15-424 Starlab

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Exploring Orbit Recircularization

Abstract:

In this paper I explore the ability to prove the safety of satellite recircularization. The basis of the proofs lies in the use of KeYmaeraX, a theorem prover created by Carnegie Mellon University's Logical Systems Lab. It gives the ability to create a model of a physical system and using differential dynamic logic, prove safety outcomes of it. We present three models as stepping stones to an overarching proof showing the ability of satellite recircularization. I attempt to prove all three, only succeeding with one. These models are preceded with an explanation of the necessary assumptions, and the justification of the assumptions, needed for the models. Using considerations of energy conservation and conservation of angular momentum, the first model that shows the existence of points in an elliptic orbit, that can be used easily to circularize an elliptic orbit (unproved). The second model shows the ability for an impulsive thrust to circularize orbits at that point (proved). With the assumptions of the first two models we can simplify the last model, which shows how with an analysis of the orbit, once an orbit, an operator can command the satellite to remain within a safe amount of circularity (unproved).

Introduction:

Contrary to popular belief, there is air in space, at least for a part of it. In low Earth orbit (LEO), the home of many satellites, there is a significant enough amount of atmosphere, that satellites do indeed experience air friction (drag). This drag, though very small, does slow down a satellite enough that it loses the properties of its deployed orbit. This means slowly falling back to Earth, and losing its circular orbit.

This is an issue for many reasons. For one, these satellites are deployed into certain orbits that are necessary to the mission that are to complete, be it imaging, communications, etc. Should these satellites lose their desired orbit, they will not be able to take the pictures they want, send and receive the data it is supposed to, or just fail to complete their mission in general. Another reason is that of safety.

Real estate in space is delicate. Scheduling an orbit is a matter of coordination with every other nearby object so that collisions do cannot and do not occur. Collisions are expensive, as the cost of launching and deploying a satellite is very high, not to mention the hardware lost. Also deployment missions are infrequent and often have queues. This means that for time sensitive missions, you really only have one shot. Collisions are also dangerous as in a collision, it is nearly impossible to control the outcome, regardless of all the controls you put into the satellite. An out of control satellite is particularly dangerous as its new and unplanned trajectory may cause it to collide with others. That collision in turn can cause more collisions and have a cascading effect. This kind of event is called Kessler Syndrome, where one collision can lead to many others and result in the destruction of many satellites in orbit, and the resulting debris can hinder future space missions. It is quite apparent that because of these reasons, being able to ensure that satellite orbits can be safely controlled is very important. It is the difference between mission success and failure. It can also be the difference of millions of dollars in hardware and planning. It can even be the difference between the continuation of everyday luxuries, or in the case of Kessler Syndrome, the end of many of modern day technologies like telecommunications. This importance justifies the need for a strong proof of the safety of satellites.

This task is not easy, as it will be shown in this paper. One reason is simply the complexity of orbital dynamics. The other reason is due to the nature of the math of orbital dynamics. It is not easy for the discrete nature of a computer to reason about it. Using KeYmaeraX helps with this process as it has the ability to use Mathematica to reason about differential equations. This, however, is also limited as to the nature of circular orbits. The mechanics of orbits are best handled in polar coordinates. This is an issue as irrational numbers, π most importantly, cannot be represented in KeYmaeraX.

My approach was to break down the necessary portions of a final proof, into several smaller models that in turn show an important step towards the safety of a satellite. By attempting to show the validity of each smaller step, we work towards a larger general proof of safety.

Assumptions:

There are several key assumptions that we make for the models. The first assumption is that we are working with a 2-dimensional system, with a point mass as the satellite and the Earth is just a point creating a central force through gravity. This simplifies the geometry significantly as we only need to either work with x and y directions in Cartesian coordinates, or one radial dimension and one polar dimension if polar coordinates, rather than worrying about three dimensions. This assumption is safe as we can always rotate the frame of reference of circular orbit to lie in a plane, and thus have one coordinate be zeroed out.

The next assumption we make is that there is no air resistance in our models. This may seem concerning as that is the thing we are trying to overcome when circularizing our orbit. This assumption is safe to make though at a small time scale (order of an orbit or so). This is because, while there is actually air resistance affecting the orbit, it is very small and has little effect in the short term. The issue that we are trying to overcome is when an orbit is dragged down by air resistance in the long run, slowly changing the orbit after a long time. To show this, let us consider some numbers provided by *Space Mission Analysis and Design* by Wertz. Considering a low Earth orbit at about 250km, we can expect to need to use somewhere between 636 and 2002 m/s of delta-v per year (delta-v being the term describing how much velocity is being added to the system). If we consider the fact that an average orbit is about 90 minutes, we find that the approximate velocity change per orbit is about 0.2 m/s. We then know that orbit velocity can be found by: $v = \sqrt{\frac{G*M}{r}}$ where G is the gravitational constant, M is the mass of the Earth, and r is the radius of the orbit. This gives us orbit velocities of about 7000m/s. This clearly shows that within a single orbit, the fraction of velocity lost, is extremely negligible and that we can ignore it in the short term. This is an extremely useful assumption to make as it allows us to assume that energy is being conserved in the system, rather than it being lost to drag. It also allows us to assume that angular momentum is being conserved, as there is no external force affecting the elliptic or circular orbit. These constants are useful for calculations and models later on.

We also assume the mass of the Earth is much greater than our satellite. This is clearly valid in the case of man-made satellites and the Earth. This allows us to say that the gravitational force of the satellite does not move the Earth any significant amount, and that our orbit can be considered that of a moving point around a fixed central force. Also, anywhere reduced mass is needed in an equation, we can assume that it is dominated by the mass of the Earth.

The last assumption we make is that thrusts are impulsive. This means that in our models, the thrusts can be modeled as a singular change in velocity rather than an accelerating force being applied for a certain amount of time. This greatly simplifies the dynamics of the situation. We use this assumption safely as it has been shown that propellant rockets, used on many satellites, can impart over 1000 m/s of delta-v in the order of minutes.

Step 1: Tangential Orbits

One difficulty of circularizing an orbit, is knowing when you are in a good position to circularize. When an orbit becomes elliptical, not all parts of the path of the orbit are perpendicular to the direction of the force of gravity (pointed to the center of the Earth) as it is in a circle. This means that a thrust in that direction at that point would just further make the orbit more elliptical. What we want to look at are the points on the orbit where we are indeed perpendicular to the force, at which point a thrust in that current direction of the path, will move the satellite towards a circular orbit. Using the assumption of energy conservation, we can show this. Taking advantage of the assumption of two dimensionality, consider the total energy of a two dimensional, planar point moving about a central force in polar coordinates.

$$E_{total} = Kinetic\ Energy + Potential\ Energy$$

$$E_{total} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + U(r)$$

Where *U(r)* is the potential energy of the central force, and where *G* is the gravitational constant, *m* is the mass of the satellite, *M* is the mass of Earth, *r* is the radius of the orbit, and φ is the polar coordinate. We observe the first term is the kinetic energy from the satellite radially, the second term is the energy from the satellite revolving about the Earth, and the last term is the potential energy due to gravity. The potential energy due to gravity is given as $-\frac{GmM}{r}$. We observe that the quantity for orbit can be simplified by using the conserved quantity of angular momentum, *L,* to the form $\frac{L^2}{2mr^2}$, thus removing the dependence on φ and leaving us:

$$E_{total} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r}$$

This form has two major benefits. For one we now have dependence on only one variable. We also avoid using the polar variable, and thus do not need to worry about the use of π. We now solve for the change of radius.

$$\dot{r} = \sqrt{\frac{2}{m}}\sqrt{E_{total} + \frac{GmM}{r} - \frac{L^2}{2mr^2}}$$

We wish to show that this equation can have the value of zero and in this way, we show that the radius is not changing and that there exists a point where the path of the point is moving perpendicular to the central force of gravity. We can show this when looking at the total energy equation. When we plug in zero for $\dot{r}$, we observe that the energy equation becomes:

$$E_{total} = \frac{L^2}{2mr^2} - \frac{GmM}{r}$$

Which contains only the terms of the circular motion energy equation. This shows that at this point we only have energy from circular motion and thus are moving such that there is no radial movement, and that we are moving perpendicular to the central force.

Now that we know what we are looking for in this model, we must now establish the initial conditions. We obviously want some sort of orbit. This means that we cannot send the satellite with a velocity directly at the Earth. We also do not want it to escape Earth, so we never want the magnitude of the kinetic energy to be more than the magnitude of the potential energy. In other words, our initial velocity should not exceed escape velocity given our initial distance. This also implies that our total energy should be less than zero as potential energy due to gravity is negative. We also know that angular momentum *L* is a conserved quantity equal to $r \times p$, where *p* is momentum.

For simplicity sake, we can start our position at a point along the positive x-axis, as we can simply rotate any reference frame such that is true, thus making the radius simply the x distance. For our model we can now just assume we start with some positive r. We now observe that velocity has both an x and y direction component. We also observe that the cross product's sign is determined by the direction of this velocity. Once again we argue that we can flip the reference frame such that the sign is positive or negative at will. For simplicity sake we assume we force it to be positive. We know it may not be zero as that would mean we do not have an orbit, but a particle moving along the x-axis (as it the radius is in the x-direction, thus there would need to be no y-direction initial movement). We also know that we have a maximum possible velocity, escape velocity. Thus we can place an upper bound on our initial angular momentum such that there is a strict, unreachable maximum of the radius (x-axis distance) multiplied by the escape velocity (in the worst case where the velocity is entirely in the y-direction). We find the escape velocity by finding where the velocity provides enough energy to overcome the potential energy binding the satellite to earth. This gives us an escape velocity of:

$$v_{escape} = \sqrt{\frac{2GM}{r}}$$

This bounds our angular momentum to be:

$$0 < L < mr\sqrt{\frac{2GM}{r}}$$

We now must consider invariants of this system. Two already established invariants of the model are the domains of both angular momentum and the total energy, as well as the fact that they are constant. However, this is not enough to prove this model. We consider the radial change equation once again:

$$\dot{r} = \sqrt{\frac{2}{m}}\sqrt{E_{total} + \frac{GmM}{r} - \frac{L^2}{2mr^2}}$$

We observe that under the second radical, we have the total energy summed with two other terms of the total energy. These terms appear with the opposite sign they have in the total energy equation. Using this, we can determine something very important about the part under the radical. We find that under the radical, we only have the radial kinetic energy, expressed without the use of $\dot{r}$. Looking at the expression for radial kinetic energy $\frac{1}{2}m\dot{r}^2$, we can see that this term must always be greater or equal to zero. This invariant helps us in two manners, it shows that we never have an imaginary answer for the radial movement, and it also provides an invariant that will let us show that there will always exist points where the velocity is perpendicular to the central force.

Unfortunately, KeYmaeraX does not support the use of existential modality, and the solution to this differential equation is very messy. I was not able to rigorously prove it using KeYmaeraX. We must rely on intuition of elliptic orbits to tell us that there does exist points where $\dot{r}$ is zero. Looking at the shape of an ellipse, we see that there is a maximal and minimal radius, where the radius goes from decreasing in length to increasing (or vice versa), and thus $\dot{r}$ must be zero.

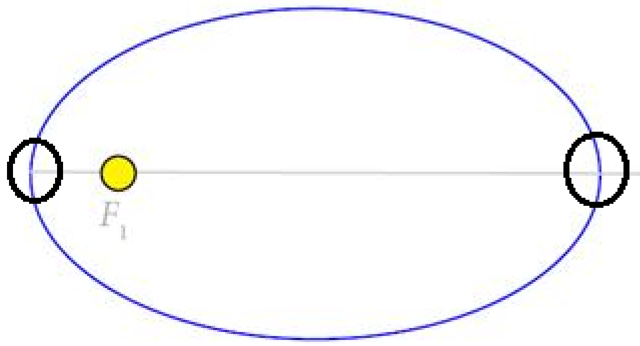

FIGURE 1 ELLIPTIC ORBIT WITH R_{MAX} AND R_{MIN} CIRCLED, WHERE THE CHANGE IN RADIUS LENGTH SHOULD BE ZERO

Step 2: Circularizing

We now have a model showing the existence of points in an elliptic orbit where the velocity is perpendicular to the central force. We now wish to show, that at these points, an impulsive thrust can circularize the orbit. This is where the assumption of thrust becomes important. Because we know we are at a point where we are moving perpendicular to the force (as in circular motion), we want to maintain exactly that. A circular orbit can be defined such that the centripetal force is equal to the central force. Mathematically it means:

$$\frac{GMm}{r} = \frac{mv^2}{r}$$

Where $\frac{mv^2}{r}$ is the centripetal force which points perpendicular to the velocity.

Once again we use the trick of being able to rotate reference frames to make this model easier, and place the critical points shown from step 1, on the x-axis, making the radius at that point simply the x direction distance. Following from step 1 we also observe that all the velocity is currently moving perpendicular to the central force that lies along x. This means that the velocity is entirely in the y-direction.

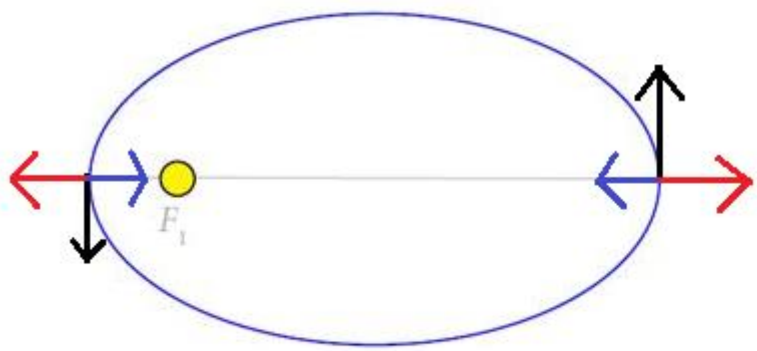

FIGURE 2: RED ARROWS SHOWING DIRECTION OF CENTRIPETAL FORCE, BLUE ARROWS SHOW DIRECTION OF GRAVITATIONAL FORCE, BLACK ARROWS SHOW DIRECTION OF VELOCITY

Using these assumptions, it is trivial to prove that finding a circular orbit at the points shown in part 1. Because we have the assumption of impulsive thrust, we can model the thrust as an instantaneous change in velocity, which is the same as simply assigning a new magnitude of velocity. Since the velocity is entirely in the y-direction, and the satellite lies on the x-axis, we know that the velocity is indeed perpendicular to the central force. Solving the force balance equation above, we find that we need to assign the velocity to be $\sqrt{\frac{GM}{r}}$.

Step 3: Putting it together

We now have models showing that within an elliptic orbit, there are key points where we can impart an impulsive thrust to circularize the orbit. We now wish to show that we can remain within a certain amount of "circle-ness" provided smart controls. This "circle-ness" can be described the eccentricity of the orbit, *e*, where the values of an elliptic orbit value from 0 to 1 non inclusive, where 0 is a perfect circle.

$$e = \sqrt{1 + \frac{2EL^2}{M(GMm)^2}}$$

For this model, we once again assume the knowledge of the outcomes of the previous models. We know that within and orbit there exists a point where we can circularize the orbit by imparting an impulsive thrust. With today's GPS technologies, it is fair to assume that a satellite can determine the portions of the orbit where $\dot{r}$ is zero and we have a turning point, and to determine velocity at that point. We can also assume that with today's knowledge of our atmosphere, and experience with space flight, we can determine a range of drag that the satellite may experience in a single orbit.

In order to look at the worst case eccentricity, we assume that the delta v from drag occurs at the maximum radius in the orbit each time, thus keeping *r* constant. At this point we know that $\dot{r}$ is zero. To prove this, we write the energy term then as:

$$E_{total} = \frac{L^2}{2mr^2} - \frac{GmM}{r}$$

Plugging this in to the eccentricity equation, and expanding out *L* we then find this equation for eccentricity:

$$e = \sqrt{1 + \frac{v^4m^2r^2}{G^2M^3m^2} - \frac{2rmv^2}{GM^2}}$$

We see that all the terms in the equation are constant except for *v*. Grouping the constants we find:

$$e = \sqrt{1 + av^4 - bv^2}$$

where *a* and *b* are positive constants. We see that as *v* approaches zero, *e* approaches 1 and thus the eccentricity moves towards its maximal value.

We must now establish our model. In our model we give the satellite control once an orbit (modeled as a loop) to determine whether or not a correcting thrust at this maximum radius is needed to maintain the desired circularity of the orbit. It will estimate by determining whether the maximal delta v possible due to drag, will lead to an overly eccentric orbit, and if it does, it will decide to circularize by establishing the velocity determined in part 2.

We must also establish initial conditions and invariants of the loop. We must of course start in a an already circularized orbit, or else all hope goes out the window to try recover it if we do not know how it is behaving at start. Given our control decisions, we also assume that on each repetition of the loop, that we maintain within the allowed eccentricity. We also must keep invariant that the maximal drag possible is always less than the current velocity. This is simply a fact of physics, as drag is opposite and proportional to your velocity. Along the same vein as this, the maximum drag delta v must be less than the velocity of a perfectly circular orbit.

My attempts at a proof did not succeed. I was able to show that the initial conditions did imply that we were safe, and that our loop invariants, did indeed imply themselves after each iteration of the loop, but could not show that the loop invariants after execution of the controls, would imply the safety conditions imposed. This is most likely due to a more complicated behavior of orbit eccentricity that I have not yet considered.

Conclusions:

The mechanics of space flight prove difficult to model. Even with my justified assumptions and trick of changing reference frames, I could only rigorously prove one my key steps towards a final overarching proof. The first model was unable to be rigorously proved as KeYmaeraX at this point does not support the proof of existentiality. However, we were able to argue by looking at the geometry of elliptic orbit, but for a fully justified safe system, should be proved more formally. Part 3, despite the simplifications from our assumptions, was too difficult for me to prove. I was unable to show that, despite our checking to see if the next orbit would be overly eccentric in order to circularize, the next orbit would indeed stay within the allowed eccentricity. One great difficulty of this task, was that the energy and angular momentum conservation assumptions could no longer be held, and those assumptions were key in the first two models.

Completing the two unfinished proofs are but a start to showing a safe control model for a satellite. Our assumptions right now assume that the drag happens instantaneously at one point. While it does provide a worst case scenario of how fast the eccentricity may increase, it does not speak to the truth of how a satellite behaves under drag. In reality, drag occurs at all points in the orbit, and should be modeled as a continuous force. This means propagating the orbit under the force of drag in a differential equation of sort for our model. It also means that our assumption of maintaining a specific maximal radius in the orbit would be invalid, and that our assumptions of conserved energy and angular momentum in the first two proofs are invalid.

Deliverables:

- Models representing all three parts
- Proof tactic for part 2 (it uses master auto solver as it mainly just real arithmetic and can be shown by algebra)
- Partial proof tactic for part 3

Works Cited:

Ley, Wilfried, Klaus Wittmann, and Willi Hallmann. Handbook of Space Technology. Chichester, U.K.: Wiley, 2009. Print.

Wertz, James Richard., and Wiley J. Larson. Space Mission Analysis and Design. El Segundo, CA: Microcosm, 1999. Print.