11: Differential Equations & Proofs

15-424: Foundations of Cyber-Physical Systems

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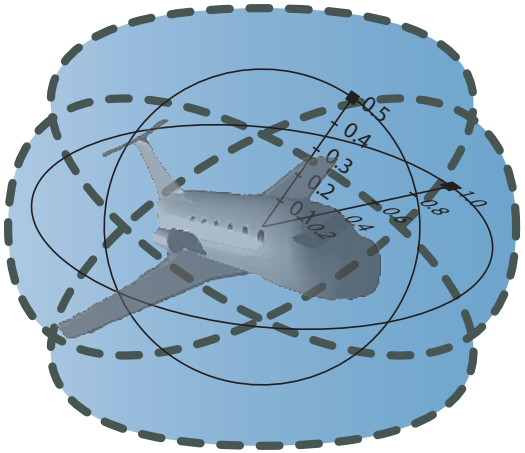

Outline

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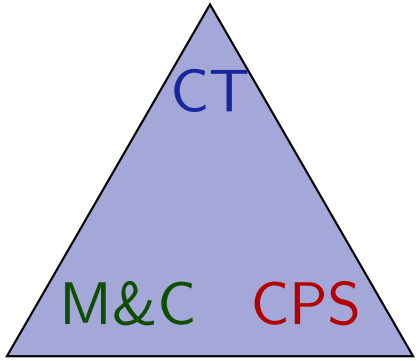
Learning Objectives

Differential Equations & Proofs

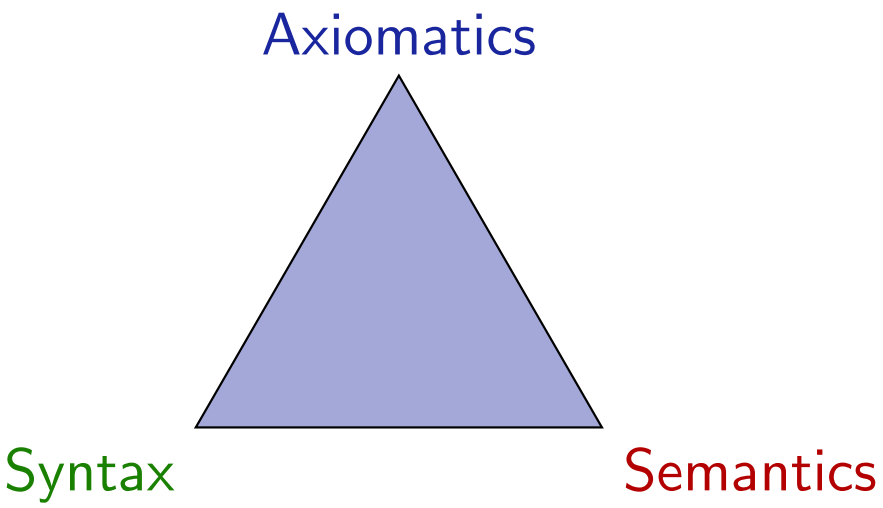
discrete vs. continuous analogy
rigorous reasoning about ODEs
beyond differential invariant terms
differential invariant formulas
cut principles for differential equations
axiomatization of ODEs
differential facet of logical trinity



understanding continuous dynamics
relate discrete+continuous

operational CPS effects
state changes along ODE

Differential Facet of Logical Trinity



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of A relate to semantics of $A \wedge B$, syntactically? If A is true, is $A \wedge B$ true, too? Conversely?

Outline

Differentials

Syntax

$$e ::= x \mid x' \mid c \mid e + k \mid e \cdot k \mid (e)'$$

Semantics

$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Axioms

$$(e+k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \text{for constants/numbers } c()$$

$$(x)' = x' \qquad \text{for variables } x \in \mathcal{V}$$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0), \varphi(r)) : \varphi \models x' = f(x) \wedge Q \text{ for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z) \qquad \ldots$$

Differential Substitution Lemmas

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r>0$, *then for all* $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \wedge Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e+k)' = (e)' + (k)'$$

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$+'$	$(e+k)' = (e)' + (k)'$
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Soundness: Proof of Derivations Lemma

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Proof.

$\omega[\![(e+k)']\!] =$

$\square$

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$$\omega[\![(e+k)']\!] = \sum_x \omega(x')\frac{\partial[\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x')\frac{\partial([\![e]\!]+[\![k]\!])}{\partial x}(\omega)$$

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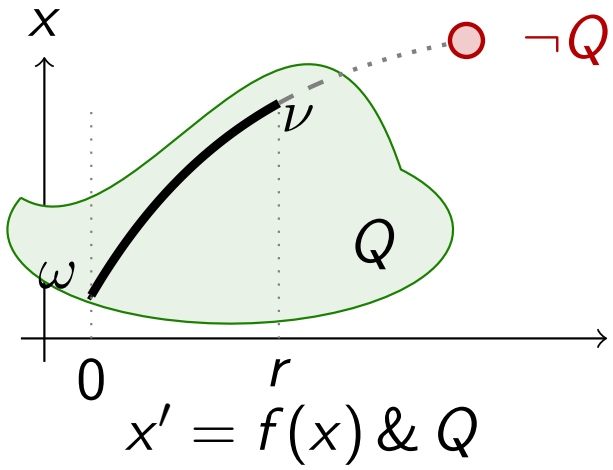
Differential Weakening



ODE

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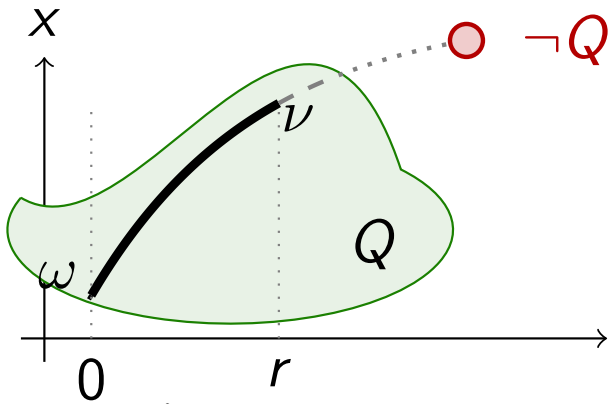

DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

$x' = f(x) \,\&\, Q$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0), \varphi(r)) : \varphi \models x' = f(x) \wedge Q \text{ for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

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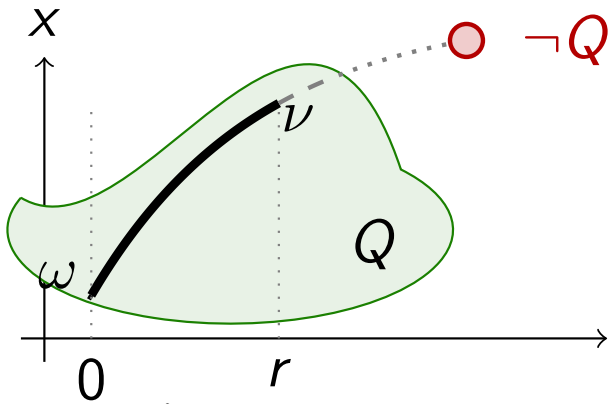
Differential equations cannot leave their domains.

Differential Weakening



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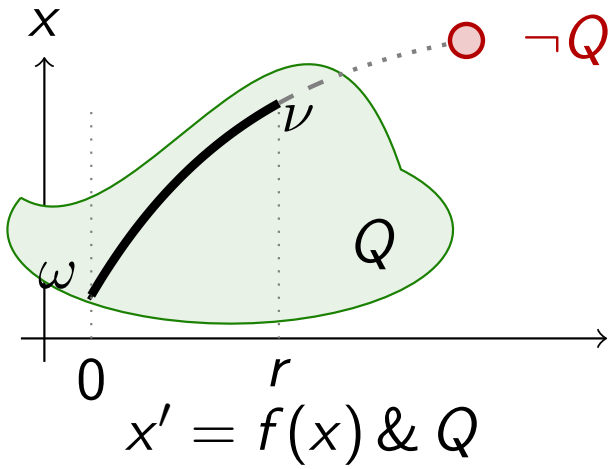
Example (Bouncing ball)

$$\text{DW}\ \frac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening



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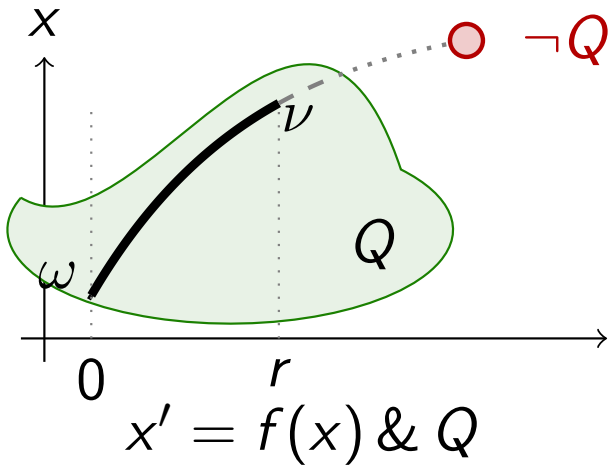

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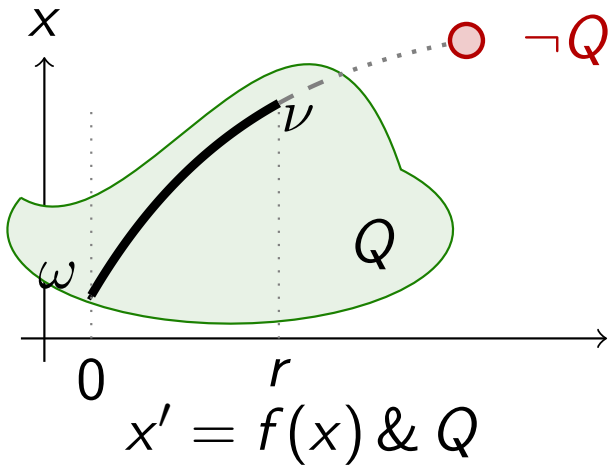

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Differential Weakening

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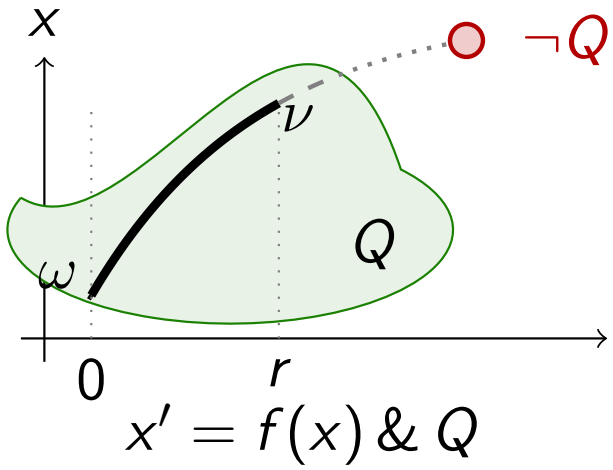
$$\text{dW}\ \frac{}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$



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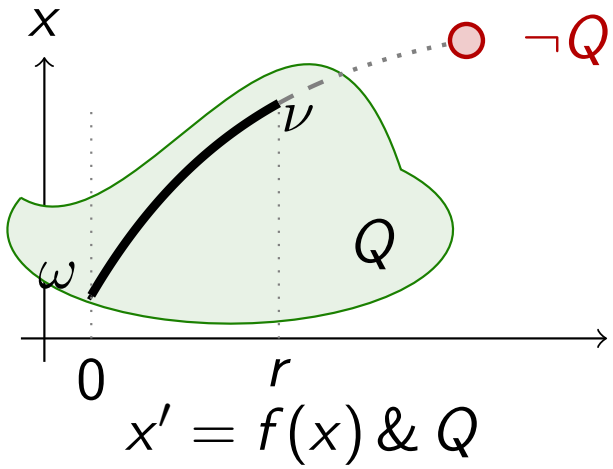

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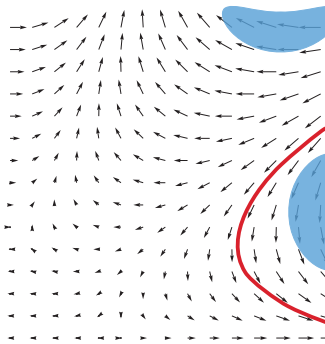
Differential Invariant Terms for Differential Equations



Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

$$\text{DI}\quad \big([x' = f(x)]\, e = 0 \leftrightarrow e = 0\big) \leftarrow [x' = f(x)]\,(e)' = 0$$

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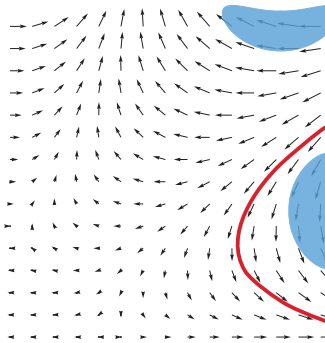

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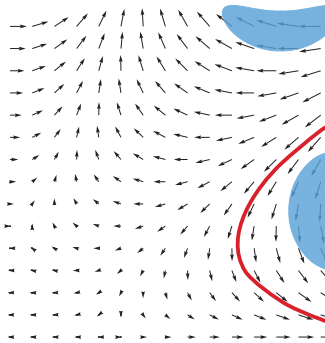

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Proof (dI is a derived rule).

$$\text{DI}\quad \frac{}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

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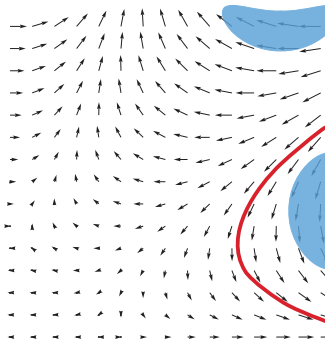

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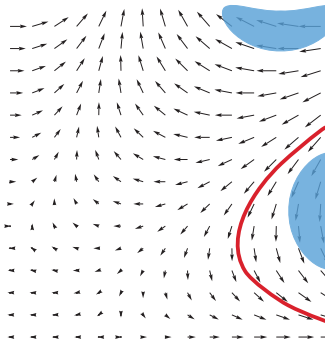

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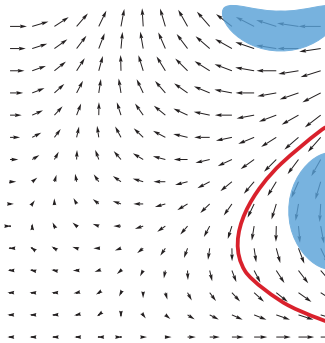

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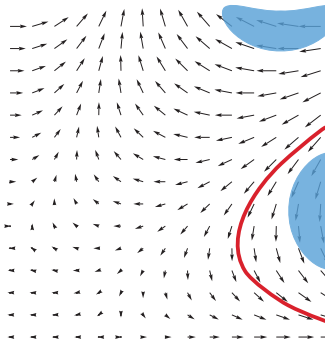
Proof (dI is a derived rule).

$$\text{DI}\;\dfrac{\text{DE}\;\dfrac{\text{DW}\;\dfrac{\text{G},\to\text{R}\;\dfrac{}{\vdash [x' = f(x) \,\&\, Q](Q \to [x' := f(x)](e)' = 0)}}{\vdash [x' = f(x) \,\&\, Q][x' := f(x)](e)' = 0}}{\vdash [x' = f(x) \,\&\, Q](e)' = 0}}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

□

Differential Invariant Terms for Differential Equations



Differential Invariant

$$\text{dI}\quad \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

$$\text{DI}\quad \big([x' = f(x) \,\&\, Q]\, e = 0 \leftrightarrow [?Q]e = 0\big) \leftarrow [x' = f(x) \,\&\, Q]\,(e)' = 0$$

$$\text{DE}\quad [x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$$

$$\text{DW}\quad [x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$$

Proof (dI is a derived rule).

$$\text{DI}\;\dfrac{\text{DE}\;\dfrac{\text{DW}\;\dfrac{\text{G},\to\text{R}\;\dfrac{Q \vdash [x' := f(x)](e)' = 0}{\vdash [x' = f(x) \,\&\, Q](Q \to [x' := f(x)](e)' = 0)}}{\vdash [x' = f(x) \,\&\, Q][x' := f(x)](e)' = 0}}{\vdash [x' = f(x) \,\&\, Q](e)' = 0}}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

$$\text{G}\;\frac{P}{[\alpha]P}$$

□

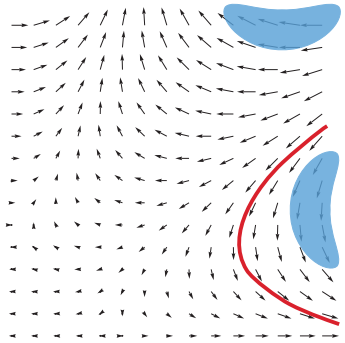
Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \textit{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{}{e = k \vdash [x' = f(x)]e = k}$$



Differential Invariant Equations

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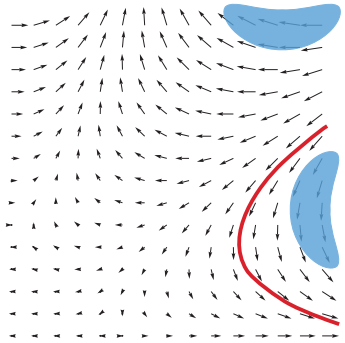
Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' = (k)'}{e = k \vdash [x' = f(x)]e = k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

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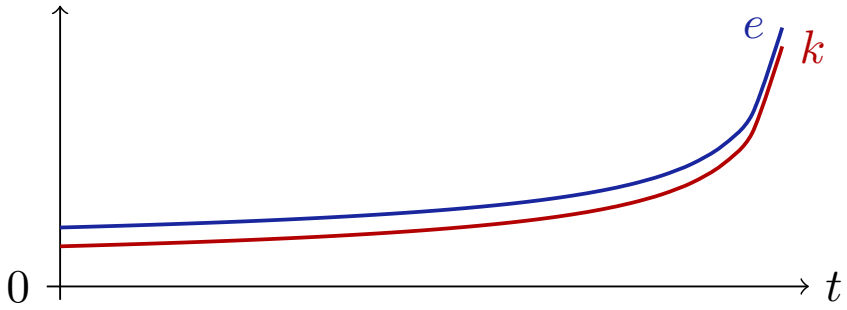


$$\text{DI} \quad \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

Proof (= rate of change from = initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

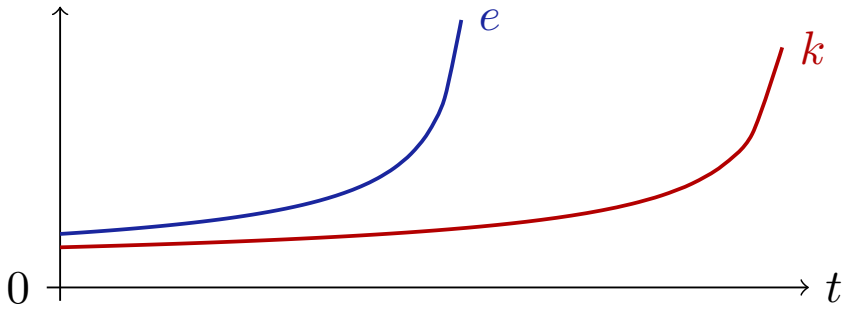
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \geq (k)'}{e \geq k \vdash [x' = f(x)]e \geq k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \geq k \leftrightarrow e \geq k\big) \leftarrow [x' = f(x)]\,(e)' \geq (k)'$$

Proof ($\geq$ rate of change from $\geq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \geq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

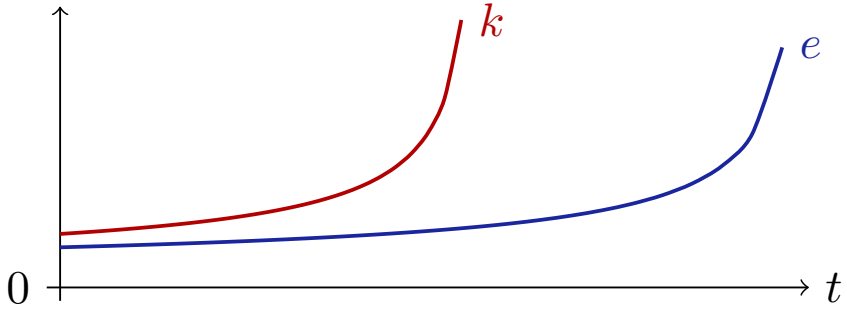
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \leq (k)'}{e \leq k \vdash [x' = f(x)]e \leq k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \leq k \leftrightarrow e \leq k\big) \leftarrow [x' = f(x)]\,(e)' \leq (k)'$$

Proof ($\leq$ rate of change from $\leq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \leq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

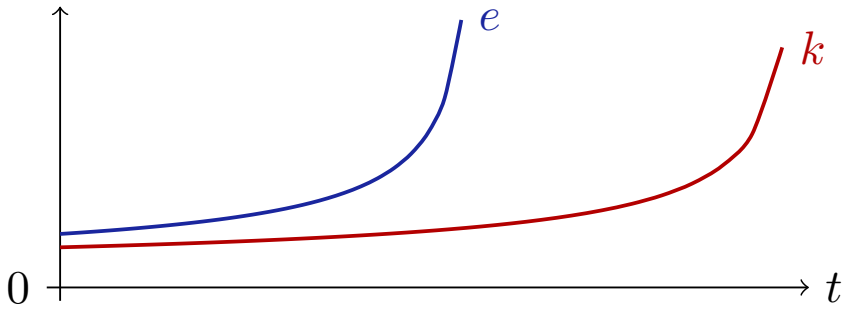
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' > (k)'}{e > k \vdash [x' = f(x)]e > k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' > (k)'$$

Proof (> rate of change from > initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] > \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

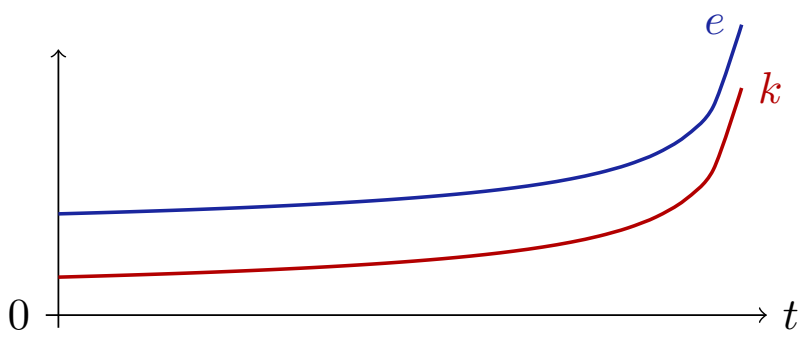
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \geq (k)'}{e > k \vdash [x' = f(x)]e > k}$$



DI $\big([x' = f(x)]\, e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' \geq (k)'$

Proof ($\geq$ rate of change from $>$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \geq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

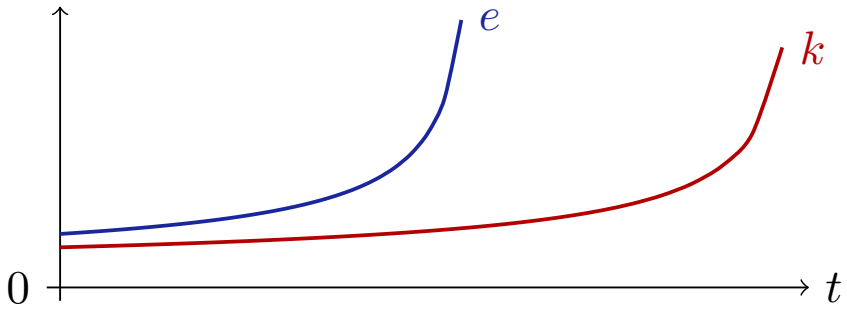
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

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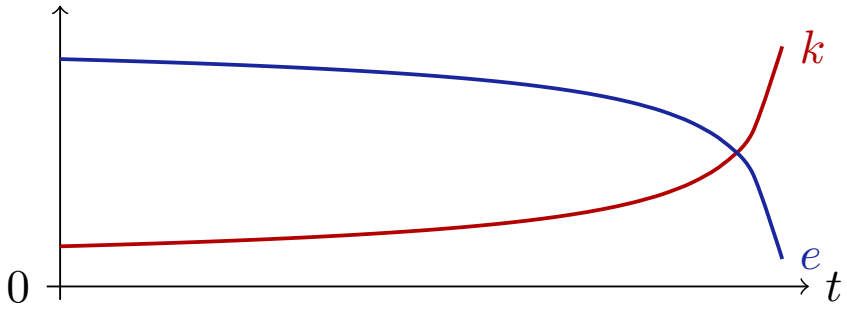

$$\text{DI} \quad \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

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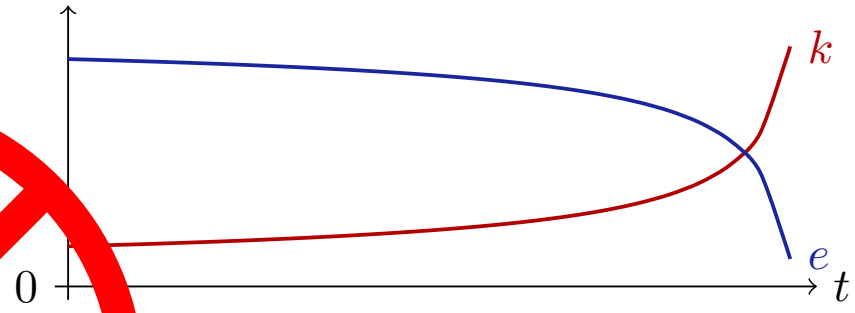

$$\text{DI} \quad ([x' = f(x)]\, e \neq k \leftrightarrow e \neq k) \leftarrow [x' = f(x)](e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

Differential Invariant Inequalities

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Differential Invariant

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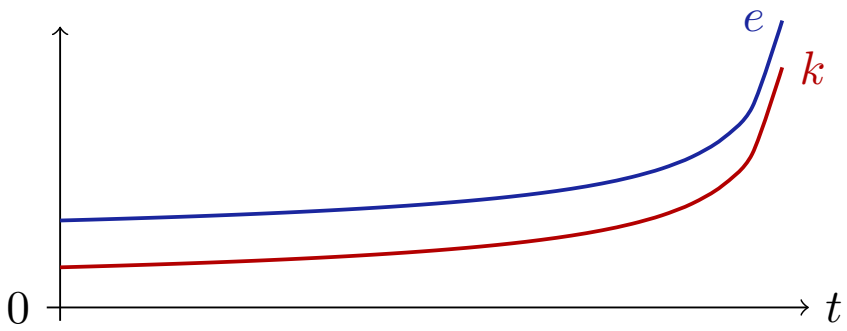


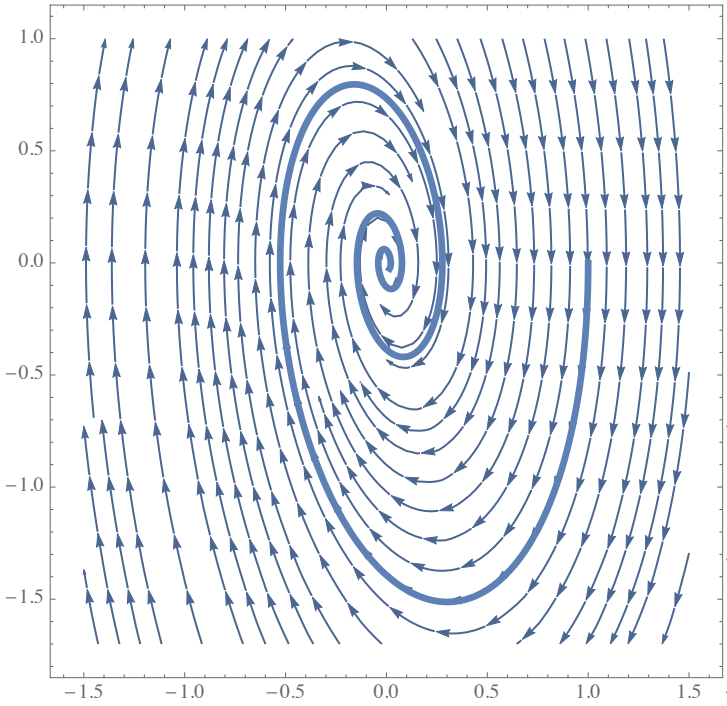
$$\text{DI} \quad \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

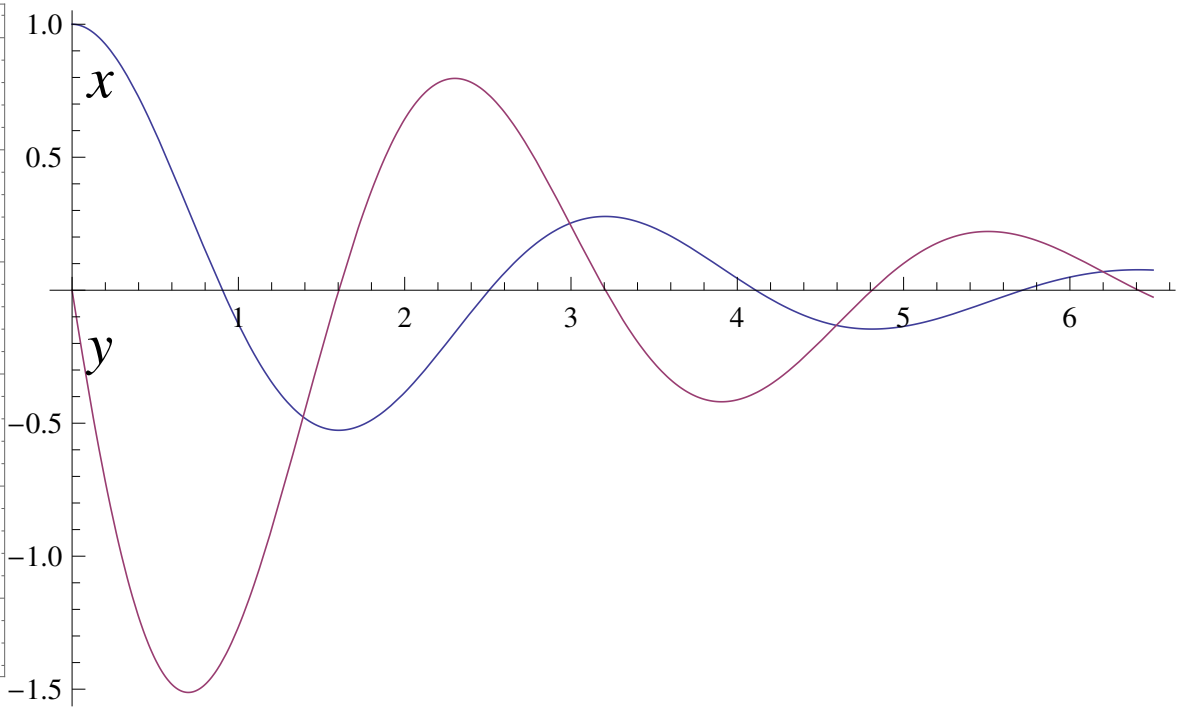
Proof (= rate of change from ≠ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

Example: Differential Invariant Inequalities

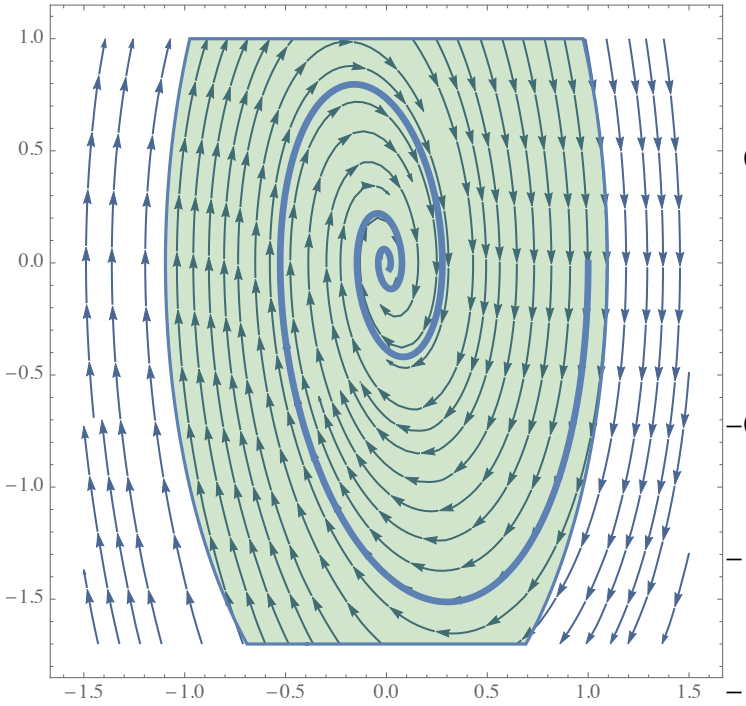
$$\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

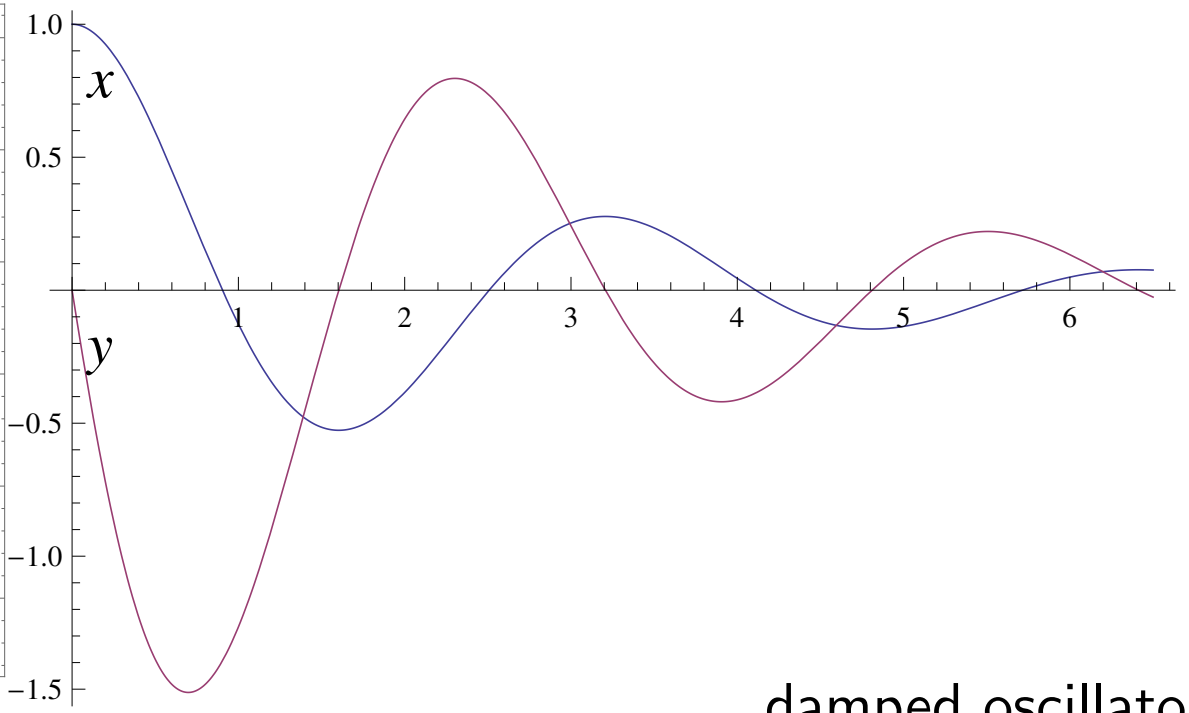




Example: Differential Invariant Inequalities: Oscillator

$$\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$



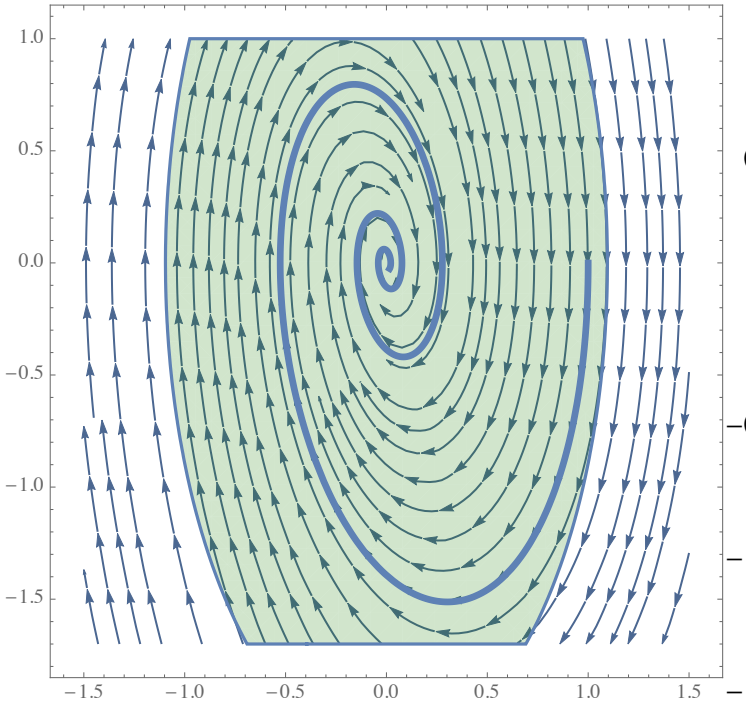


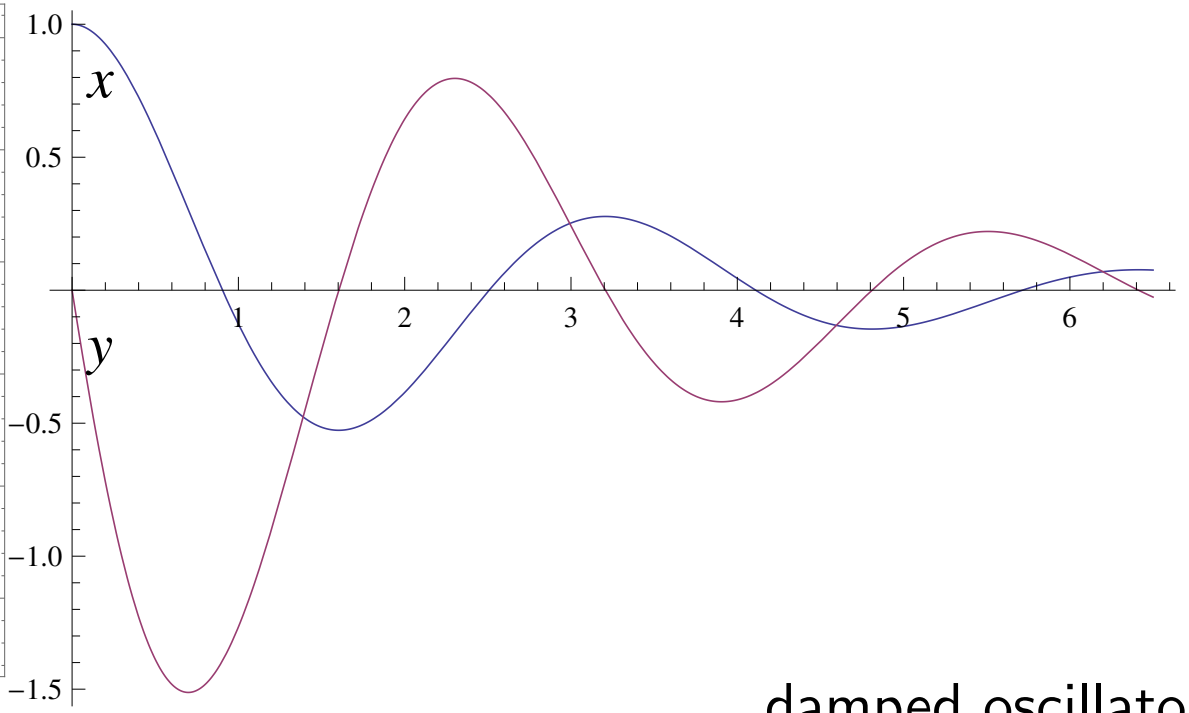
damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\omega{\geq}0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}{-}\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0$$

$$\omega^2 x^2 + y^2 {\leq} c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2 x^2 + y^2 {\leq} c^2$$





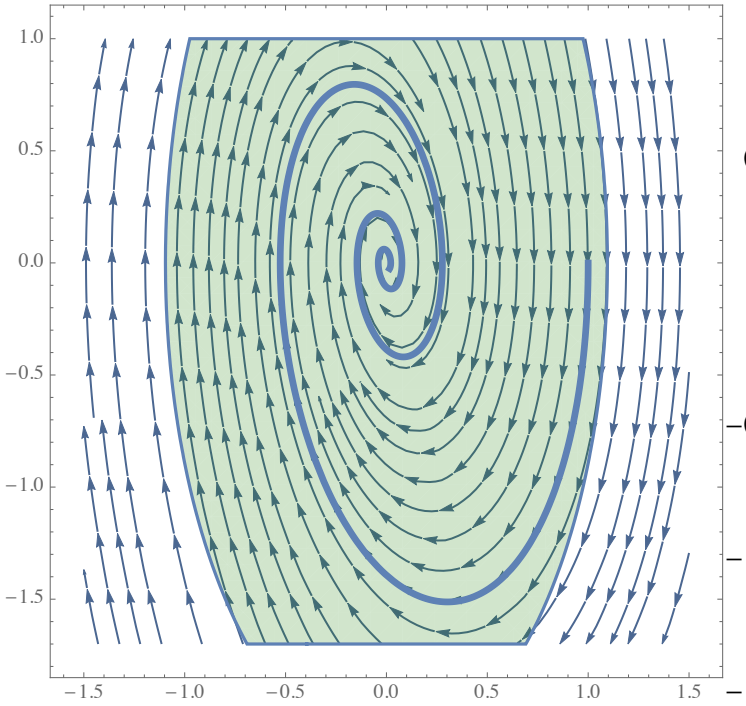
damped oscillator

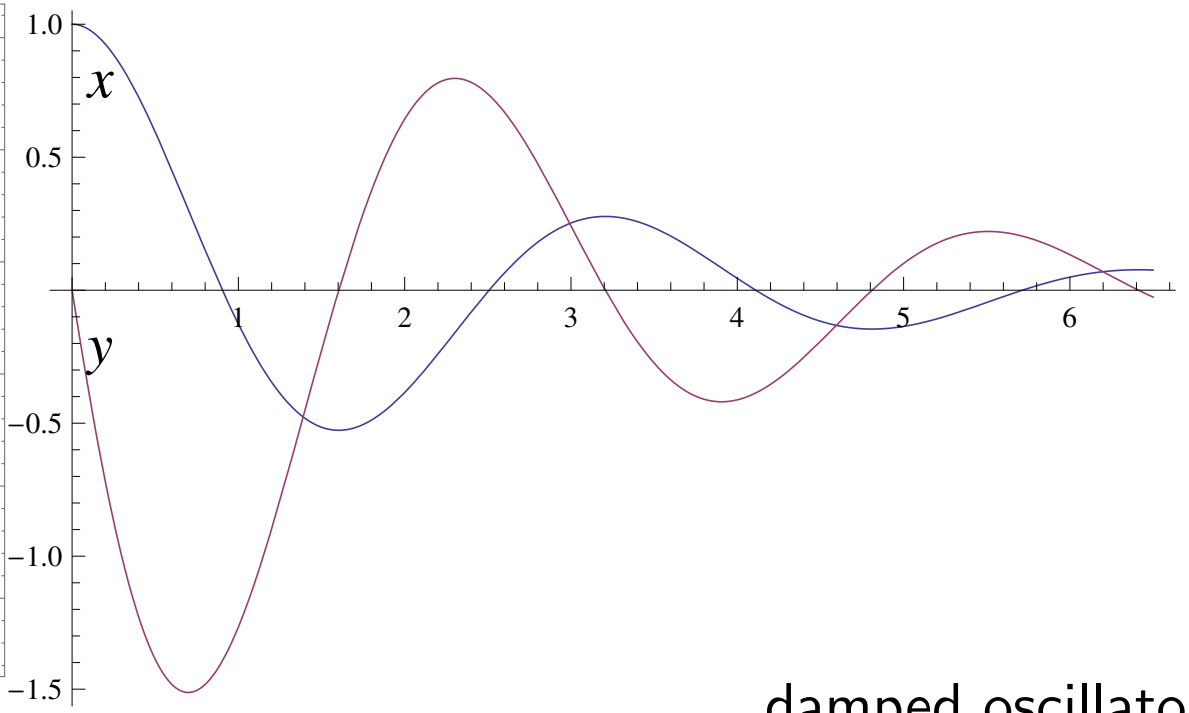
Example: Differential Invariant Inequalities: Oscillator

$$\omega \geq 0 \wedge d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega \geq 0 \wedge d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 xx' + 2yy' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega \geq 0 \wedge d \geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$





damped oscillator

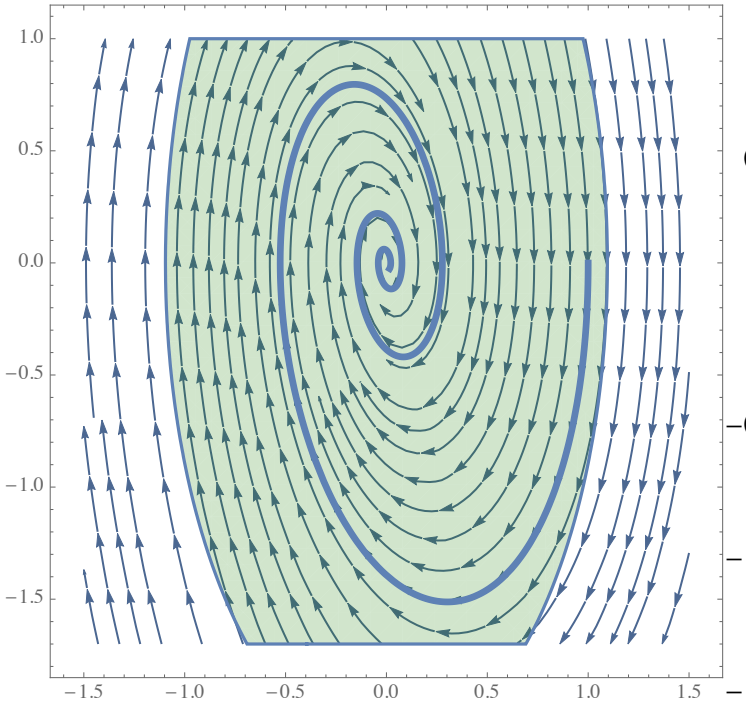
Example: Differential Invariant Inequalities: Oscillator

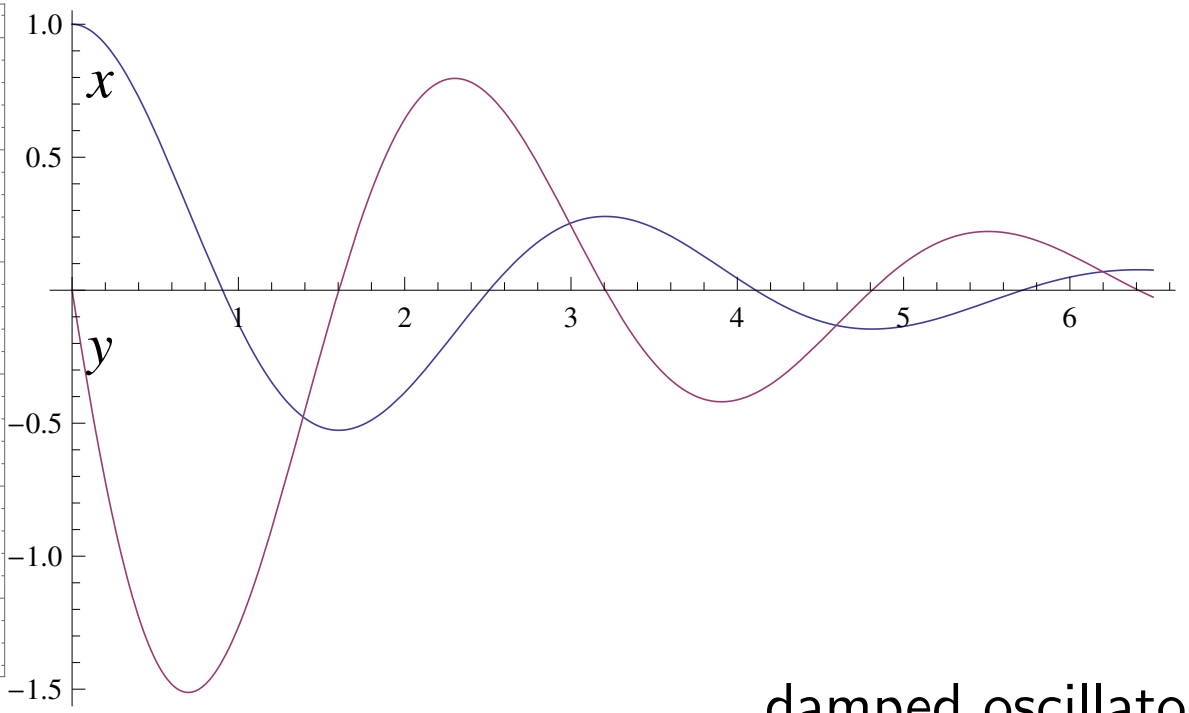
$$*$$

$$\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0$$

$$\omega{\geq}0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}{-}\omega^2x - 2d\omega y]\, 2\omega^2xx' + 2yy' \leq 0$$

$$\omega^2x^2{+}y^2{\leq}c^2 \vdash [x' = y, y' = -\omega^2x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2x^2{+}y^2{\leq}c^2$$





damped oscillator

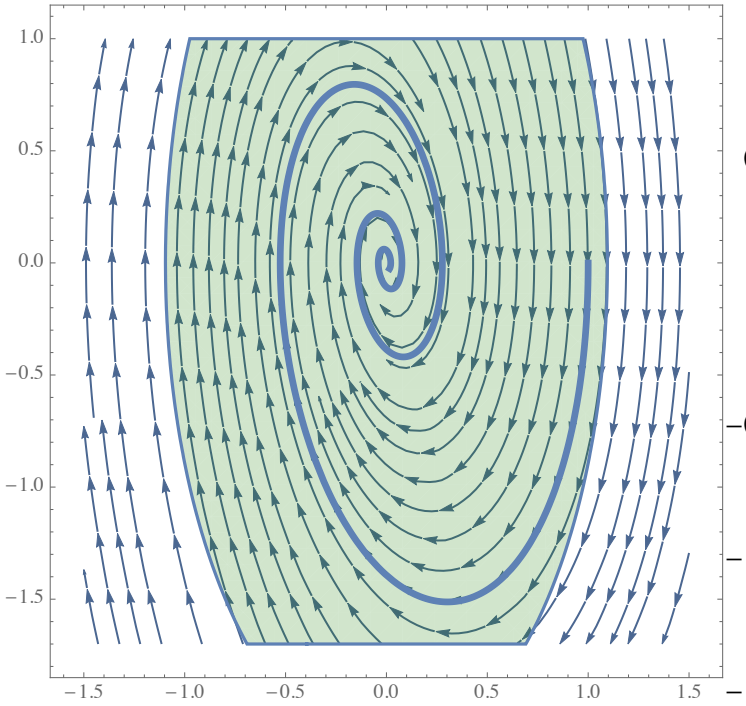
Example: Differential Invariant Inequalities: Oscillator

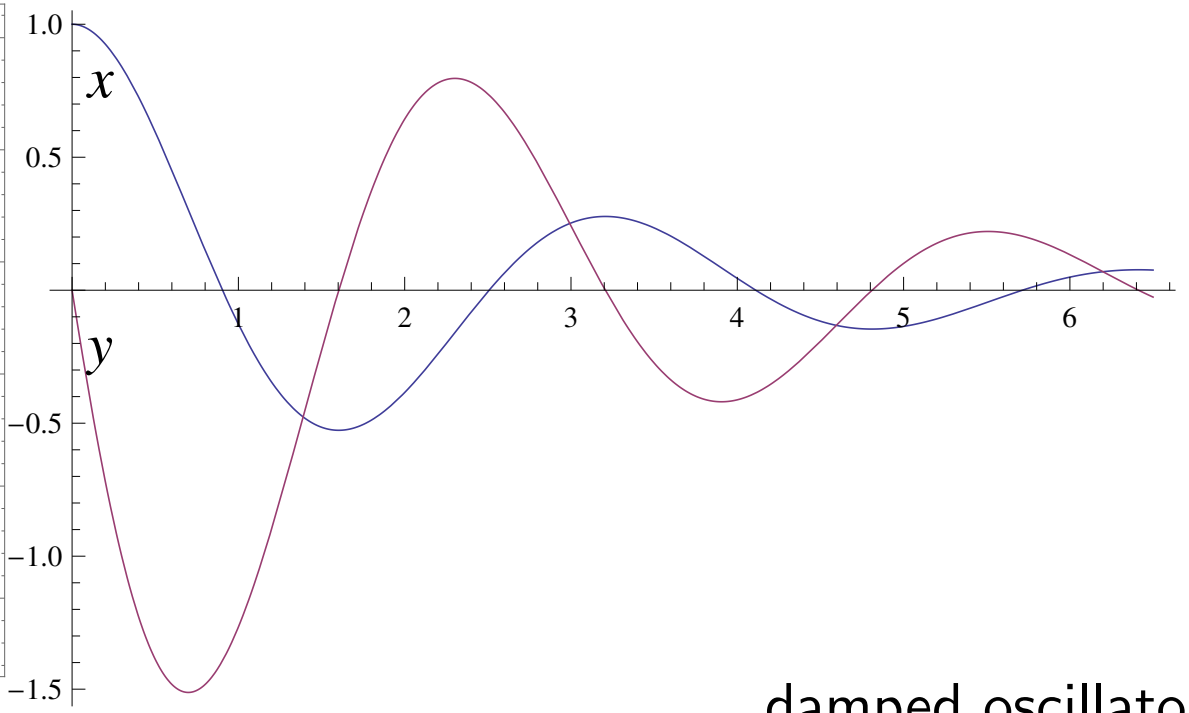
$$*$$

$$\omega \geq 0 \wedge d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega \geq 0 \wedge d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega \geq 0 \wedge d \geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$



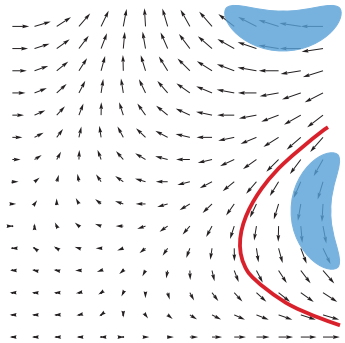


damped oscillator

Differential Invariant Conjunctions

Differential Invariant

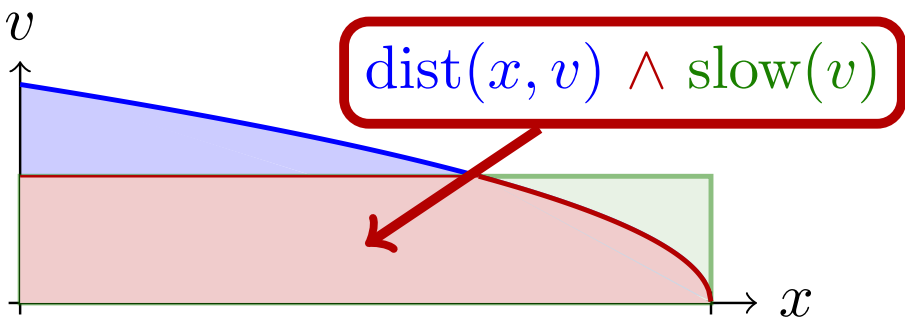
$$\text{dI}\quad \frac{}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$



Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \wedge (B)')}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$

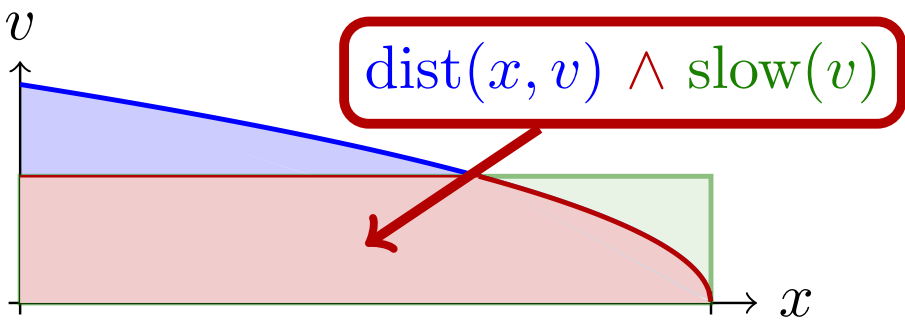


$$\text{DI}\quad \big([x' = f(x)](A \wedge B) \leftrightarrow (A \wedge B)\big) \leftarrow [x' = f(x))]((A)' \wedge (B)')$$

Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \wedge (B)')}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \wedge B) \leftrightarrow (A \wedge B)\big) \leftarrow [x' = f(x))]((A)' \wedge (B)')$$

Proof (separately).

$$[]\wedge,\text{WL}\ \dfrac{\text{DI}\ \dfrac{\vdash [x' = f(x)](A)'}{A \vdash [x' = f(x)]A} \qquad \text{DI}\ \dfrac{\vdash [x' = f(x)](B)'}{B \vdash [x' = f(x)]B}}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$

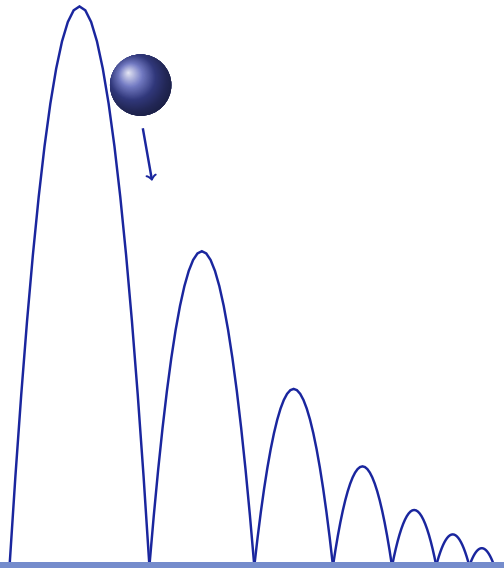
□

$$[]\wedge\quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

Quantum's Back for a Differential Invariant Proof

$$2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \land x{\geq}0)$$

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$[]\wedge \quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$

$$[]\wedge \frac{\dfrac{}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \wedge x{\geq}0)}$$

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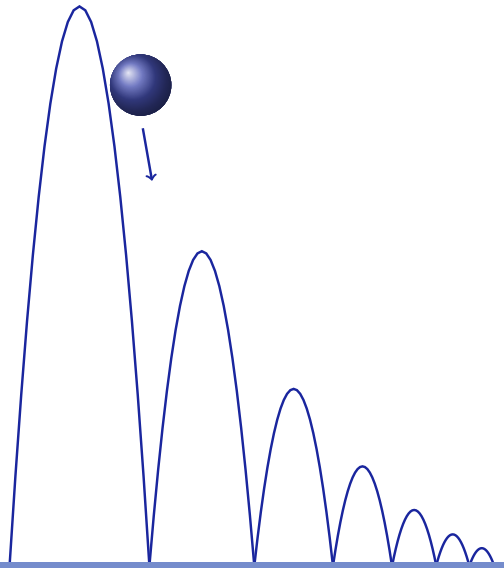

Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{}{x{\geq}0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2}\,\mathsf{dI} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \land x{\geq}0)}\,[]\land
$$

No solutions but still a proof.
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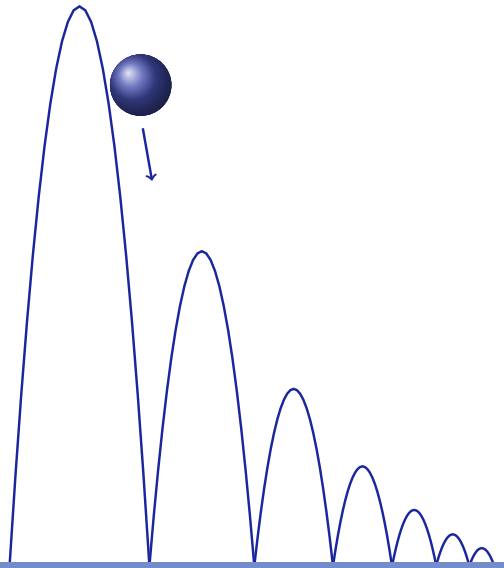
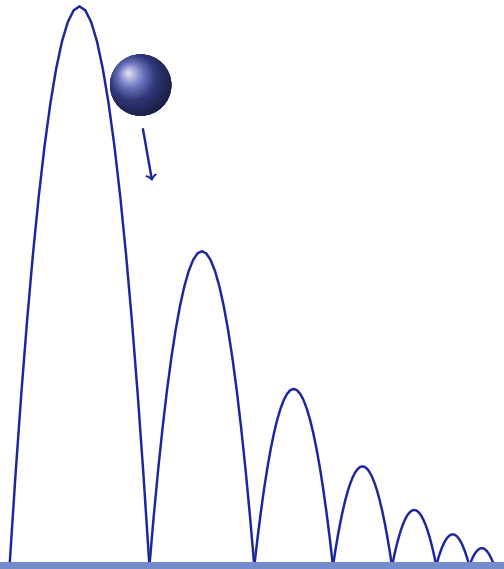


Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{x{\geq}0 \vdash 2gv = -2v(-g)}}{x{\geq}0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}\,[']}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2}\,\mathsf{dI} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \land x{\geq}0)}\,[]\land
$$

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.



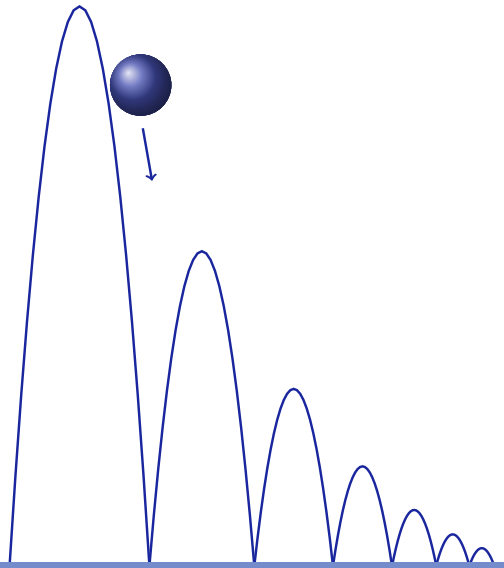
Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}\,[':=]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-g}\,\&\,x\geq 0]2gx{=}2gH{-}v^2}\,\mathsf{dI} \qquad \dfrac{}{\vdash [x''{=}{-g}\,\&\,x\geq 0]x\geq 0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g\,\&\,x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}\,[]\wedge
$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



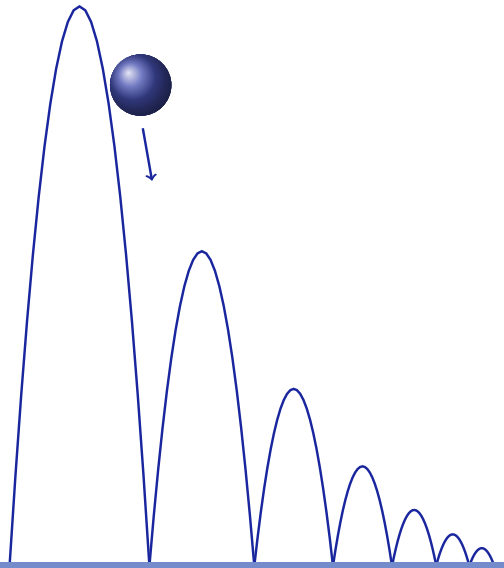
Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq 0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}\,[{}'{:=}]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x\geq 0]2gx{=}2gH{-}v^2}\,\mathrm{dI} \quad \dfrac{\dfrac{}{x\geq 0 \vdash x\geq 0}}{\vdash [x''{=}{-}g \,\&\, x\geq 0]x\geq 0}\,\mathrm{dW}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}\,[]\wedge
$$

No solutions but still a proof.

Simple proof with simple arithmetic.

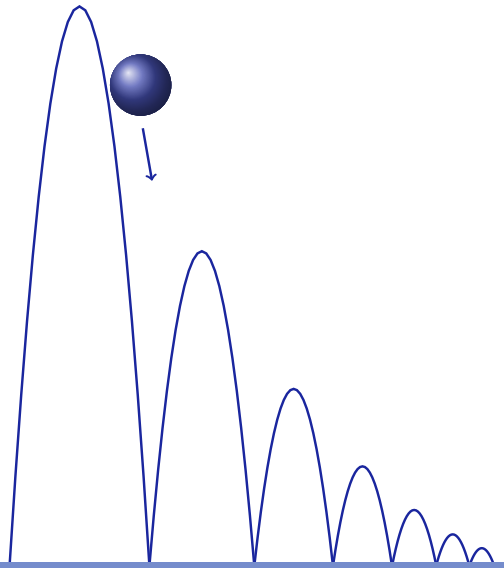
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq 0 \vdash [x'{:=}v][v'{:=}-g]2gx' = -2vv'}\,[':=]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x\geq 0]2gx{=}2gH{-}v^2}\,\mathrm{dI} \quad \dfrac{\dfrac{*}{x\geq 0 \vdash x\geq 0}\,\mathrm{id}}{\vdash [x''{=}{-}g \,\&\, x\geq 0]x\geq 0}\,\mathrm{dW}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx{=}2gH{-}v^2 \wedge x\geq 0)}\,[]\wedge
$$

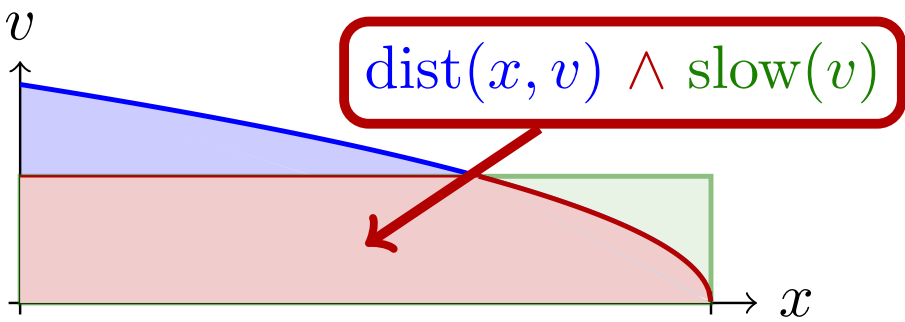
No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.



Differential Invariant Conjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \wedge (B)')}{A \wedge B \vdash [x' = f(x)](A \wedge B)}$$

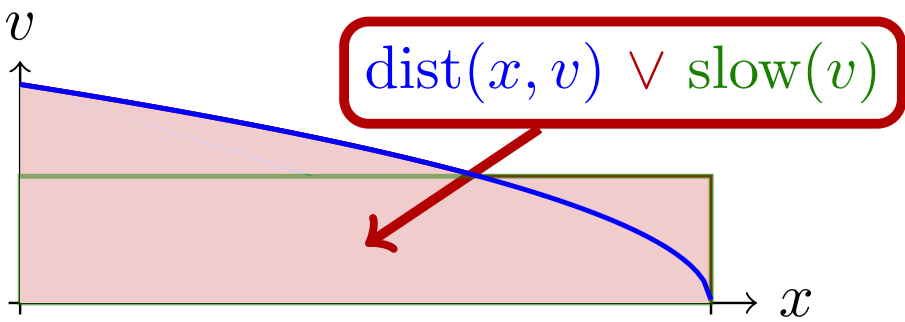


$$\text{DI}\quad \big([x' = f(x)](A \wedge B) \leftrightarrow (A \wedge B)\big) \leftarrow [x' = f(x))]((A)' \wedge (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \vee (B)')}{A \vee B \vdash [x' = f(x)](A \vee B)}$$



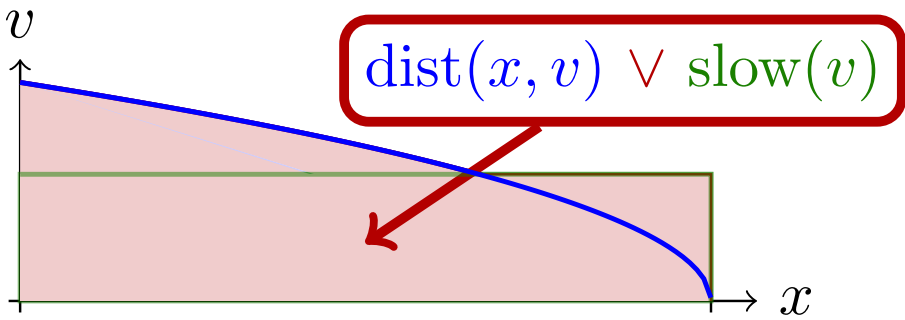
$$\text{DI}\quad \big([x' = f(x)](A \vee B) \leftrightarrow (A \vee B)\big) \leftarrow [x' = f(x))]((A)' \vee (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \vee (B)')}{A \vee B \vdash [x' = f(x)](A \vee B)}$$

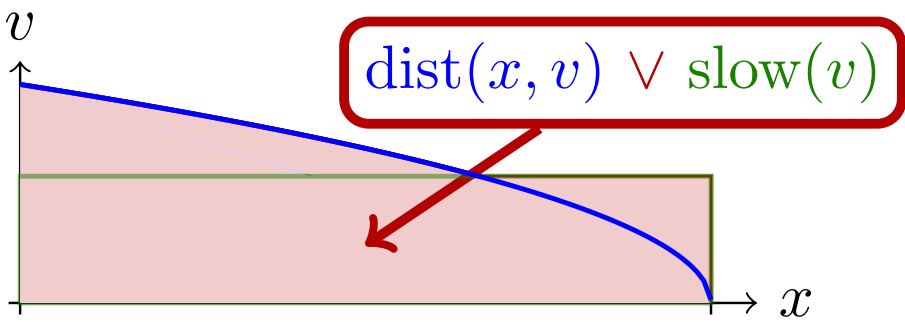
$$\text{DI}\quad \big([x' = f(x)](A \vee B) \leftrightarrow (A \vee B)\big) \vdash [x' = f(x))]((A)' \vee (B)')$$



Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

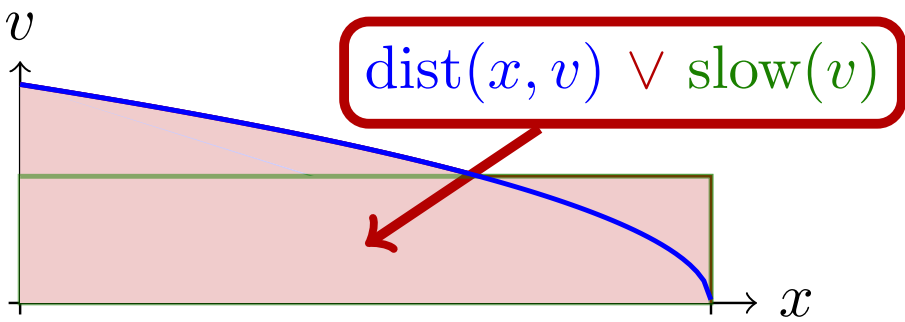


$$\text{DI}\quad \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \wedge (B)')}{A \vee B \vdash [x' = f(x)](A \vee B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \vee B) \leftrightarrow (A \vee B)\big) \leftarrow [x' = f(x))]((A)' \wedge (B)')$$

Proof (separately).

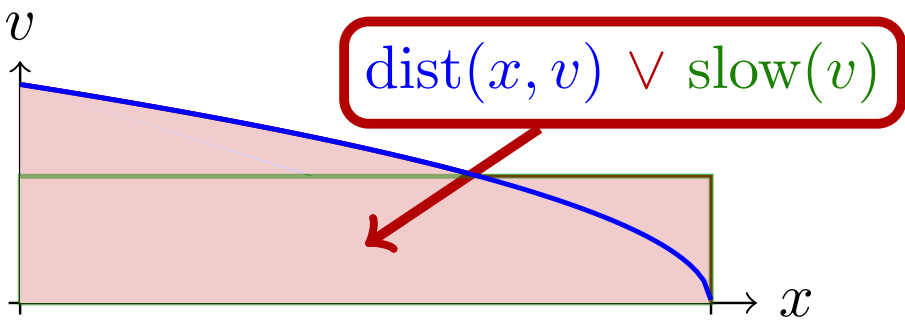
$$\vee\text{L}\,\dfrac{\text{MR}\,\dfrac{\dfrac{*}{A \vdash A \vee B}\quad \text{DI}\,\dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \vee B)}\qquad \text{MR}\,\dfrac{\dfrac{*}{B \vdash A \vee B}\quad \text{DI}\,\dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \vee B)}}{A \vee B \vdash [x' = f(x)](A \vee B)}$$

□

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \wedge (B)')}{A \vee B \vdash [x' = f(x)](A \vee B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \vee B) \leftrightarrow (A \vee B)\big) \leftarrow [x' = f(x))]((A)' \wedge (B)')$$

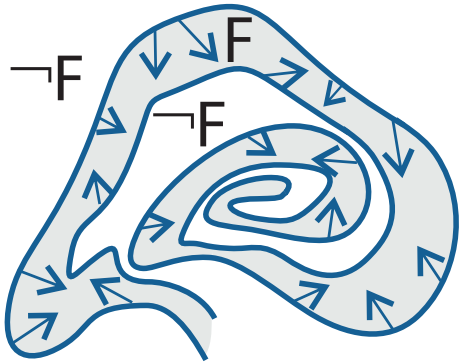
Proof (separately).

$$\vee\text{L}\,\dfrac{\text{MR}\,\dfrac{\dfrac{*}{A \vdash A \vee B}\quad \text{DI}\,\dfrac{\vdash [x'{=}f(x)](A)'}{A \vdash [x'{=}f(x)]A}}{A \vdash [x' = f(x)](A \vee B)}\qquad \text{MR}\,\dfrac{\dfrac{*}{B \vdash A \vee B}\quad \text{DI}\,\dfrac{\vdash [x'{=}f(x)](B)'}{B \vdash [x'{=}f(x)]B}}{B \vdash [x' = f(x)](A \vee B)}}{A \vee B \vdash [x' = f(x)](A \vee B)}$$

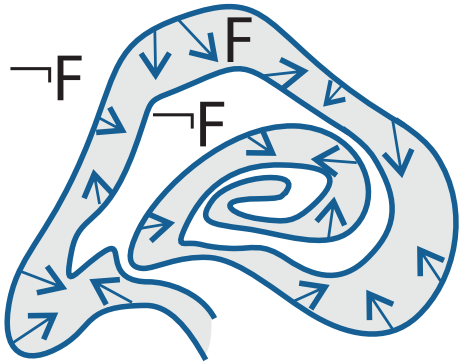
□

$$[]\wedge\quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

Assuming Differential Invariance



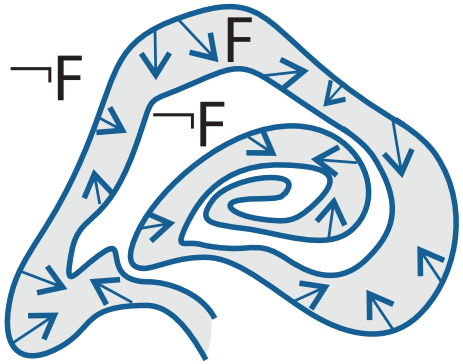
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



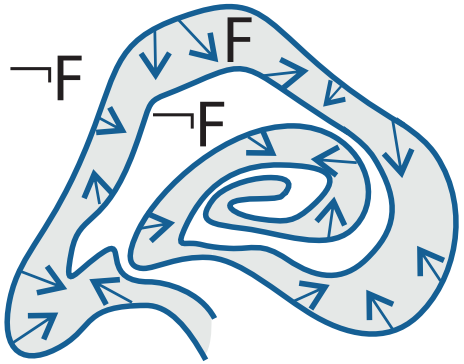
$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{loop} \quad \frac{F \vdash [\alpha]F}{F \vdash [\alpha^*]F}$$

Assuming Differential Invariance



$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

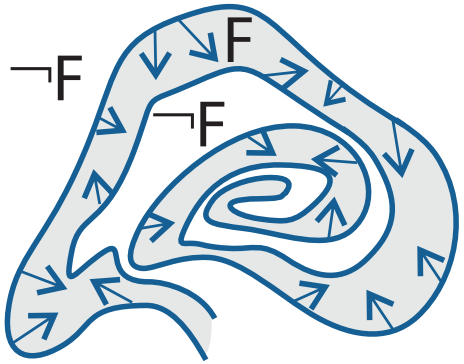


$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

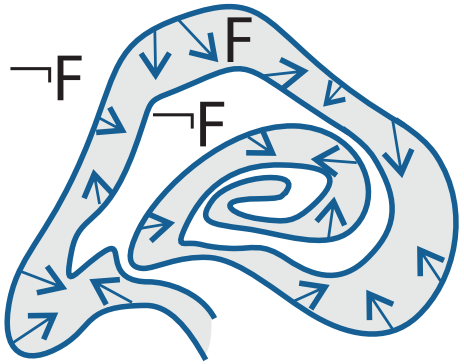
Example (Restrictions)

$$\frac{}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance



$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

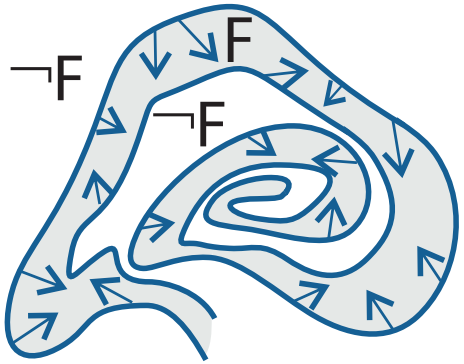


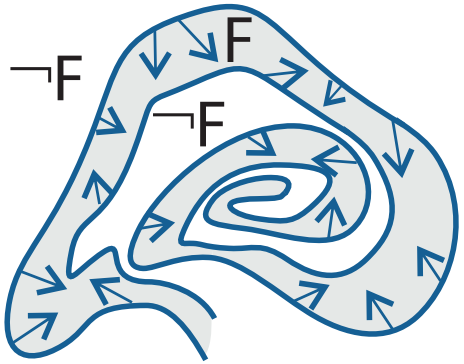
$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

$$\frac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance





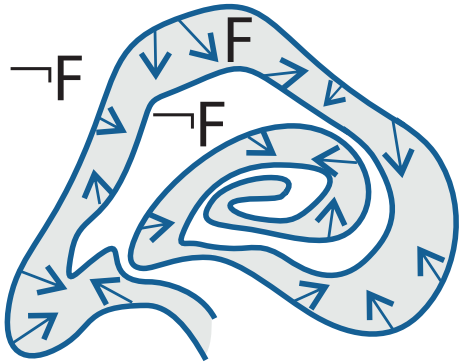
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

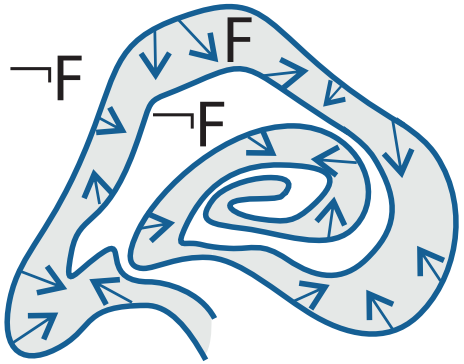
Example (Restrictions)

$$\frac{\dfrac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance



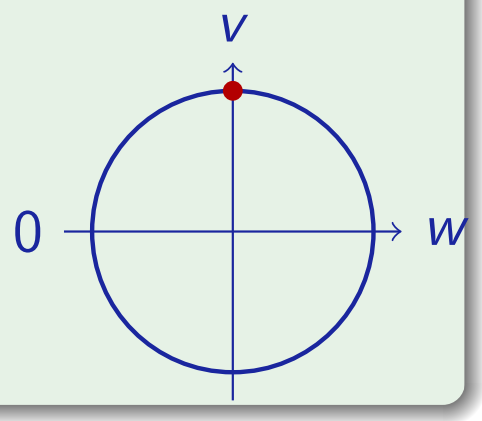
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



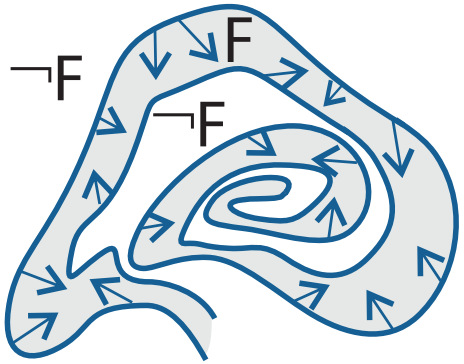
$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

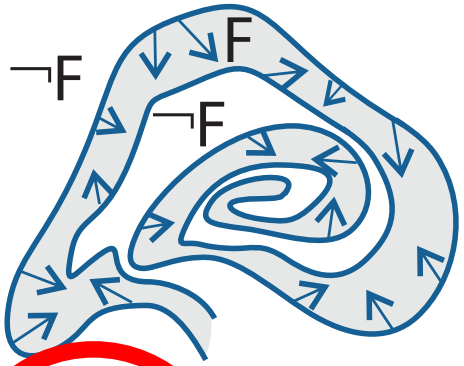
$$\frac{\dfrac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



Assuming Differential Invariance



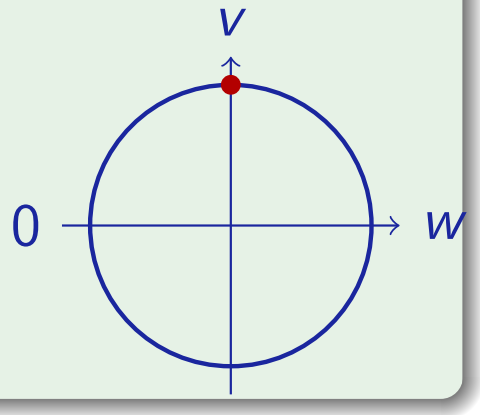
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



$$\frac{F \wedge Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions are unsound!)

$$\frac{\dfrac{\dfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}\,(\text{unsound})}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

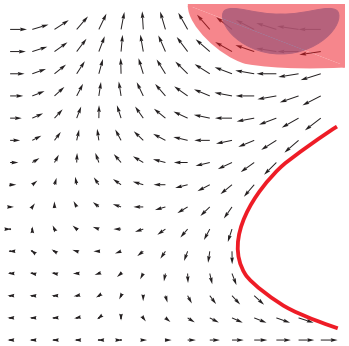


Outline

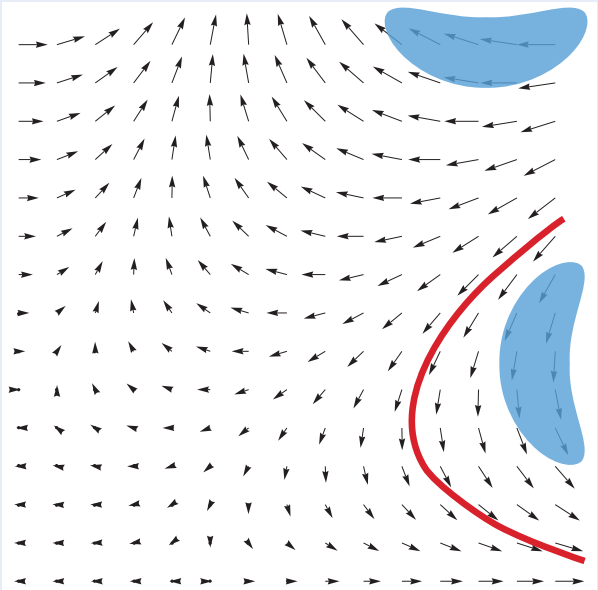
Differential Cuts

Differential Cut

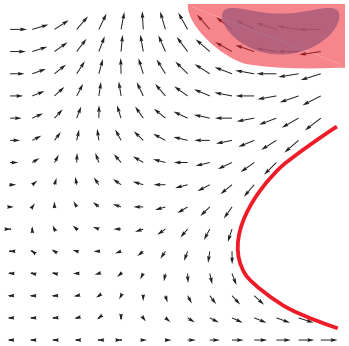
$$\frac{}{F \vdash [x' = f(x)]F}$$



Differential Cut



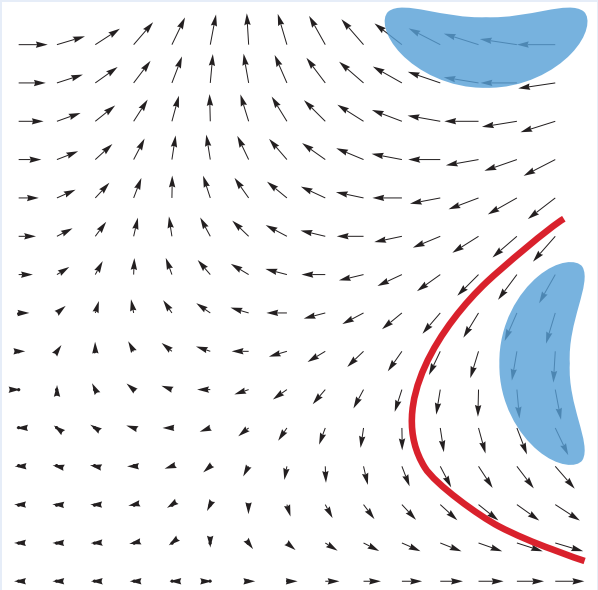
Differential Cuts



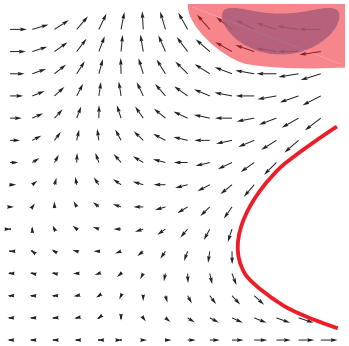
Differential Cut

$$\frac{F \vdash [x' = f(x)]C}{F \vdash [x' = f(x)]F}$$

Differential Cut



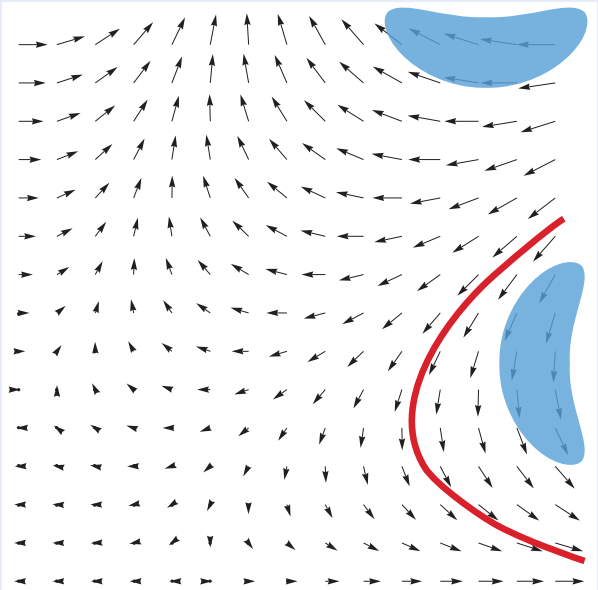
Differential Cuts



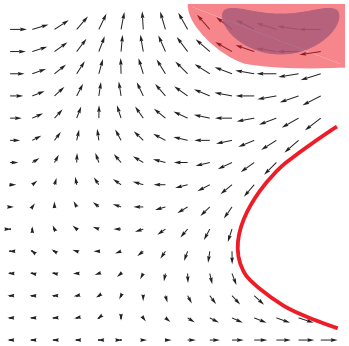
Differential Cut

$$\frac{F \vdash [x' = f(x)]C \qquad F \vdash [x' = f(x) \,\&\, C]F}{F \vdash [x' = f(x)]F}$$

Differential Cut



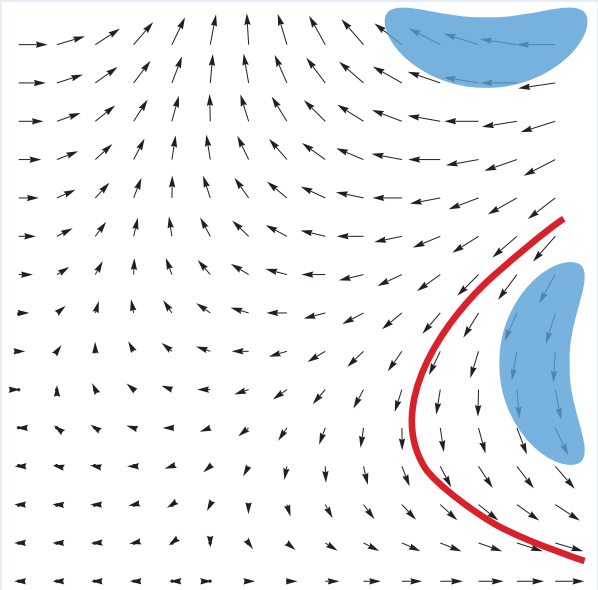
Differential Cuts



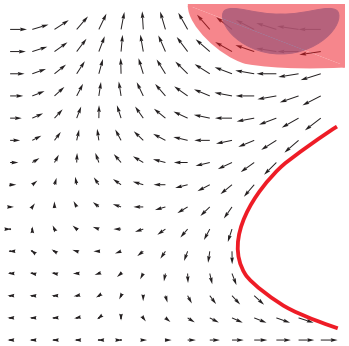
Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



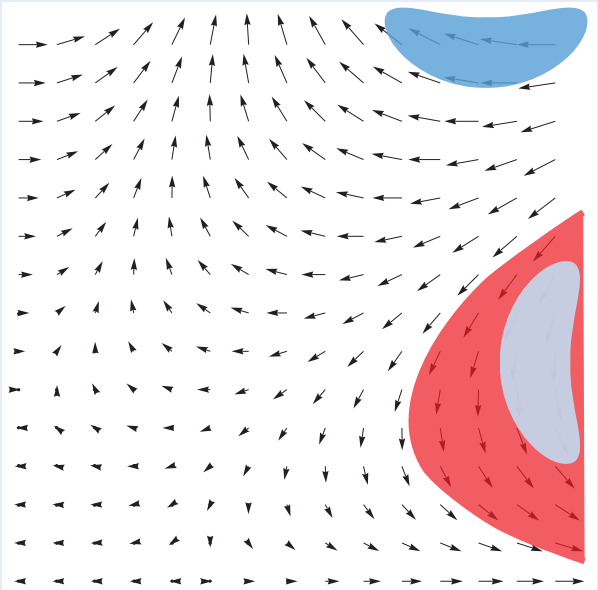
Differential Cuts



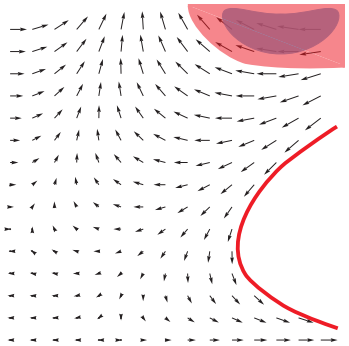
Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



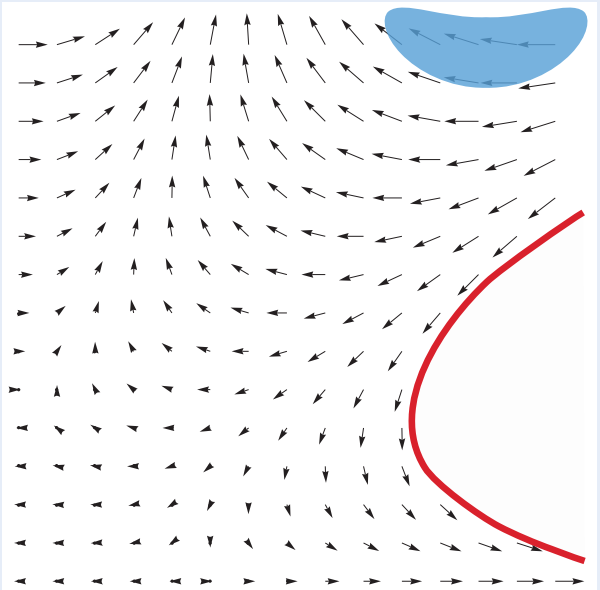
Differential Cuts



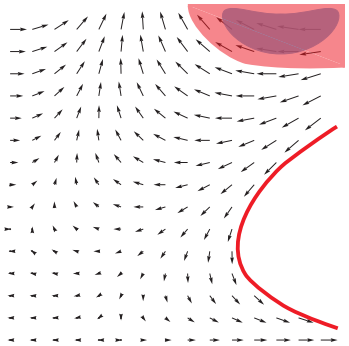
Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



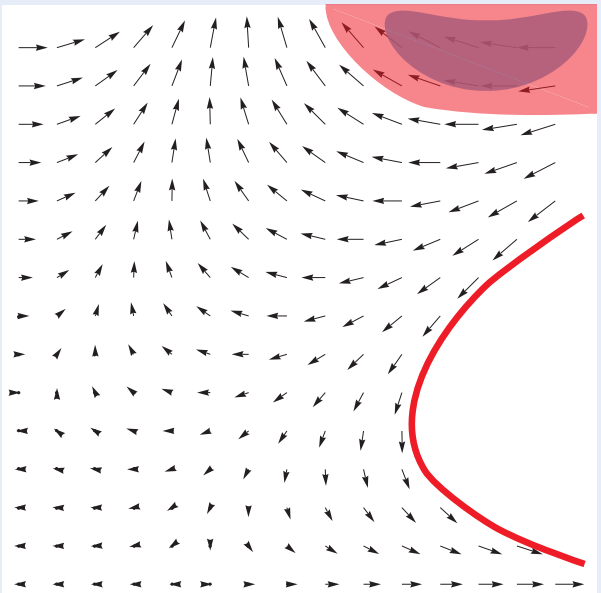
Differential Cuts



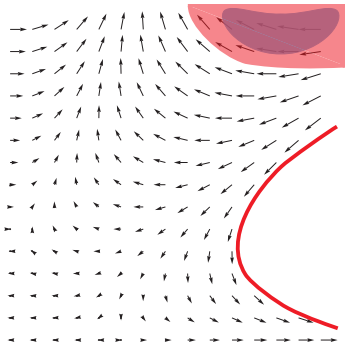
Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



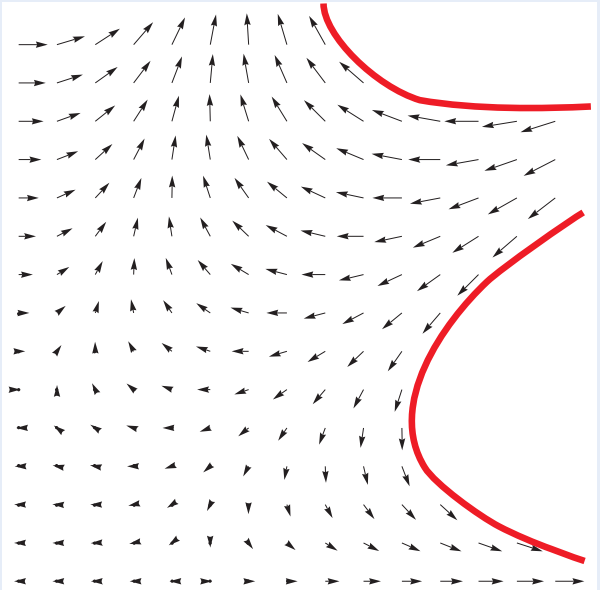
Differential Cuts



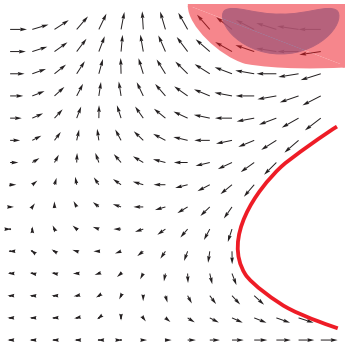
Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



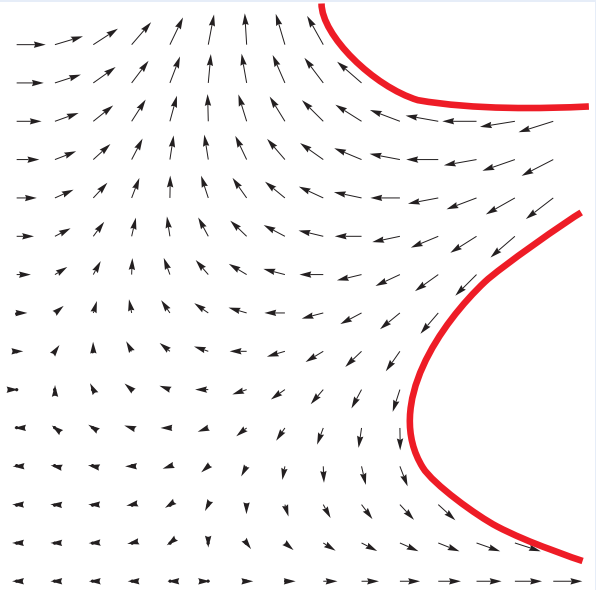
Differential Cuts



Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut



Proof (Soundness).

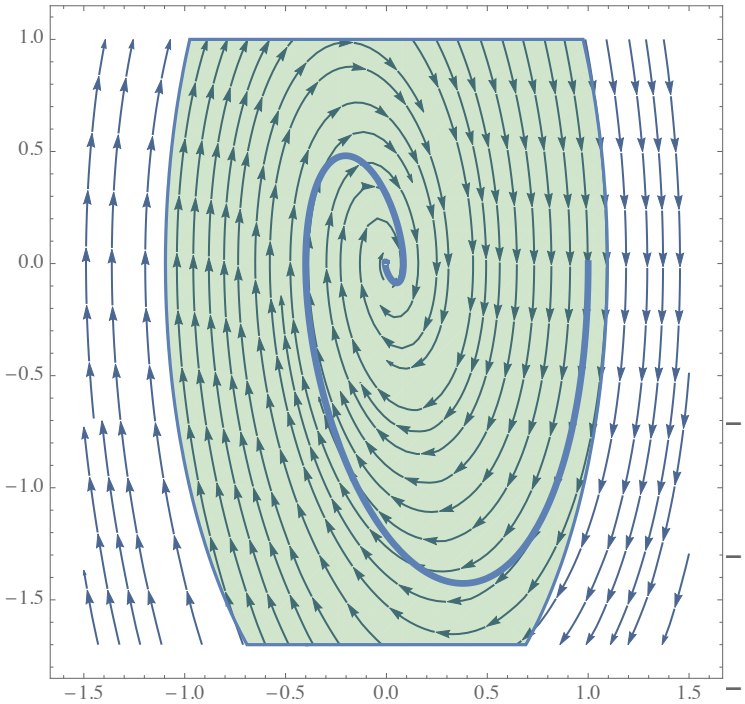
Let $\varphi \models x' = f(x) \wedge Q$ starting in $\omega \in [\![F]\!]$.
$\omega \in [\![[x' = f(x) \,\&\, Q]C]\!]$ by left premise.
Thus, $\varphi \models x' = f(x) \wedge Q \wedge C$.
Thus, $\varphi(r) \in [\![F]\!]$ by second premise. $\square$

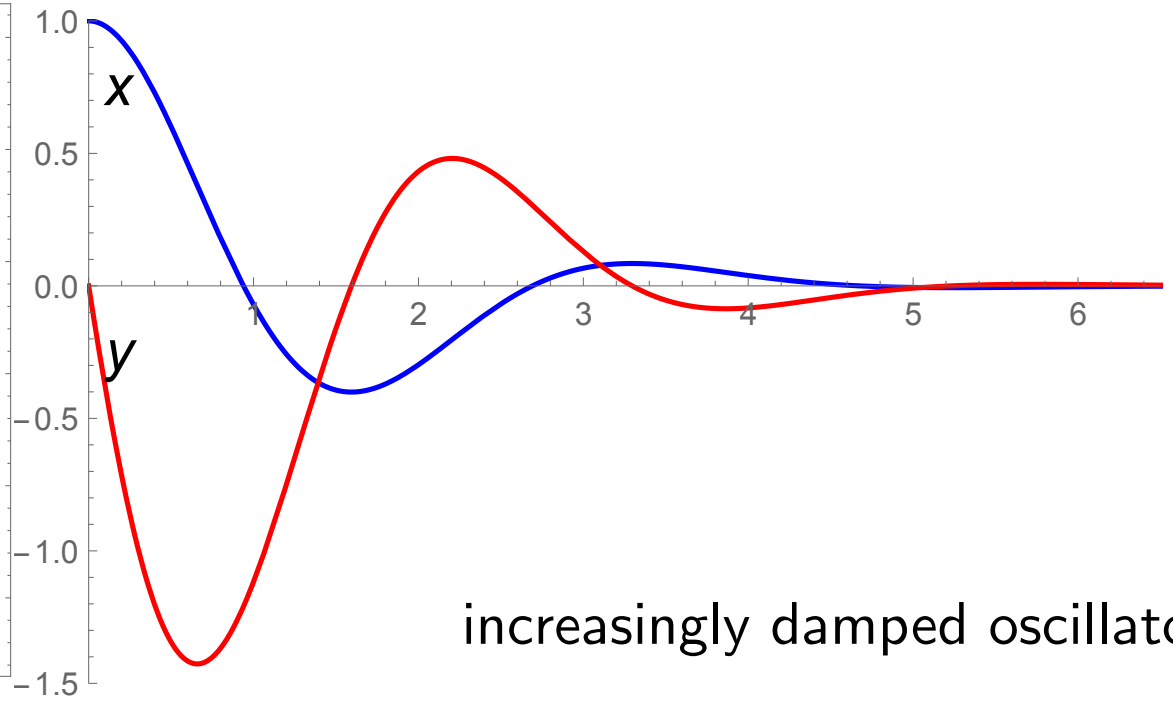
Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\ \overline{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\ \omega^2x^2+y^2\leq c^2}$$





increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0\wedge d\geq 0]\,\omega^2x^2+y^2\leq}$$

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0\wedge d\geq 0]\ \omega^2x^2+y^2\leq}$$

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\ \omega^2x^2+y^2\leq c^2}$$

$$\text{dI}\ \frac{}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\ d\geq 0}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0\wedge d\geq 0]\,\omega^2x^2+y^2\leq}$$

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

$$\text{dI}\ \frac{[':=]\ \dfrac{}{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0\wedge d\geq 0]\ \omega^2x^2+y^2\leq}$$

$$\text{dC}\ \frac{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\ \omega^2x^2+y^2\leq c^2}{}$$

$$\mathbb{R}\ \frac{}{\omega\geq 0 \vdash 7\geq 0}$$

$$[':=]\ \frac{\omega\geq 0 \vdash [d':=7]\ d'\geq 0}{}$$

$$\text{dI}\ \frac{}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\ d\geq 0}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0\wedge d\geq 0]\,\omega^2x^2+y^2\leq}$$

$$\text{dC}\ \frac{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}{}$$

DC

$$\mathbb{R}\ \frac{*}{\omega\geq 0 \vdash 7\geq 0}$$

$$[':=]\ \frac{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}{}$$

$$\text{dI}\ \frac{}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\begin{array}{ll}
[':=] & \omega\geq 0 \wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0 \\
\hline
\text{dI} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2 \\
\hline
\text{dC} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \omega\geq 0 \vdash 7\geq 0 \\
\hline
[':=] & \omega\geq 0 \vdash [d':=7]\, d'\geq 0 \\
\hline
\text{dI} & d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0
\end{array}
$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\begin{array}{ll}
\mathbb{R} & \omega\geq 0\wedge d\geq 0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0 \\
\hline
[':=] & \omega\geq 0\wedge d\geq 0 \vdash [x':=y][y':=-\omega^2x - 2d\omega y]\, 2\omega^2xx' + 2yy' \leq 0 \\
\hline
\text{dI} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2 \\
\hline
\text{dC} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \omega\geq 0 \vdash 7\geq 0 \\
\hline
[':=] & \omega\geq 0 \vdash [d':=7]\, d'\geq 0 \\
\hline
\text{dI} & d\geq 0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0
\end{array}
$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \omega\geq 0 \wedge d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0 \\
\hline
[':=] & \omega\geq 0 \wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 xx' + 2yy' \leq 0 \\
\hline
\text{dI} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq \\
\hline
\text{dC} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2
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\begin{array}{ll}
 & * \\
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\hline
[':=] & \omega\geq 0 \vdash [d':=7]\, d'\geq 0 \\
\hline
\text{dI} & d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0
\end{array}
$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \omega\geq 0 \wedge d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0 \\
[':=] & \omega\geq 0 \wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 xx' + 2yy' \leq 0 \\
\text{dI} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq \\
\text{dC} & \omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \omega\geq 0 \vdash 7\geq 0 \\
[':=] & \omega\geq 0 \vdash [d':=7]\, d'\geq 0 \\
\text{dI} & d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0
\end{array}
$$

Could repeatedly diffcut in formulas to help the proof

Ex: Differential Cuts

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

Ex: Differential Cuts

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{\text{['}:=\text{]}\ \dfrac{}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{\text{[}'{:=}\text{]}\ \dfrac{\mathbb{R}\ \dfrac{}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x'{:=}(x-2)^4 + y^5][y'{:=}y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$
\begin{array}{ll}
\text{dC} & x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1 \\
\hline
\mathbb{R} & \qquad * \\
\hline
 & \qquad \vdash 5y^4 y^2 \geq 0 \\
\hline
[':=] & \qquad \vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0 \\
\hline
\text{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
$$

Ex: Differential Cuts

$$\text{dI}\ \frac{}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \ \triangleright}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{\text{[}'{:=}\text{]}\ \frac{\mathbb{R}\ \frac{*}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x'{:=}(x-2)^4 + y^5][y'{:=}y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$
\begin{array}{ll}
[':=] & y^5 \geq 0 \vdash [x':=(x-2)^4+y^5][y':=y^2]2x^2x' \geq 0 \\
\hline
\text{dI} & x^3 \geq -1 \vdash [x' = (x-2)^4+y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright \\
\hline
\text{dC} & x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\hline
\mathbb{R} & \vdash 5y^4y^2 \geq 0 \\
\hline
[':=] & \vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0 \\
\hline
\text{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]y^5 \geq 0
\end{array}
$$

Ex: Differential Cuts

$$
\begin{array}{ll}
\mathbb{R} & y^5 \geq 0 \vdash 2x^2((x-2)^4 + y^5) \geq 0 \\
\hline
[':=] & y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0 \\
\hline
\text{dI} & x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright \\
\hline
\text{dC} & x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\hline
\mathbb{R} & \vdash 5y^4y^2 \geq 0 \\
\hline
[':=] & \vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0 \\
\hline
\text{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
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Ex: Differential Cuts

$$
\begin{array}{ll}
 & * \\
\hline
\mathbb{R} & y^5 \geq 0 \vdash 2x^2((x-2)^4 + y^5) \geq 0 \\
\hline
[':=] & y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0 \\
\hline
\text{dI} & x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright \\
\hline
\text{dC} & x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1
\end{array}
$$

$$
\begin{array}{ll}
 & * \\
\hline
\mathbb{R} & \vdash 5y^4y^2 \geq 0 \\
\hline
[':=] & \vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0 \\
\hline
\text{dI} & y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
$$

Outline

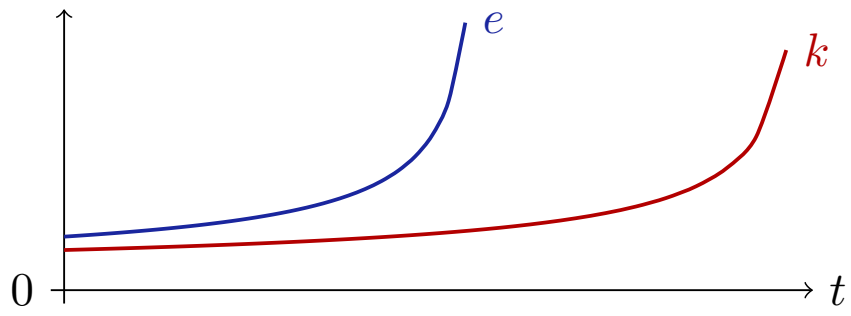
Soundness Proof: Differential Invariants

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \leq z \leq r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{DI} \quad \begin{array}{l} ([x' = f(x)]e \geq 0 \leftrightarrow e \geq 0) \\ \leftarrow [x' = f(x)](e)' \geq 0 \end{array}$$



Proof ($\geq$ rate of change from $\geq$ initial value. Case $r = 0$ is easier.)

$h(t) \stackrel{\text{def}}{=} \varphi(t)[\![e]\!]$ is differentiable on $[0, r]$ if $r > 0$ by diff. lemma.

$\frac{\mathrm{d}h(t)}{\mathrm{d}t}(z) = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \geq 0$ by lemma + assume for all z.

$h(r) - \underbrace{h(0)}_{\geq 0} = \underbrace{(r - 0)}_{>0} \underbrace{\frac{\mathrm{d}h(t)}{\mathrm{d}t}(\xi)}_{\geq 0} \geq 0$ by mean-value theorem for some ξ. $\square$

Outline

Differential Invariants for Differential Equations

Differential Weakening

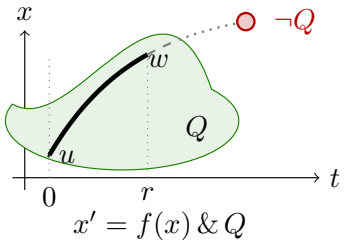
$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$

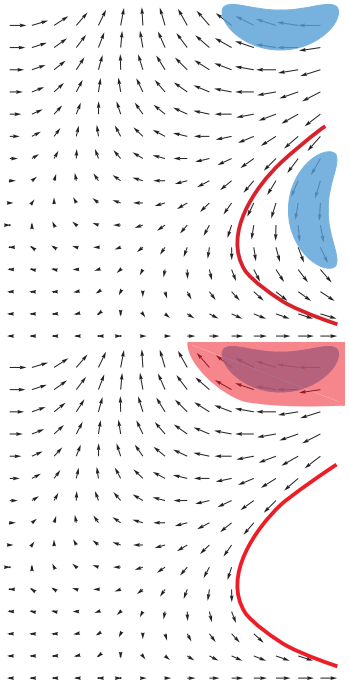
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





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