02: Differential Equations & Domains
15-424: Foundations of Cyber-Physical Systems

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Outline

1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
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1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
Learning Objectives
Differential Equations & Domains

semantics of differential equations
descriptive power of differential equations
syntax versus semantics

CT
M&C
CPS

continuous dynamics
differential equations
evolution domains
first-order logic

continuous operational effects
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
    y'(t) &= f(t, y) \\
    y(t_0) &= y_0
\end{align*}
\]

Intuition:
Example (Vector field and one solution of a differential equation)

\[
\begin{pmatrix}
y'(t) = f(t, y) \\
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Intuition:
1. At each point in space, plot the value of \( f(t, y) \) as a vector.
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1. At each point in space, plot the value of \( f(t, y) \) as a vector
2. Start at initial state \( y_0 \) at initial time \( t_0 \)
3. Follow the direction of the vector
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4. The diagram should show infinitely many vectors . . .
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Your car’s ODE: \( x' = v, v' = a \)
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Your car’s ODE \( x' = v, v' = a \)

Well it’s a wee bit more complicated
\[
\begin{align*}
\left(
\begin{array}{c}
x'(t) = \frac{1}{4}x(t) \\
x(0) = 1
\end{array}
\right)
\end{align*}
\]
Intuition for Differential Equations

\[
\begin{align*}
\frac{dx}{dt} &= \frac{1}{4}x(t) \\
x(0) &= 1
\end{align*}
\]

\[
\xrightarrow{\Delta = 4}
\]

\[
\begin{align*}
x(t + \Delta) &:= x(t) + \frac{1}{4}x(t)\Delta \\
x(t_0) &:= 1
\end{align*}
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The Meaning of Differential Equations

1. What exactly is a vector field?
2. What does it mean to describe directions of evolution at every point in space?
3. Could directions possibly contradict each other?

**Importance of meaning**

The physical impacts of CPSs do not leave much room for failure, so we immediately want to get into the mood of consistently studying the behavior and exact meaning of all relevant aspects of CPS.
Definition (Ordinary Differential Equation, ODE)

Let $f : D \to \mathbb{R}^n$ be defined on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \to \mathbb{R}^n$ is a solution of the initial value problem (IVP)

\[
\begin{align*}
  y'(t) &= f(t, y) \\
  y(t_0) &= y_0
\end{align*}
\]

on interval $I \subseteq \mathbb{R}$, if for all times $t \in I$, $Y(t)$ is defined and $Y'(t) = f(t, Y(t))$.
Definition (Ordinary Differential Equation, ODE)

\( f : D \rightarrow \mathbb{R}^n \) on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) (i.e., open connected). Then \( Y : I \rightarrow \mathbb{R}^n \) is solution of initial value problem (IVP)

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    y'(t) &= f(t, y) \\
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on interval \( I \subseteq \mathbb{R} \), iff, for all times \( t \in I \),

1. defined \( (t, Y(t)) \in D \)
Definition (Ordinary Differential Equation, ODE)

Each function $f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is solution of initial value problem (IVP)

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\begin{align*}
    y'(t) &= f(t, y) \\
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on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

1. defined $(t, Y(t)) \in D$
2. time-derivative $Y'(t)$ exists and $Y'(t) = f(t, Y(t))$. 

If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$. If $f$ continuous, then $Y$ continuously differentiable.
Definition (Ordinary Differential Equation, ODE)

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3. initial value \( Y(t_0) = y_0 \)
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Example: A Constant Differential Equation

Example (Initial value problem)

\[
\begin{pmatrix}
    x'(t) = 5 \\
    x(0) = 2
\end{pmatrix}
\]

has a solution

Check by inserting solution into ODE+IVP.

\[
\begin{align*}
    (x(t))' &= (5t + 2)' \\
    x(0) &= 5 \cdot 0 + 2 = 2
\end{align*}
\]
Example (Initial value problem)

\[
\begin{pmatrix}
  x'(t) &=& 5 \\
  x(0) &=& 2
\end{pmatrix}
\]

has a solution \( x(t) = 5t + 2 \)
Example: A Constant Differential Equation

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  (x(t))' & = & (5t + 2)' = 5 \\
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\end{pmatrix}
\]
Example (Initial value problem)

\[
\begin{pmatrix}
  x'(t) = \frac{1}{4}x(t) \\
  x(0) = 1
\end{pmatrix}
\]

has a solution
Example: A Linear Differential Equation from before

Example (Initial value problem)

\[
\begin{pmatrix}
  x'(t) = \frac{1}{4}x(t) \\
  x(0) = 1
\end{pmatrix}
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has a solution \( x(t) = e^{\frac{t}{4}} \)
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x(0) &= e^{\frac{0}{4}} = 1
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### ODE Examples

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### ODE Examples

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Solutions of differential equations can be much more involved than the differential equations themselves.

Representational and descriptive power of differential equations!

Simple differential equations can describe quite complicated physical processes.

Local description as the direction into which the system evolves.
Outline

1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
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Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain $Q$ is denoted by

$$x' = f(x) \& Q$$

conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain $Q$. 

[Diagram showing a trajectory in the $x-t$ plane with $x'$ as the differential equation and $Q$ as the evolution domain.]
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\[x' = f(x) & Q\]
Definition (Semantics of differential equations)

A function \( \varphi : [0, r] \rightarrow S \) of some duration \( r \geq 0 \) satisfies the differential equation \( x' = f(x) \& Q \), written \( \varphi \models x' = f(x) \& Q \), iff:

1. \( \varphi(\zeta)(x') = \frac{d\varphi(t)(x)}{dt}(\zeta) \) exists at for all times \( 0 \leq \zeta \leq r \)
2. \( \varphi(\zeta) \in [x' = f(x) \& Q] \) for all times \( 0 \leq \zeta \leq r \)
3. \( \varphi(\zeta)(z) = \varphi(0)(z) \) for all variables \( z \neq x \)
Developed on the board:

1. First-order logic of real arithmetic
2. The meaning of terms
3. The meaning of formulas

See lecture notes for details [1].
André Platzer.
Foundations of cyber-physical systems.
URL: http://www.cs.cmu.edu/~aplatzer/course/fcps17.html.

André Platzer.
Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.
doi:10.1007/978-3-642-14509-4.