02: Differential Equations & Domains

15-424: Foundations of Cyber-Physical Systems

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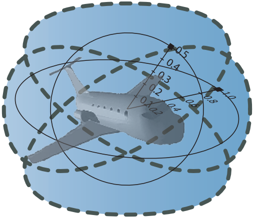

Outline

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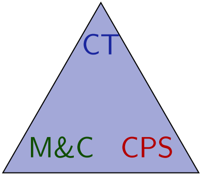
Learning Objectives

Differential Equations & Domains

semantics of differential equations
descriptive power of differential equations
syntax versus semantics



continuous dynamics
differential equations
evolution domains
first-order logic

continuous operational effects

Outline

Example (Vector field and one solution of a differential equation)

$$\begin{pmatrix} y'(t) = & f(t,y) \\ y(t_0) = & y_0 \end{pmatrix}$$

Intuition:

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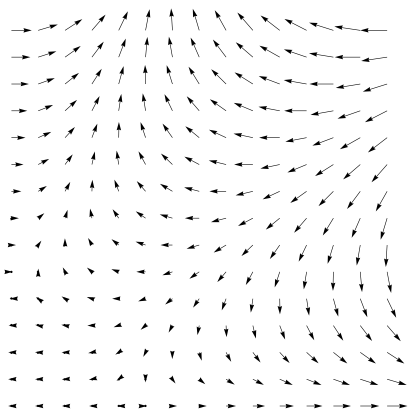

Intuition:

1. At each point in space, plot the value of $f(t,y)$ as a vector

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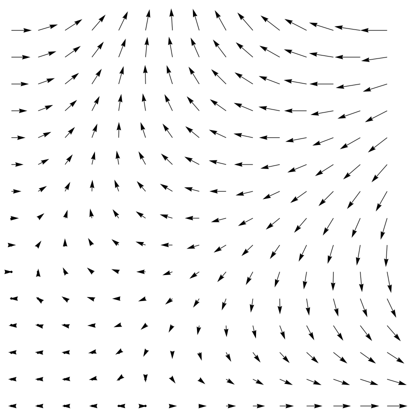

Intuition:

1. At each point in space, plot the value of $f(t, y)$ as a vector
2. Start at initial state y_0 at initial time t_0

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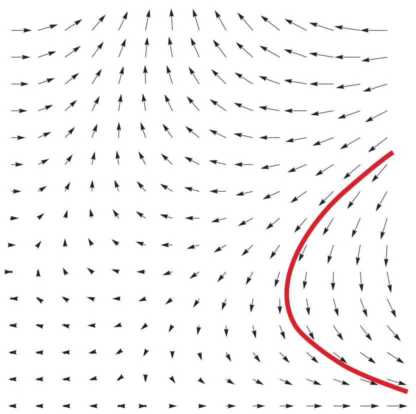

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1. At each point in space, plot the value of $f(t,y)$ as a vector
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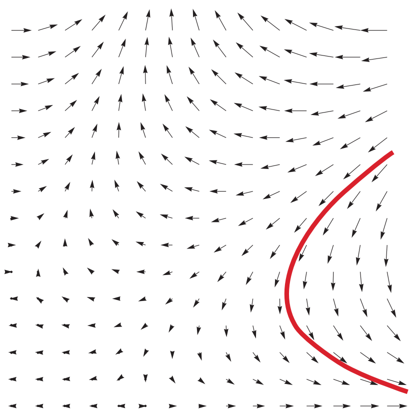

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↯ The diagram should show infinitely many vectors ...

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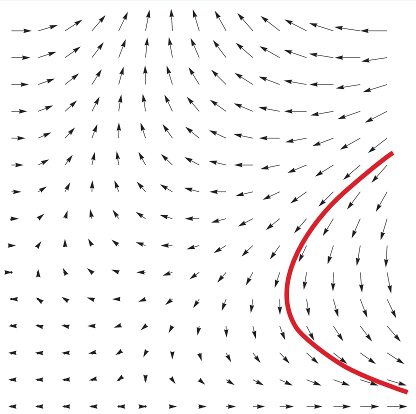

Your car's ODE $x' = v, v' = a$

Differential Equations as Models of Continuous Processes

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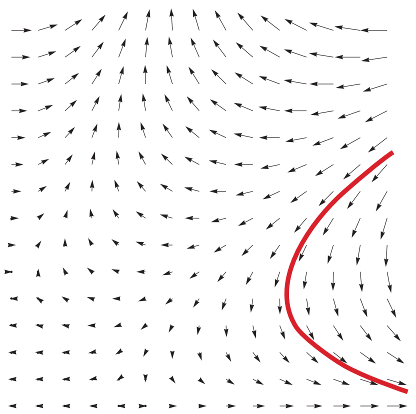

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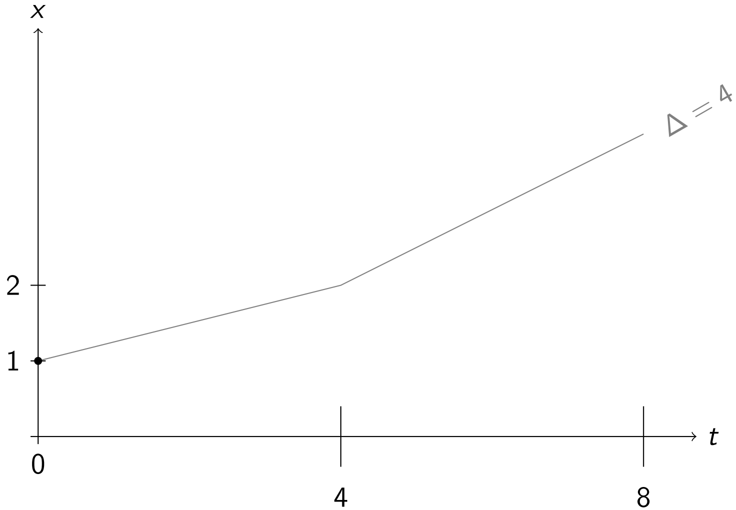
Well it's a wee bit more complicated

Intuition for Differential Equations



$$\begin{pmatrix} x'(t) = & \frac{1}{4}x(t) \\ x(0) = & 1 \end{pmatrix}$$

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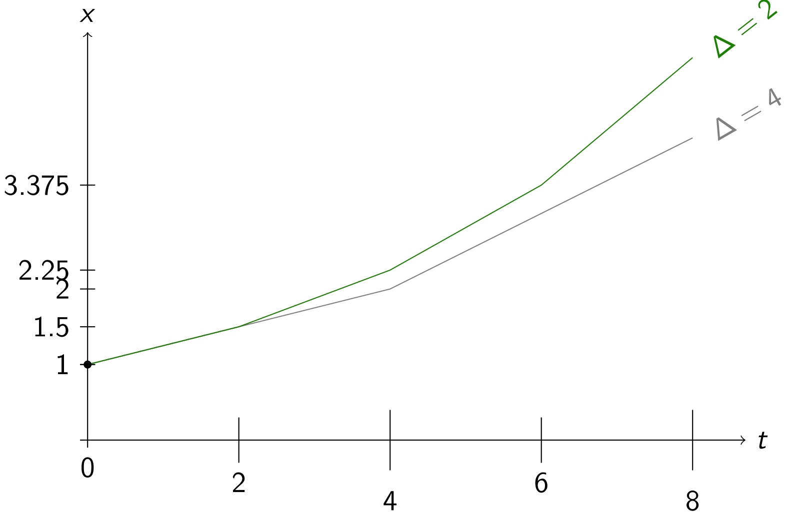

$$\begin{pmatrix} x'(t) = & \frac{1}{4}x(t) \\ x(0) = & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x(t+\Delta) := & x(t) + \frac{1}{4}x(t)\Delta \\ x(t_0) := & 1 \end{pmatrix}$$

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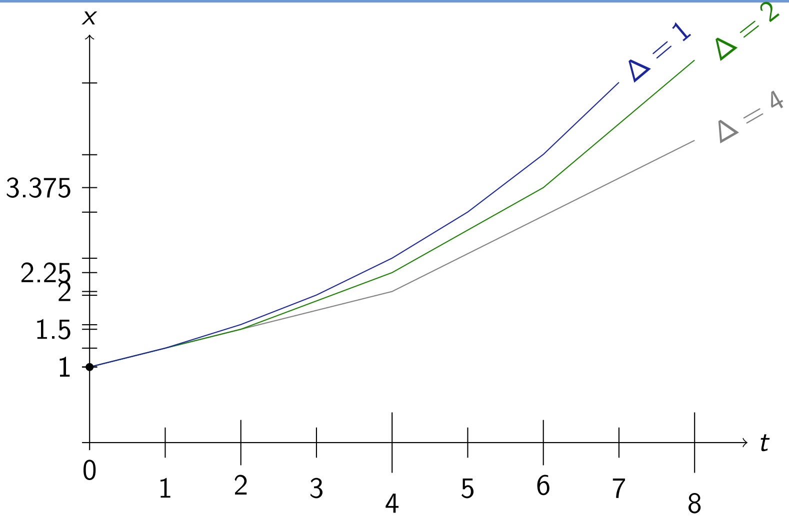

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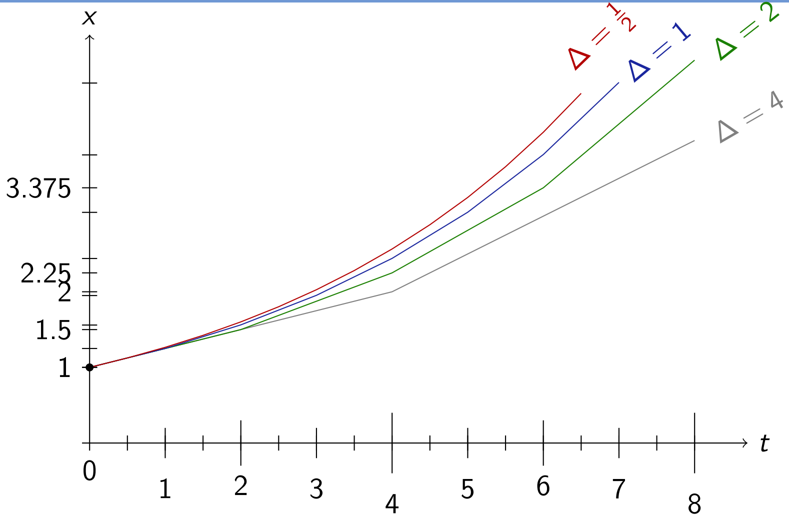

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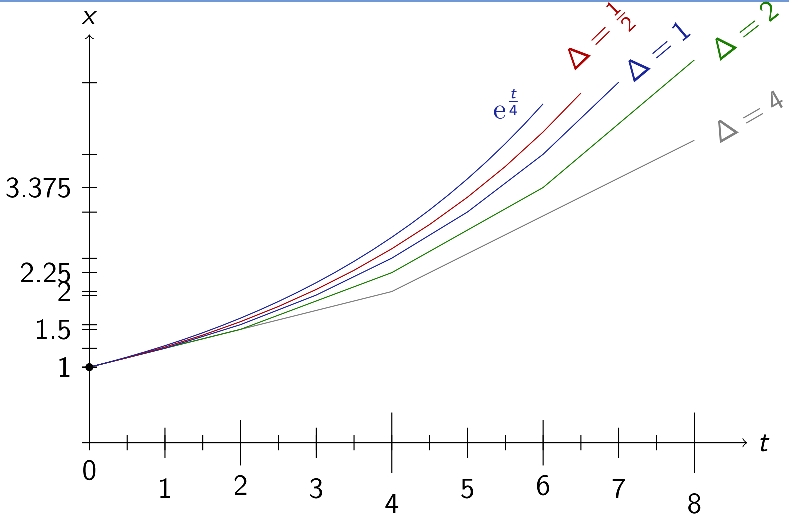

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Outline

The Meaning of Differential Equations

1. What exactly is a vector field?
2. What does it mean to describe directions of evolution at every point in space?
3. Could directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure, so we immediately want to get into the mood of consistently studying the behavior and exact meaning of all relevant aspects of CPS.

Differential Equations & Initial-Value Problems

Definition (Ordinary Differential Equation, ODE)

$f : D \to \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \to \mathbb{R}^n$ is *solution* of initial value problem (IVP)

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on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.
If f continuous, then Y continuously differentiable.

Outline

Example: A Constant Differential Equation

Example (Initial value problem)

$$\begin{pmatrix} x'(t) = & 5 \\ x(0) = & 2 \end{pmatrix}$$

has a solution

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Check by inserting solution into ODE+IVP.

$$\begin{pmatrix} (x(t))' = & (5t+2)' = 5 \\ x(0) = & 5 \cdot 0 + 2 = 2 \end{pmatrix}$$

□

Example: A Linear Differential Equation from before

Example (Initial value problem)

$$\left(\begin{array}{rl} x'(t) = & \frac{1}{4}x(t) \\ x(0) = & 1 \end{array}\right)$$

has a solution

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$$\left(\begin{array}{rl} x'(t) = & \frac{1}{4}x(t) \\ x(0) = & 1 \end{array}\right)$$

has a solution $x(t) = e^{\frac{t}{4}}$

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□

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$

ODE Examples

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$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
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$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$

ODE Examples

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$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1+2t} \ldots$

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$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$

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$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic

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$x' = x^2 + x^4$	???

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$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

Solutions more complicated than ODE

ODE	Solution
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$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

Takeaway Message

Descriptive power of differential equations

1. Solutions of differential equations can be much more involved than the differential equations themselves.
2. Representational and descriptive power of differential equations!
3. Simple differential equations can describe quite complicated physical processes.
4. Local description as the direction into which the system evolves.

Outline

Evolution Domain Constraints

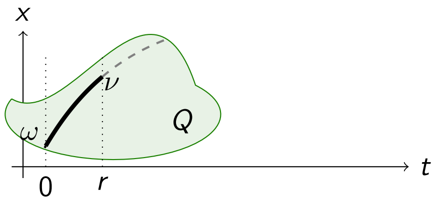
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A *differential equation* $x' = f(x)$ *with evolution domain* Q is denoted by

$$x' = f(x) \,\&\, Q$$

conjunctive notation (&) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain Q.



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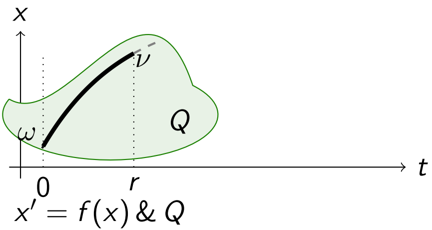

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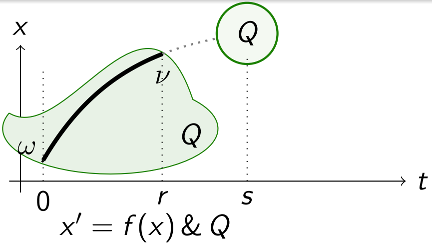
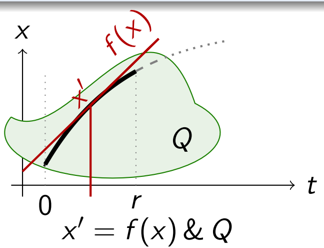


Semantics of ODE with Evolution Constraints

Definition (Semantics of differential equations)

A function $\varphi : [0, r] \to \mathcal{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \,\&\, Q$, written $\varphi \models x' = f(x) \land Q$, iff:

1. $\varphi(\zeta)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(\zeta)$ exists at for all times $0 \leq \zeta \leq r$
2. $\varphi(\zeta) \in [\![x' = f(x) \land Q]\!]$ for all times $0 \leq \zeta \leq r$
3. $\varphi(\zeta)(z) = \varphi(0)(z)$ for all variables $z \neq x$



Developed on the board:

1. First-order logic of real arithmetic
2. The meaning of terms
3. The meaning of formulas

See lecture notes for details [1].

André Platzer.
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