Differential Equations via Temporal Logic and Infinitesimals

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model evolution of state using
temporal logic over a time domain

model differential time using
time domain with infinitesimals

Temporal Logic Foundation

1. modal operator $\bigcirc_t \varphi$: φ holds after time t
2. two kinds of variable:
   - constant variables A – static over time
     e.g. $A = 5 \leftrightarrow \bigcirc_t (A = 5)$
   - differentiable variables x – change over time

$$
\begin{aligned}
e &::= x \mid A \mid 0 \mid 1 \mid \varepsilon \mid e + e \mid e \cdot e \\
\varphi &::= e = e \mid e \leq e \mid \varphi \wedge \varphi \mid \neg \varphi \mid \forall x . \varphi \mid \forall A . \varphi \mid \bigcirc_e \varphi
\end{aligned}
$$

Infinitesimals

two common approaches:

- Non-Standard Analysis
  - model-theoretic
  - non-constructive
  - *invertible* ($\varepsilon > 0$ very small, $1/\varepsilon$ very large)
- Smooth Infinitesimal Analysis
  - algebra / algebraic geometry
  - *nilpotent* ($\varepsilon > 0$, $\varepsilon^2 = 0$)

Ring of Dual Numbers

$$\mathbb{R}[\varepsilon] = \mathbb{R}[x]/(x^2) = \{a + b\varepsilon : a, b \in \mathbb{R}\}$$

1. $(a_1 + b_1\varepsilon) + (a_2 + b_2\varepsilon) = (a_1 + a_2) + (b_1 + b_2)\varepsilon$
2. $(a_1 + b_1\varepsilon)(a_2 + b_2\varepsilon) = a_1 a_2 + (a_1 b_2 + a_2 b_1)\varepsilon$
3. lexicographic ordering
4. $P(a + \varepsilon) = P(a) + P'(a) \cdot \varepsilon$

Syntax & Axiomatics

$$(a \approx b) :\equiv (a \cdot \varepsilon = b \cdot \varepsilon)$$

$$(x + y) + z = x + (y + z)$$
$$x + y = y + x$$
$$x \cdot y = y \cdot x$$
$$x + 0 = x$$
$$x \cdot 1 = x$$
$$\varepsilon^2 = 0$$

"$\leq$ is a total order"

$$x \leq y \rightarrow y + z \leq x + z$$
$$0 \leq x \wedge 0 \leq y \rightarrow 0 \leq x \cdot y$$
$$0 < \varepsilon$$

$$\exists y . x + y = 0$$
$$x \not\approx 0 \rightarrow \exists y . x \cdot y = 1$$
$$x \approx 0 \rightarrow \exists y . x = y \cdot \varepsilon$$
$$\exists A . A = x$$

Syntax & Axiomatics

$$\bigcirc_t(\varphi \wedge \psi) \leftrightarrow (\bigcirc_t\varphi) \wedge (\bigcirc_t\psi)$$
$$(\bigcirc_t\neg\varphi) \leftrightarrow \neg \bigcirc_t \varphi$$
$$(\bigcirc_t\forall A.\varphi) \leftrightarrow \forall A.(\bigcirc_t\varphi)$$
$$(\bigcirc_{t_1} \bigcirc_{t_2} \varphi) \leftrightarrow \exists A.(\bigcirc_{t_1} t_2 = A) \wedge \bigcirc_{t_1+A}\varphi$$

Syntax & Axiomatics

$$\exists\tilde{!}A.\forall X.\ x = X \rightarrow \forall B. \bigcirc_{B\cdot\varepsilon} (x = X + A \cdot B \cdot \varepsilon)$$

(Kock-Lawvere axiom)

$$\exists\tilde{!}X_f.\forall x.\ t \geq 0 \wedge x \approx x_i \rightarrow$$
$$(\forall 0 < A < t. \bigcirc_A (\forall X.x = X \rightarrow \bigcirc_\varepsilon x = X + e(X))) \rightarrow \bigcirc_t (x \approx X_f)$$

(uniqueness of solutions)

Semantics

$$x \mapsto D_0(x), D_1(x) : \mathbb{R} \to \mathbb{R}, D_0(x) \text{ differentiable}$$

$$\begin{aligned}
[\![x]\!]_u^{D;C} &= D_0(x)(u_0) + (D_1(x)(u_0) + (D_0(x))'(u_0) \cdot u_1) \cdot \varepsilon \\
[\![A]\!]_u^{D;C} &= C(A) \\
[\![0]\!]_u^{D;C} &= 0 \\
[\![1]\!]_u^{D;C} &= 1 \\
[\![\varepsilon]\!]_u^{D;C} &= \varepsilon \\
[\![e_1 + e_2]\!]_u^{D;C} &= [\![e_1]\!]_u^{D;C} + [\![e_2]\!]_u^{D;C} \\
[\![e_1 \cdot e_2]\!]_u^{D;C} &= [\![e_1]\!]_u^{D;C} \cdot [\![e_2]\!]_u^{D;C}
\end{aligned}$$

Differential Equations

$[x' = \theta(x)]\varphi$ becomes (almost)

$$\forall r \geq 0.(\forall 0 < t < r. \bigcirc_t (x' \textit{ is } \theta)) \to \bigcirc_r \varphi$$

where $x' \textit{ is } \theta$ is shorthand for

$$\forall X.x = X \to \bigcirc_\varepsilon x = X + \theta(X) \cdot \varepsilon$$

Why bother?

1. Good question...
2. Derivative facts for free: x' *is* θ is shorthand for

$$\forall X . x = X \rightarrow \bigcirc_{\varepsilon} x = X + \theta(X) \cdot \varepsilon$$

If x'_1 *is* θ_1 and x'_2 *is* θ_2 then

$$\bigcirc_{\varepsilon} x_1 x_2 = (X_1 + \theta_1(X_1) \cdot \varepsilon)(X_2 + \theta(X_2) \cdot \varepsilon)$$

$$(X_1 + \theta_1(X_1) \cdot \varepsilon)(X_2 + \theta(X_2) \cdot \varepsilon) = X_1 X_2 + (X_1 \theta(X_2) + X_2 \theta(X_1)) \cdot \varepsilon$$