25: Distributed Systems & Hybrid Systems
15-424: Foundations of Cyber-Physical Systems

André Platzer
Carnegie Mellon University, Pittsburgh, PA
1 Motivation

2 Quantified Differential Dynamic Logic QdL
   - Design
   - Syntax
   - Semantics

3 Proof Calculus for Distributed Hybrid Systems
   - Compositional Verification Calculus
   - Deduction Modulo with Free Variables & Skolemization
   - Actual Existence and Creation
   - Soundness and Completeness
   - Quantified Differential Invariants

4 Applications

5 Conclusions
Q: I want to verify my car

Challenge
Q: I want to verify my car  
A: Hybrid systems

Challenge (Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
Q: I want to verify my car  
A: Hybrid systems  
Q: But there’s a lot of cars!

**Challenge (Hybrid Systems)**

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
Q: I want to verify a lot of cars

Challenge

Distributed systems

Q: But they move!

Local computation (finite state automaton)
Remote communication (network graph)
Q: I want to verify a lot of cars
A: Distributed systems

Challenge (Distributed Systems)

- Local computation (finite state automaton)
- Remote communication (network graph)
Complex Physical Systems:  Distributed Systems

Q: I want to verify a lot of cars  
A: Distributed systems

Q: But they move!

Challenge (Distributed Systems)

- Local computation  
  (finite state automaton)
- Remote communication  
  (network graph)

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Q: I want to verify lots of moving cars

Challenge

Continuous dynamics (differential equations)
Discrete dynamics (control decisions)
Structural dynamics (remote communication)
Dimensional dynamics (appearance)
Q: I want to verify lots of moving cars
A: Distributed hybrid systems

Challenge (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Structural dynamics (remote communication)
Q: I want to verify lots of moving cars. A: Distributed hybrid systems.

Challenge (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Structural dynamics (remote communication)
- Dimensional dynamics (appearance)
Q: I want to verify lots of moving cars  
A: Distributed hybrid systems  
Q: How?

**Challenge (Distributed Hybrid Systems)**

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Structural dynamics (remote communication)
- Dimensional dynamics (appearance)
State of the Art: Modeling and Simulation

No formal verification of distributed hybrid systems


Hybrid CSP [6] Semantics in Extended Duration Calculus


χ process algebra [8] Simulation, translation of fragments to PHAVER, UPPAAL


Φ-calculus [9] Semantics in rich set theory

ACP \textit{hs} \textit{srt} [10] Modeling language proposal

Contributions

1. System model and semantics for distributed hybrid systems: QHP
2. Specification and verification logic: QdL
3. Proof calculus for QdL
4. First verification approach for distributed hybrid systems
5. Sound and complete axiomatization relative to differential equations
6. Prove collision freedom in a (simple) distributed car control system, where new cars may appear dynamically on the road
7. Logical foundation for analysis of distributed hybrid systems
8. Fundamental extension: first-order $x(i)$ versus primitive $x$
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   • Actual Existence and Creation
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4 Applications

5 Conclusions
Outline (Conceptual Approach)

1 Motivation

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   - Design
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   - Semantics

3 Proof Calculus for Distributed Hybrid Systems
   - Compositional Verification Calculus
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   - Actual Existence and Creation
   - Soundness and Completeness
   - Quantified Differential Invariants

4 Applications

5 Conclusions
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)
- Structural dynamics (communication/coupling)

\[ n := \text{new Car} \]
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ x'' = a \]

- Discrete dynamics (control decisions)

- Structural dynamics (communication/coupling)

\( n \) := new Car
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ x'' = a \]

- Discrete dynamics (control decisions)
  \[ a := \text{if} \ldots \text{then} \, A \, \text{else} \, -b \]

- Structural dynamics (communication/coupling)
Model for Distributed Hybrid Systems

Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ x'' = a \]
- Discrete dynamics (control decisions)
  \[ a := \text{if } \ldots \text{then } A \text{ else } -b \]
- Structural dynamics (communication/coupling)

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Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ x'' = a \]
- Discrete dynamics (control decisions)
  \[ a := \text{if } .. \text{ then } A \text{ else } -b \]
- Structural dynamics (communication/coupling)
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ x(i)'' = a(i) \]
- Discrete dynamics (control decisions)
  \[ a(i) := \text{if} \ldots \text{then} A \text{else} -b \]
- Structural dynamics (communication/coupling)
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ \forall i \, x(i)'' = a(i) \]
- Discrete dynamics (control decisions)
  \[ \forall i \, a(i) := \text{if } \ldots \text{then } A \text{ else } -b \]
- Structural dynamics (communication/coupling)
Q: How to model distributed hybrid systems

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ \forall i \, x(i)'' = a(i) \]

- Discrete dynamics (control decisions)
  \[ \forall i \, a(i) := \text{if} \ldots \text{then} \ A \text{else} \ -b \]

- Structural dynamics (communication/coupling)
  \[ \ell(i) := \text{carInFrontOf}(i) \]
Q: How to model distributed hybrid systems

A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ \forall i \; x(i)'' = a(i) \]
- Discrete dynamics (control decisions)
  \[ \forall i \; a(i) := \text{if} \ldots \text{then} A \text{else} -b \]
- Structural dynamics (communication/coupling)
  \[ \ell(i) := \text{carInFrontOf}(i) \]
- Dimensional dynamics (appearance)

André Platzer (CMU)
Q: How to model distributed hybrid systems

A: Quantified Hybrid Programs

Model (Distributed Hybrid Systems)

- Continuous dynamics (differential equations)
  \[ \forall i \ x(i)'' = a(i) \]

- Discrete dynamics (control decisions)
  \[ \forall i \ a(i) := \text{if .. then A else } -b \]

- Structural dynamics (communication/coupling)
  \[ \ell(i) := \text{carInFrontOf}(i) \]

- Dimensional dynamics (appearance)
  \[ n := \text{new Car} \]
Definition (Quantified hybrid program $\alpha$)

- $\forall i : C \ x(i)' = \theta$ (quantified ODE)
- $\forall i : C \ x(i) := \theta$ (quantified assignment)
- $\text{jump & test}$
- $\alpha; \beta$ (seq. composition)
- $\alpha \cup \beta$ (nondet. choice)
- $\alpha^*$ (nondet. repetition)

$\text{Kleene algebra}$
Definition (Quantified hybrid program $\alpha$)

\[
\begin{align*}
\forall i : C & \; x(s)' = \theta \\
\forall i : C & \; x(s) := \theta \\
?Q & \\
\alpha; \beta & \\
\alpha \cup \beta & \\
\alpha^* &
\end{align*}
\]

(quantified ODE)
(quantified assignment)
(conditional execution)
(seq. composition)
(nondet. choice)
(nondet. repetition)

\{ jump & test \}
\{ Kleene algebra \}
Quantified Differential Dynamic Logic QdL: Syntax

**Definition (Quantified hybrid program \( \alpha \))**

\[
\begin{align*}
\forall i : C \ x(s)' & = \theta \quad \text{(quantified ODE)} \\
\forall i : C \ x(s) & := \theta \quad \text{(quantified assignment)} \\
\ ? Q & \text{ (conditional execution)} \\
\alpha ; \beta & \text{ (seq. composition)} \\
\alpha \cup \beta & \text{ (nondet. choice)} \\
\alpha^* & \text{ (nondet. repetition)}
\end{align*}
\]

\( \{ \text{jump \\& test} \} \) \hspace{1cm} \( \text{Kleene algebra} \)

**DCCS** \( \equiv (\text{ctrl};\text{drive})^* \)

\[
\begin{align*}
\text{ctrl} & \equiv \ \forall i : C \ a(i) := \text{if } \forall j : C \ \text{far}(i,j) \text{ then } A \text{ else } -b \\
\text{drive} & \equiv \ \forall i : C \ x(i)'' = a(i)
\end{align*}
\]
Definition (Quantified hybrid program $\alpha$)

$$\forall i : C \ x(s)' = \theta$$  \hspace{1cm} \text{(quantified ODE)}

$$\forall i : C \ x(s) := \theta$$  \hspace{1cm} \text{(quantified assignment)}

$$?Q$$  \hspace{1cm} \text{(conditional execution)}

$$\alpha; \beta$$  \hspace{1cm} \text{(seq. composition)}

$$\alpha \cup \beta$$  \hspace{1cm} \text{(nondet. choice)}

$$\alpha^*$$  \hspace{1cm} \text{(nondet. repetition)}

$$\{ \ \}$$  \hspace{1cm} \text{jump & test}

Kleene algebra

$$DCCS \equiv (\text{appear}; \text{ctrl}; \text{drive})^*$$

$$\text{appear} \equiv n := \text{new} \ C; \ ?(\forall j : C \ \text{far}(j, n))$$

$$\text{ctrl} \equiv \forall i : C \ a(i) := \text{if} \ \forall j : C \ \text{far}(i, j) \ \text{then} \ A \ \text{else} \ -b$$

$$\text{drive} \equiv \forall i : C \ x(i)'' = a(i)$$
Definition (Quantified hybrid program $\alpha$)

\[
\begin{align*}
\forall i : C & \ x(s)' = \theta & \text{(quantified ODE)} \\
\forall i : C & \ x(s) := \theta & \text{(quantified assignment)} \\
?Q & & \text{(conditional execution)} \\
\alpha ; \beta & & \text{(seq. composition)} \\
\alpha & \cup \beta & \text{(nondet. choice)} \\
\alpha^* & & \text{(nondet. repetition)}
\end{align*}
\]

\[DCCS \equiv (\text{appear} ; \text{ctrl} ; \text{drive})^* \]

appear \equiv n := \text{new } C; \ ?(\forall j : C \ \text{far}(j, n))

ctrl \equiv \forall i : C \ a(i) := \text{if } \forall j : C \ \text{far}(i, j) \text{ then } A \text{ else } -b

drive \equiv \forall i : C \ x(i)'' = a(i)

new C is definable!
Quantified Differential Dynamic Logic QdŁ: Syntax

Definition (QdŁ Formula $\phi$)

$\neg, \land, \lor, \rightarrow, \forall x, \exists x, =, \geq, +, \cdot$  

(\mathbb{R}-first-order part)

$[\alpha]\phi, \langle \alpha \rangle \phi$  

@dynamic part)

$[(\text{appear}; \text{ctrl}; \text{drive})^*] \forall i \neq j : C \ x(i) \neq x(j)$
Quantified Differential Dynamic Logic QdŁ: Syntax

Definition (QdŁ Formula $\phi$)

$\neg, \land, \lor, \rightarrow, \forall x, \exists x, =, \geq, +, \cdot$ (\(\mathbb{R}\)-first-order part)

$[\alpha] \phi, \langle \alpha \rangle \phi$ (dynamic part)

\[
\forall i, j : C \ far(i, j) \rightarrow [(appear ; ctrl ; drive)^*] \ \forall i \neq j : C \ x(i) \neq x(j)
\]
Quantified Differential Dynamic Logic QdŁ: Syntax

Definition (QdŁ Formula $\phi$)

$\neg, \land, \lor, \rightarrow, \forall x, \exists x, =, \geq, +, \cdot$ (ℝ-first-order part)

$[\alpha]\phi, \langle \alpha \rangle \phi$ (dynamic part)

$\forall i, j : C \ far(i, j) \rightarrow [(appear; ctrl; drive)^*] \ \forall i \neq j : C \ x(i) \neq x(j)$

$\text{far}(i, j) \equiv i \neq j \rightarrow x(i) < x(j) \land v(i) \leq v(j) \land a(i) \leq a(j)$

$\lor x(i) > x(j) \land v(i) \geq v(j) \land a(i) \geq a(j) \ldots$
Definition (Quantified hybrid program $\alpha$: transition semantics)

$$\forall i : C \; x(s) := \theta$$

if $w(x)(v_i^e[s]) = v_i^e[\theta]$ (for all $e$)
and otherwise unchanged
Definition (Quantified hybrid program $\alpha$: transition semantics)

$\forall i : C \ x(s)' = \theta$

$\forall i \ x(s)' = \theta$

$\frac{d \varphi(t)^e_i [x(s)]}{dt}(\zeta) = \varphi(\zeta)^e_i [\theta]$ (for all $e$)
Quantified Differential Dynamic Logic QdL: Semantics

Definition (Quantified hybrid program $\alpha$: transition semantics)

$$\alpha; \beta$$

$\alpha$ transition from $v$ to $s$

$\beta$ transition from $s$ to $w$
Definition (Quantified hybrid program $\alpha$: transition semantics)
Definition (Quantified hybrid program $\alpha$: transition semantics)

$\alpha; \beta$

$\alpha$

$\beta$

$x$

$t$

$\nu$

$\nu$

$\nu$

$\nu$

$\nu$

$\nu$

$\nu$

$\nu$

$\nu$

$\nu$
Definition (Quantified hybrid program $\alpha$: transition semantics)

$\alpha^*$

$V \xrightarrow{\alpha} S_1 \xrightarrow{\alpha} S_2 \xrightarrow{\ldots} S_n \xrightarrow{\alpha} W$
Definition (Quantified hybrid program $\alpha$: transition semantics)

$\alpha^*$

$v \xrightarrow{\alpha} s_1 \xrightarrow{\alpha} s_2 \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} s_n \xrightarrow{\alpha} w$

$X \quad v \quad w \quad t$
Definition (Quantified hybrid program $\alpha$: transition semantics)
Definition (Quantified hybrid program $\alpha$: transition semantics)

$\begin{align*}
\alpha & \rightarrow w_1 \\
\beta & \rightarrow w_2 \\
\alpha \cup \beta & 
\end{align*}$

![Diagram showing transition semantics](image.png)
Definition (Quantified hybrid program $\alpha$: transition semantics)

$\text{if } v \models Q \Rightarrow \text{no change}$

$\text{if } v \not\models Q \Rightarrow \text{otherwise no transition}$
Definition (Quantified hybrid program $\alpha$: transition semantics)

$$\begin{align*}
\text{if } \nu \models Q & \quad \text{no change if } \nu \models Q \\
\text{otherwise no transition}
\end{align*}$$
Definition (QdŁ Formula $\phi$)

$[\alpha] \phi$

Composition semantics $\Rightarrow$ compositional calculus
Definition (QdL Formula $\phi$)
Definition (QdŁ Formula $\phi$)

$$[\alpha]\phi$$

$$\alpha$$-span

Composition semantics $\Rightarrow$ compositional calculus!
Definition (QdŁ Formula $\phi$)

- $[\alpha]\phi$
- $\langle \beta \rangle \phi$
- $\beta$-span

$\alpha$-span

Compositional semantics $\Rightarrow$ Compositional calculus!
Definition (Qd\(\mathcal{L}\) Formula \(\phi\))

\[ \langle \beta \rangle [\alpha] \phi \]

\[ \langle \beta \rangle \phi \]

\[ [\alpha] \phi \]

\[ \beta\text{-span} \]

\[ \alpha\text{-span} \]

Compositional semantics \(\Rightarrow\) Compositional calculus
Quantified Differential Dynamic Logic QdŁ: Semantics

Definition (QdŁ Formula $\phi$)

compositional semantics $\Rightarrow$ compositional calculus!
Outline (Verification Approach)

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5 Conclusions
\[ \phi([\forall i x(i) := \theta]x(u)) \]
∀i (i = u → \phi(\theta))

\frac{\phi([\forall i x(i) := \theta] x(u))}{\phi(\forall i x(i) := \theta)}
\( \forall i \ (i = [\forall i \ x(i) := \theta] u \rightarrow \phi(\theta)) \)

\[ \phi([\forall i \ x(i) := \theta] x(u)) \]
∀i (i = [∀i x(i) := θ]u → φ(θ))

\[
\phi([∀i x(i) := θ]x(u))
\]

\[
\phi([∀i x(s) := θ]x(u))
\]

∀i x(s) := θ

ϕ

V

W
∀i (i = [∀i x(i) := θ] u → φ(θ))

\[ \phi([∀i x(i) := θ] x(u)) \]

if \( \exists i s = u \) then \( ∀i (s = u → φ(θ)) \) else \( φ(x(u)) \)

\[ \phi([∀i x(s) := θ] x(u)) \]
∀ i (i = [∀ i x(i) := θ] u → φ(θ))


φ([∀ i x(i) := θ] x(u))

\[ \text{if } \exists i s = u \text{ then } \forall i (s = u \rightarrow φ(θ)) \text{ else } φ(x(u)) \]

\[ \phi([∀ i x(s) := θ] x(u)) \]
∀ i (i = [∀i x(i) := θ] u → φ(θ))

∀(∀i x(i) := θ]x(u))

if ∃ i s = u then ∀i (s = u → φ(θ)) else φ(x(u))

φ([∀i x(s) := θ] x(u))
∀i (i = [∀i x(i) := θ]u → φ(θ))

\[ \phi([∀i x(i) := θ]x(u)) \]

if \( \exists i s = u \) then \( ∀i (s = u → φ(θ)) \) else \( φ(x(u)) \)

\[ \phi([∀i x(s) := θ]x(u)) \]
∀i (i = [∀i x(i) := θ] u → φ(θ))

\[ \phi([∀i x(i) := θ] x(u)) \]

if \( \exists i s = [A]u \) then \( ∀i (s = [A]u → φ(θ)) \) else \( ϕ(x([A]u)) \)

\[ \phi([∀i x(s) := θ] x(u)) \]

∀i x(s) := θ → ϕ

∀i x(s) := θ

∀i x(s) := θ
\[
\forall i (i = [\forall i x(i) := \theta]u \rightarrow \phi(\theta))
\]

\[
\phi([\forall i x(i) := \theta]x(u))
\]

\[
\text{if } \exists i s = [A]u \text{ then } \forall i (s = [A]u \rightarrow \phi(\theta)) \text{ else } \phi(x([A]u))
\]

\[
\phi([\forall i x(s) := \theta]x(u))
\]

\[
\forall t \geq 0 [\forall i x(i) := x_i(t)] \phi
\]

\[
[\forall i x(i)' = \theta] \phi
\]
Proof Calculus for Quantified Differential Dynamic Logic

\[
\forall i (i = [\forall i x(i) := \theta] u \rightarrow \phi(\theta)) \\
\phi([\forall i x(i) := \theta] x(u))
\]

\[
\text{if } \exists i s = [A] u \text{ then } \forall i (s = [A] u \rightarrow \phi(\theta)) \text{ else } \phi(x([A] u))
\]

\[
\phi([\forall i x(s) := \theta] x(u))
\]

\[
\forall t \geq 0 [\forall i x(i) := x_i(t)] \phi
\]

\[
[\forall i x(i)\prime = \theta] \phi
\]
Proof Calculus for Quantified Differential Dynamic Logic

compositional semantics $\Rightarrow$ compositional rules!
\[
\frac{[\alpha] \phi \land [\beta] \phi}{[\alpha \cup \beta] \phi}
\]
Proof Calculus for Quantified Differential Dynamic Logic

\[
\begin{align*}
&\frac{[\alpha] \phi \land [\beta] \phi}{[\alpha \cup \beta] \phi} \\
&\frac{[\alpha][\beta] \phi}{[\alpha; \beta] \phi}
\end{align*}
\]
\[
\frac{[\alpha] \phi \land [\beta] \phi}{[\alpha \cup \beta] \phi}
\]

\[
\frac{[\alpha][\beta] \phi}{[\alpha; \beta] \phi}
\]

\[
\phi \quad (\phi \rightarrow [\alpha] \phi) \quad [\alpha^*] \phi
\]
∀i ≠ j x(i) ≠ x(j) → [∀i x(i)'' = −b] ∀j ≠ k x(j) ≠ x(k)
∀i ≠ j x(i) ≠ x(j) → [∀i x(i)′ = v(i), v(i)′ = −b] ∀j ≠ k x(j) ≠ x(k)

∀i ≠ j x(i) ≠ x(j) → [∀i x(i)″ = −b] ∀j ≠ k x(j) ≠ x(k)
∀ \( i \neq j \) \( x(i) \neq x(j) \) → \( \forall t \geq 0 \left[ \forall i \ x(i) := -\frac{b}{2} t^2 + v(i) t + x(i) \right] \forall j \neq k \ x(j) \neq x(k) \)

∀ \( i \neq j \) \( x(i) \neq x(j) \) → \( \forall t \geq 0 \left[ \forall i \ x(i) = v(i), \ v(i)' = -b \right] \forall j \neq k \ x(j) \neq x(k) \)

∀ \( i \neq j \) \( x(i) \neq x(j) \) → \( \left[ \forall i \ x(i)'' = -b \right] \forall j \neq k \ x(j) \neq x(k) \)
\[\forall i \neq j \ x(i) \neq x(j) \rightarrow s \geq 0 \rightarrow [\forall i \ x(i) := -\frac{b}{2}s^2 + v(i)s + x(i)] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow \forall t \geq 0 [\forall i \ x(i) := -\frac{b}{2}t^2 + v(i)t + x(i)] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)' = v(i), \ v(i)' = -b] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)'' = -b] \forall j \neq k \ x(j) \neq x(k)\]
\[
\forall i \neq j \; x(i) \neq x(j), s \geq 0 \rightarrow [\forall i \; x(i) := -\frac{b}{2} s^2 + v(i) s + x(i)] \forall j \neq k \; x(j) \neq x(k)
\]

\[
\forall i \neq j \; x(i) \neq x(j) \rightarrow s \geq 0 \rightarrow [\forall i \; x(i) := -\frac{b}{2} s^2 + v(i) s + x(i)] \forall j \neq k \; x(j) \neq x(k)
\]

\[
\forall i \neq j \; x(i) \neq x(j) \rightarrow \forall t \geq 0 \; [\forall i \; x(i) := -\frac{b}{2} t^2 + v(i) t + x(i)] \forall j \neq k \; x(j) \neq x(k)
\]

\[
\forall i \neq j \; x(i) \neq x(j) \rightarrow [\forall i \; x(i)' = v(i), v(i)' = -b] \forall j \neq k \; x(j) \neq x(k)
\]

\[
\forall i \neq j \; x(i) \neq x(j) \rightarrow [\forall i \; x(i)'' = -b] \forall j \neq k \; x(j) \neq x(k)
\]
∀i ≠ j x(i) ≠ x(j), s ≥ 0 → ∀j ≠ k \(-\frac{b}{2} s^2 + v(j)s + x(j) \neq -\frac{b}{2} s^2 + v(k)s + x(k)\)

∀i ≠ j x(i) ≠ x(j), s ≥ 0 → [∀i x(i) := −\(\frac{b}{2} s^2 + v(i)s + x(i)\)] ∀j ≠ k x(j) ≠ x(k)

∀i ≠ j x(i) ≠ x(j) → ∀t ≥ 0 [∀i x(i) := −\(\frac{b}{2} t^2 + v(i)t + x(i)\)] ∀j ≠ k x(j) ≠ x(k)

∀i ≠ j x(i) ≠ x(j) → ∀i x(i)′ = v(i), v(i)' = −b] ∀j ≠ k x(j) ≠ x(k)

∀i ≠ j x(i) ≠ x(j) → [∀i x(i)'' = −b] ∀j ≠ k x(j) ≠ x(k)
\[ \forall i \neq j \ x(i) \neq x(j) \rightarrow \forall j \neq k \quad \forall s \geq 0 \left( -\frac{b}{2}s^2 + v(j)s + x(j) \neq -\frac{b}{2}s^2 + v(k)s + x(k) \right) \]

\[ \forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow \forall j \neq k \left( -\frac{b}{2}s^2 + v(j)s + x(j) \neq -\frac{b}{2}s^2 + v(k)s + x(k) \right) \]

\[ \forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow [\forall i \ x(i) := -\frac{b}{2}s^2 + v(i)s + x(i)] \forall j \neq k \ x(j) \neq x(k) \]

\[ \forall i \neq j \ x(i) \neq x(j) \rightarrow \forall t \geq 0 \left[ \forall i \ x(i) := -\frac{b}{2}t^2 + v(i)t + x(i) \right] \forall j \neq k \ x(j) \neq x(k) \]

\[ \forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)' = v(i), v(i)' = -b] \forall j \neq k \ x(j) \neq x(k) \]

\[ \forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)'' = -b] \forall j \neq k \ x(j) \neq x(k) \]
\begin{align*}
\forall i \neq j \ x(i) \neq x(j) & \rightarrow \forall j \neq k \ QE\forall s \geq 0 ( - \frac{b}{2} s^2 + v(j) s + x(j) \neq - \frac{b}{2} s^2 + v(k) s + x(k)) \\
\forall i \neq j \ x(i) \neq x(j), s \geq 0 & \rightarrow \forall j \neq k ( - \frac{b}{2} s^2 + v(j) s + x(j) \neq - \frac{b}{2} s^2 + v(k) s + x(k)) \\
\forall i \neq j \ x(i) \neq x(j), s \geq 0 & \rightarrow [\forall i \ x(i) := - \frac{b}{2} s^2 + v(i) s + x(i)] \forall j \neq k \ x(j) \neq x(k) \\
\forall i \neq j \ x(i) \neq x(j) & \rightarrow s \geq 0 \rightarrow [\forall i \ x(i) := - \frac{b}{2} s^2 + v(i) s + x(i)] \forall j \neq k \ x(j) \neq x(k) \\
\forall i \neq j \ x(i) \neq x(j) & \rightarrow [\forall i \ x(i)' = v(i), v(i)' = -b] \forall j \neq k \ x(j) \neq x(k) \\
\forall i \neq j \ x(i) \neq x(j) & \rightarrow [\forall i \ x(i)'' = -b] \forall j \neq k \ x(j) \neq x(k)
\end{align*}
\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow \forall j \neq k \ (x(j) \leq x(k) \land v(j) \leq v(k) \lor x(j) \geq x(k) \land v(j) \geq v(k))
\]

\[
\forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow \forall j \neq k \ (-\frac{b}{2} s^2 + v(j)s + x(j) \neq -\frac{b}{2} s^2 + v(k)s + x(k))
\]

\[
\forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow [\forall i \ x(i) := -\frac{b}{2} s^2 + v(i)s + x(i)] \forall j \neq k \ x(j) \neq x(k)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow s \geq 0 \rightarrow [\forall i \ x(i) := -\frac{b}{2} s^2 + v(i)s + x(i)] \forall j \neq k \ x(j) \neq x(k)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i) := -\frac{b}{2} t^2 + v(i)t + x(i)] \forall j \neq k \ x(j) \neq x(k)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)' = v(i), v(i)' = -b] \forall j \neq k \ x(j) \neq x(k)
\]

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\[
\forall X, Y, V, W \ (X \neq Y \rightarrow X \leq Y \land V \leq W \lor X \geq Y \land V \geq W)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow \forall j \neq k \ (x(j) \leq x(k) \land v(j) \leq v(k) \lor x(j) \geq x(k) \land v(j) \geq v(k))
\]

\[
\forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow \forall j \neq k \left( -\frac{b}{2} s^2 + v(j) s + x(j) \neq -\frac{b}{2} s^2 + v(k) s + x(k) \right)
\]

\[
\forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow [\forall i \ x(i) := -\frac{b}{2} s^2 + v(i) s + x(i)] \forall j \neq k \ x(j) \neq x(k)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow s \geq 0 \rightarrow [\forall i \ x(i) := -\frac{b}{2} s^2 + v(i) s + x(i)] \forall j \neq k \ x(j) \neq x(k)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)' = v(i), v(i)' = -b] \forall j \neq k \ x(j) \neq x(k)
\]

\[
\forall i \neq j \ x(i) \neq x(j) \rightarrow [\forall i \ x(i)'' = -b] \forall j \neq k \ x(j) \neq x(k)
\]
\[\forall X, Y, V, W \ (X \neq Y \rightarrow X \leq Y \wedge V \leq W \vee X \geq Y \wedge V \geq W)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow \forall j \neq k \ (x(j) \leq x(k) \wedge v(j) \leq v(k) \vee x(j) \geq x(k) \wedge v(j) \geq v(k))\]

\[\forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow \forall j \neq k \left(-\frac{b}{2} s^2 + v(j) s + x(j) \neq -\frac{b}{2} s^2 + v(k) s + x(k)\right)\]

\[\forall i \neq j \ x(i) \neq x(j), s \geq 0 \rightarrow \left[\forall i \ x(i) := -\frac{b}{2} s^2 + v(i) s + x(i)\right] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow s \geq 0 \rightarrow \left[\forall i \ x(i) := -\frac{b}{2} s^2 + v(i) s + x(i)\right] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow \forall t \geq 0 \left[\forall i \ x(i) := -\frac{b}{2} t^2 + v(i) t + x(i)\right] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow \forall t \geq 0 \left[\forall i \ x(i) := -\frac{b}{2} t^2 + v(i) t + x(i)\right] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow \left[\forall i \ x(i)' = v(i), v(i)' = -b\right] \forall j \neq k \ x(j) \neq x(k)\]

\[\forall i \neq j \ x(i) \neq x(j) \rightarrow \left[\forall i \ x(i)'' = -b\right] \forall j \neq k \ x(j) \neq x(k)\]
Actual Existence Function $E(\cdot)$

$$E(i) = \begin{cases} 
0 & \text{if } i \text{ denotes a possible object} \\
1 & \text{if } i \text{ denotes an actively existing objects}
\end{cases}$$
Actual Existence Function $E(i)$

$$E(i) = \begin{cases} 0 & \text{if } i \text{ denotes a possible object} \\ 1 & \text{if } i \text{ denotes an actively existing object} \end{cases}$$

$$[n := \text{new } C] \phi$$
Actual Existence and Creation

Actual Existence Function $E(\cdot)$

$$E(i) = \begin{cases} 
0 & \text{if } i \text{ denotes a possible object} \\
1 & \text{if } i \text{ denotes an actively existing objects}
\end{cases}$$

$$[(\forall j : C \ n := j); \ [n := \text{new } C]\phi]$$
Actual Existence and Creation

**Actual Existence Function** $E(·)$

$$E(i) = \begin{cases} 
0 & \text{if } i \text{ denotes a possible object} \\
1 & \text{if } i \text{ denotes an actively existing object}
\end{cases}$$

$$\left[ (\forall j : C ~ n := j); ~(E(n) = 0); \quad [n := \text{new } C] \phi \right]$$
Actual Existence and Creation

Actual Existence Function $E(\cdot)$

$$E(i) = \begin{cases} 
0 & \text{if } i \text{ denotes a possible object} \\
1 & \text{if } i \text{ denotes an actively existing objects}
\end{cases}$$

$$\left( \forall j : C \ n := j \right); \ ?(E(n) = 0); \ E(n) := 1 \phi$$

$$[n := \text{new } C] \phi$$
Actual Existence and Creation

Actual Existence Function $E(\cdot)$

$$E(i) = \begin{cases} 
0 & \text{if } i \text{ denotes a possible object} \\
1 & \text{if } i \text{ denotes an actively existing object} 
\end{cases}$$

$$[(\forall j : C \ n := j); \ ?(E(n) = 0); \ E(n) := 1] \phi$$

$$[n := \text{new } C] \phi$$

$$\forall i : C! \ \phi \ \equiv$$
$$\forall i : C! \ f(s) := \theta \ \equiv$$
$$\forall i : C! \ f(s)' = \theta \ \equiv$$

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FCPS / 25: Distributed Systems & Hybrid Systems
Actual Existence and Creation

Actual Existence Function \( \mathcal{E}(\cdot) \)

\[
\mathcal{E}(i) = \begin{cases} 
0 & \text{if } i \text{ denotes a possible object} \\
1 & \text{if } i \text{ denotes an actively existing objects}
\end{cases}
\]

\[
[(\forall j : C \ n := j); \ ?(\mathcal{E}(n) = 0); \ \mathcal{E}(n) := 1] \phi \\
\quad [n := \text{new } C] \phi
\]

\[\forall i : C! \ \phi \quad \equiv \quad \forall i : C \ (\mathcal{E}(i) = 1 \rightarrow \phi)\]

\[\forall i : C! \ f(s) := \theta \quad \equiv \quad \forall i : C \ f(s) := (\text{if } \mathcal{E}(i) = 1 \text{ then } \theta \text{ else } f(s))\]

\[\forall i : C! \ f(s)' = \theta \quad \equiv \quad \forall i : C \ f(s)' = \mathcal{E}(i)\theta\]
Soundness and Completeness

Theorem (Relative Completeness)

QdL calculus is a sound & complete axiomatisation of distributed hybrid systems relative to quantified differential equations.

Proof 16p.

Corollary (Proof-theoretical Alignment)

proving distributed hybrid systems = proving dynamical systems!

Corollary (Yes, we can!)

distributed hybrid systems can be verified by recursive decomposition
Soundness and Completeness

Theorem (Relative Completeness)

QdŁ calculus is a sound & complete axiomatisation of distributed hybrid systems relative to quantified differential equations.

Proof 16p.

Corollary (Proof-theoretical Alignment)

proving distributed hybrid systems = proving dynamical systems!
Quantified Differential Invariants

Theorem (Quantified Differential Invariant) \((QdI)\)

\[
\frac{Q \rightarrow [\forall i : C f(i)' := \theta]F'}{F \rightarrow [\forall i : C f(i)' = \theta \& Q]F}
\]

is sound
∀i : C \ 2x(i)^3 ≥ 1 → [∀i : C \ x(i)' = x(i)^2 + x(i)^4 + 2]∀i : C \ 2x(i)^3 ≥ 1
A Simple Proof with Quantified Differential Invariants

\[
\forall i : C \ 2x(i)^3 \geq 1 \rightarrow [\forall i : C \ x(i)' = x(i)^2 + x(i)^4 + 2])\forall i : C \ 2x(i)^3 \geq 1
\]
A Simple Proof with Quantified Differential Invariants

\[
\forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2 \forall i : C \ (2x(i)^3)' \geq 0
\]

\[
\forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2 \rightarrow \forall i : C \ (2x(i)^3) \geq 1
\]
\[\forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2\] \[\forall i : C \ 6x(i)^2x(i) \geq 0\]

\[\forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2\] \[\forall i : C \ (2x(i)^3)' \geq 0\]

\[\forall i : C \ 2x(i)^3 \geq 1 \rightarrow [\forall i : C \ x(i)' = x(i)^2 + x(i)^4 + 2]\forall i : C \ 2x(i)^3 \geq 1\]
\[ \forall i : C \ 6x(i)^2(x(i)^2 + x(i)^4 + 2) \geq 0 \]

\[ [\forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2] \forall i : C \ 6x(i)^2x(i)' \geq 0 \]

\[ \forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2 \]

\[ [\forall i : C \ (2x(i)^3)' \geq 0] \forall i : C \ 2x(i)^3 \geq 0 \]

\[ \forall i : C \ 2x(i)^3 \geq 1 \rightarrow [\forall i : C \ x(i)' = x(i)^2 + x(i)^4 + 2] \forall i : C \ 2x(i)^3 \geq 1 \]
true

\( \forall i : C \ 6x(i)^2(x(i)^2 + x(i)^4 + 2) \geq 0 \)

\[ \forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2 \forall i : C \ 6x(i)^2 x(i)' \geq 0 \]

\[ \forall i : C \ x(i)' := x(i)^2 + x(i)^4 + 2 \forall i : C \ (2x(i)^3)' \geq 0 \]

\[ \forall i : C \ 2x(i)^3 \geq 1 \rightarrow \forall i : C \ x(i)' = x(i)^2 + x(i)^4 + 2 \forall i : C \ 2x(i)^3 \geq 1 \]
1 Motivation

2 Quantified Differential Dynamic Logic QdL
   • Design
   • Syntax
   • Semantics

3 Proof Calculus for Distributed Hybrid Systems
   • Compositional Verification Calculus
   • Deduction Modulo with Free Variables & Skolemization
   • Actual Existence and Creation
   • Soundness and Completeness
   • Quantified Differential Invariants

4 Applications

5 Conclusions
Driver’s License Test for Robotic Cars?
Driver’s License Test for Robotic Cars?
Driver’s License Test for Robotic Cars? Proof!
Challenge: Local lane dynamics

- A car controller for a differential equation respects separation of local lane.
Car Control: Local Lane Control Challenge

Challenge: Local lane dynamics

- A car controller for a differential equation respects separation of local lane.
- Follower car maintains safe distance to leader:

\[
\text{follower} \ll \text{leader} \rightarrow \text{ctrl} \; x''_i = \text{ctrl} \]

\[
\text{follower} \ll \text{leader} \equiv \left( x_f \leq x_\ell \right) \land \left( f \neq \ell \right) \rightarrow \left( x_\ell > x_f + v_f^2 f^2 b - v_\ell^2 \ell^2 B \right) \land \left( x_\ell > x_f \right) \land v_f \geq 0 \land v_\ell \geq 0
\]
Car Control: Local Lane Control Challenge

Challenge: Local lane dynamics

- A car controller for a differential equation respects separation of local lane.
- Follower car maintains safe distance to leader:
  \[ f \ll \ell \rightarrow [(a_i := \text{ctrl}; \ x_i'' = a_i)^*] f \ll \ell \]
A car controller for a differential equation respects separation of local lane.

Follower car maintains safe distance to leader:

\[ f \ll \ell \rightarrow [(a_i := ctrl; \ x_i'' = a_i)^*] f \ll \ell \]

\[
f \ll \ell \equiv (x_f \leq x_\ell) \land (f \neq \ell) \rightarrow \\
(x_\ell > x_f + \frac{v_f^2}{2b} - \frac{v_\ell^2}{2B} \\
\land x_\ell > x_f \land v_f \geq 0 \land v_\ell \geq 0)
\]
Challenge: Global lane dynamics

- All controllers for arbitrarily many differential equations respect separation globally on lane.
Challenge: Global lane dynamics

- All controllers for arbitrarily many differential equations respect separation globally on lane.
- Each car safe behind all others
Car Control: Global Lane Control Challenge

Challenge: Global lane dynamics

- All controllers for arbitrarily many differential equations respect separation globally on lane.
- Each car safe behind all others

$$[(\forall i \ a(i) := \text{ctrl}; \ \forall i \ x(i)'' = a(i)) \wedge \forall i, j \ i \ll j]$$
Challenge: Local highway dynamics

- All controllers for arbitrarily many differential equations respect separation locally on highway.

\[ (n := \text{new } C; \forall i a(i) := \text{ctrl}; \forall i x(i)' = a(i)) \]

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Challenge: Local highway dynamics

- All controllers for arbitrarily many differential equations respect separation locally on highway.
- For each lane: all controllers for the differential equations respect separation even if cars appear or disappear.
All controllers for arbitrarily many differential equations respect separation locally on highway.

For each lane: all controllers for the differential equations respect separation even if cars appear or disappear.

Each car safe behind all others, even if new cars appear or disappear.
Challenge: Local highway dynamics

- All controllers for arbitrarily many differential equations respect separation locally on highway.
- For each lane: all controllers for the differential equations respect separation even if cars appear or disappear.
- Each car safe behind all others, even if new cars appear or disappear.

\[(n := \text{new } C; \ \forall i \ a(i) := \text{ctrl}; \ \forall i \ x(i)'' = a(i))^* \] \ \forall i, j \ i \ll j
All controllers for arbitrarily many differential equations respect separation globally on highway.
All controllers for arbitrarily many differential equations respect separation globally on highway.

All controllers for the differential equations respect separation even if cars switch lanes.
All controllers for arbitrarily many differential equations respect separation globally on highway.

All controllers for the differential equations respect separation even if cars switch lanes.

On all lanes, all car safe behind all others on their lanes, even if cars switch lanes.
Challenge: Global highway dynamics

- All controllers for arbitrarily many differential equations respect separation globally on highway.
- All controllers for the differential equations respect separation even if cars switch lanes.
- On all lanes, all car safe behind all others on their lanes, even if cars switch lanes.

$$\forall l (n := \text{new } C; \ \forall i a(i) := \text{ctrl}; \ \forall i x(i)'' = a(i))^* \ \forall l \forall i, j \ i \ll j$$
Outline

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Conclusions

quantified differential dynamic logic

\[ QdL = \text{FOL} + \text{DL} + \text{QHP} \]

- Distributed hybrid systems everywhere
- System model and semantics
- Logic for distributed hybrid systems
- Compositional proof calculus
- First verification approach
- Sound & complete / diff. eqn.
- Quantified differential invariants
- Distributed car control verified
- Distributed aircraft control verified
Conclusions

- Distributed hybrid systems everywhere
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- Quantified differential invariants
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quantified differential dynamic logic

\[ \text{QdL} = \text{FOL} + \text{DL} + \text{QHP} \]
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