline
\langle:=\rangle & \forall x\, (0 \leq x < 1 \lor \langle x := x-1\rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x)) \\ \hline
\text{US} & \forall x\, (0 \leq x < 1 \lor \langle x := x-1\rangle \langle \alpha^*\rangle 0 \leq x < 1 \rightarrow \langle \alpha^*\rangle 0 \leq x < 1) \rightarrow (x \geq 0 \rightarrow \langle \alpha^*\rangle 0 \leq x < 1) \\ \hline
\langle * \rangle,??,?? & x \geq 0 \rightarrow \langle \alpha^*\rangle 0 \leq x < 1
\end{array}
$$

Figure 2: dGL Angel proof for non-game system Example 6
$x \geq 0 \rightarrow \langle (x := x-1)^*\rangle 0 \leq x < 1$

assumption for **??** near the bottom of the proof inFig. 2 is provable easily using $\langle * \rangle$,**??**:

$$
\begin{array}{ll}
 & * \\ \hline
\langle * \rangle & 0 \leq x < 1 \lor \langle x := x-1\rangle \langle \alpha^*\rangle 0 \leq x < 1 \rightarrow \langle \alpha^*\rangle 0 \leq x < 1 \\ \hline
\forall\text{R} & \forall x\, (0 \leq x < 1 \lor \langle x := x-1\rangle \langle \alpha^*\rangle 0 \leq x < 1 \rightarrow \langle \alpha^*\rangle 0 \leq x < 1)
\end{array}
$$

Example 7 (Choice game). The dGL formula

$$x = 1 \land a = 1 \rightarrow \langle (x := a; a := 0 \cap x := 0)^*\rangle x \neq 1$$

is provable as shown in Fig. 3, where $\beta \cap \gamma$ is short for $x := a; a := 0 \cap x := 0$ and $\langle (\beta \cap \gamma)^*\rangle x \neq 1$ short for $\langle (x := a; a := 0 \cap x := 0)^*\rangle x \neq 1$:

Example 8 (2-Nim-type game). The dGL formula

$$x \geq 0 \rightarrow \langle (x := x-1 \cap x := x-2)^*\rangle 0 \leq x < 2$$

is provable as shown in Fig. 3, where $\beta \cap \gamma$ is short for $x := x-1 \cap x := x-2$ and $\langle (\beta \cap \gamma)^*\rangle 0{\leq}x{<}2$ short for $\langle (x := x-1 \cap x := x-2)^*\rangle 0 \leq x < 2$:

$$
\begin{array}{rr}
 & * \\
\hline
\mathbb{R} & \forall x\,(x \neq 1 \lor p(a,0) \land p(0,a) \to p(x,a)) \to (\mathit{true} \to p(x,a)) \\
\hline
\langle ;\rangle,\langle :=\rangle & \forall x\,(x \neq 1 \lor \langle\beta\rangle p(x,a) \land \langle\gamma\rangle p(x,a) \to p(x,a)) \to (\mathit{true} \to p(x,a)) \\
\hline
\langle\cup\rangle,\langle^{d}\rangle & \forall x\,(x \neq 1 \lor \langle\beta\cap\gamma\rangle p(x,a) \to p(x,a)) \to (\mathit{true} \to p(x,a)) \\
\hline
\text{US} & \forall x\,(x \neq 1 \lor \langle\beta\cap\gamma\rangle\langle(\beta\cap\gamma)^*\rangle x \neq 1 \to \langle(\beta\cap\gamma)^*\rangle x \neq 1) \to (\mathit{true} \to \langle(\beta\cap\gamma)^*\rangle x \neq 1) \\
\hline
\langle^*\rangle,??,?? & \mathit{true} \to \langle(\beta\cap\gamma)^*\rangle x \neq 1 \\
\hline
\mathbb{R} & x = 1 \land a = 1 \to \langle(\beta\cap\gamma)^*\rangle x \neq 1
\end{array}
$$

Figure 3: dG$\mathcal{L}$ Angel proof for choice game Example 7
$x = 1 \land a = 1 \to \langle(x := a; a := 0 \cap x := 0)^*\rangle x \neq 1$

$$
\begin{array}{rr}
 & * \\
\hline
\mathbb{R} & \forall x\,(0{\leq}x{<}2 \lor p(x-1) \land p(x-2) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline
\langle :=\rangle & \forall x\,(0{\leq}x{<}2 \lor \langle\beta\rangle p(x) \land \langle\gamma\rangle p(x) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline
\langle\cup\rangle,\langle^{d}\rangle & \forall x\,(0{\leq}x{<}2 \lor \langle\beta\cap\gamma\rangle p(x) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline
\text{US} & \forall x\,(0{\leq}x{<}2 \lor \langle\beta\cap\gamma\rangle\langle(\beta\cap\gamma)^*\rangle 0{\leq}x{<}2 \to \langle(\beta\cap\gamma)^*\rangle 0{\leq}x{<}2) \to (\mathit{true} \to \langle(\beta\cap\gamma)^*\rangle 0{\leq}x{<}2) \\
\hline
\langle^*\rangle,??,?? & \mathit{true} \to \langle(\beta\cap\gamma)^*\rangle 0{\leq}x{<}2 \\
\hline
\mathbb{R} & x \geq 0 \to \langle(\beta\cap\gamma)^*\rangle 0{\leq}x{<}2
\end{array}
$$

Figure 4: dG$\mathcal{L}$ Angel proof for 2-Nim-type game Example 8
$x \geq 0 \to \langle(x := x - 1 \cap x := x - 2)^*\rangle 0 \leq x < 2$

Example 9 (Hybrid game). The dG$\mathcal{L}$ formula

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle 0 \leq x < 1$$

is provable as shown in Fig. 5, where the notation $\langle(\beta\cup\gamma)^*\rangle 0 \leq x < 1$ is short for $\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$: The proof steps for β use in $\langle ' \rangle$ that $t \mapsto x + t$

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \hline \forall x\, (0 \leq x < 1 \lor \forall t{\geq}0\, p(1+t) \lor p(x-1) \to p(x)) \to (\mathit{true} \to p(x)) \\
\langle := \rangle & \hline \forall x\, (0 \leq x < 1 \lor \langle x := 1\rangle \neg\exists t{\geq}0\, \langle x := x+t\rangle \neg p(x) \lor p(x-1) \to p(x)) \to (\mathit{true} \to p(x)) \\
\langle ' \rangle & \hline \forall x\, (0 \leq x < 1 \lor \langle x := 1\rangle \neg \langle x' = 1\rangle \neg p(x) \lor p(x-1) \to p(x)) \to (\mathit{true} \to p(x)) \\
\langle ; \rangle, \langle ^d \rangle & \hline \forall x\, (0 \leq x < 1 \lor \langle \beta \rangle p(x) \lor \langle \gamma \rangle p(x) \to p(x)) \to (\mathit{true} \to p(x)) \\
\langle \cup \rangle & \hline \forall x\, (0 \leq x < 1 \lor \langle \beta \cup \gamma \rangle p(x) \to p(x)) \to (\mathit{true} \to p(x)) \\
\text{US} & \hline \forall x\, (0{\leq}x{<}1 \lor \langle \beta\cup\gamma\rangle\langle(\beta\cup\gamma)^*\rangle 0 \leq x < 1 \to \langle(\beta\cup\gamma)^*\rangle 0 \leq x < 1) \to (\mathit{true} \to \langle(\beta\cup\gamma)^*\rangle 0 \leq x < 1) \\
\langle ^* \rangle, ??, ?? & \hline \mathit{true} \to \langle(\beta\cup\gamma)^*\rangle 0 \leq x < 1
\end{array}
$$

Figure 5: dG$\mathcal{L}$ Angel proof for hybrid game Example 9
$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle 0 \leq x < 1$

is the solution of the differential equation, so the subsequent use of $\langle := \rangle$ substitutes 1 in for x to obtain $t \mapsto 1 + t$. Recall from Lecture 22 that the winning regions for this formula need $>\omega$ iterations to converge. It is still provable easily.

Exercises

Exercise 1 (***). The following formula was proved using dG$\mathcal{L}$'s hybrid games type proof rules in Fig. 2

$$x \geq 0 \to \langle (x := x - 1)^* \rangle 0 \leq x < 1$$

Try to prove it using the convergence rule con instead.

References

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