17: Winning Strategies & Regions

15-424: Foundations of Cyber-Physical Systems

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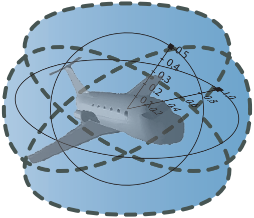

Outline

1. Learning Objectives
2. Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics
3. Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly
4. Summary

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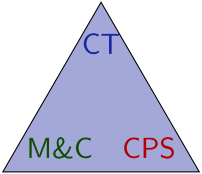
fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics



adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$x := e \mid ?Q \mid x' = f(x) \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$p(e_1, \ldots, e_n) \mid e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals | Some Reals | Angel Wins | Demon Wins

TOCL'15

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Differential Game Logic: Denotational Semantics

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} \;:\; \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \quad \{\omega \;:\; \nu \in [\![P]\!] \text{ for some } \nu \text{ with } (\omega,\nu) \in [\![\alpha]\!]\} \text{ ???} \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Definition (Hybrid game α: denotational semantics)

$\varsigma_{x:=e}(X) =$

Differential Game Logic: Denotational Semantics

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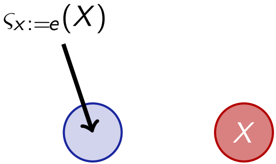
$$\varsigma_{x:=e}(X) = \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\}$$



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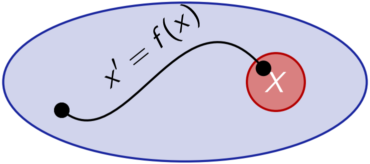

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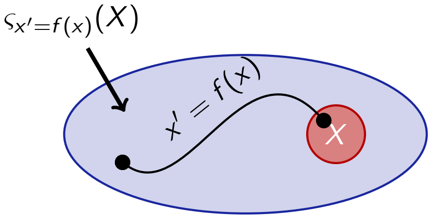
Definition (Hybrid game α: denotational semantics)

$$\varsigma_{x'=f(x)}(X) = \{\varphi(0) \in \mathcal{S} \; : \; \varphi(r) \in X, \frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(z) = \varphi(z)[\![f(x)]\!] \text{ for all } z\}$$



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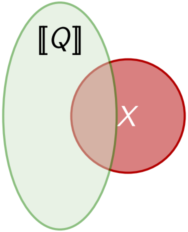

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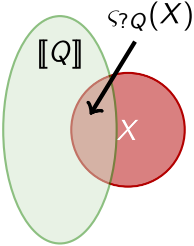

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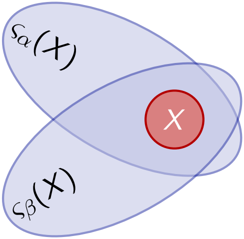
$\varsigma_{\alpha\cup\beta}(X) =$



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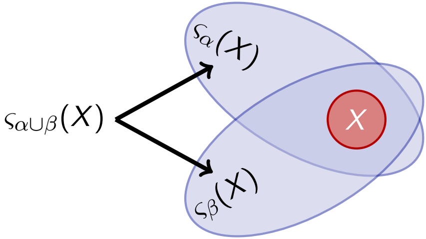

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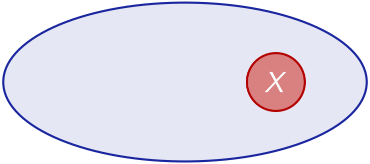

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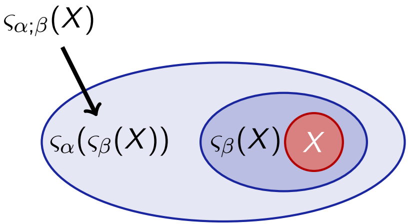

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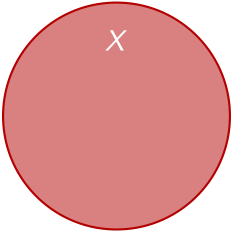

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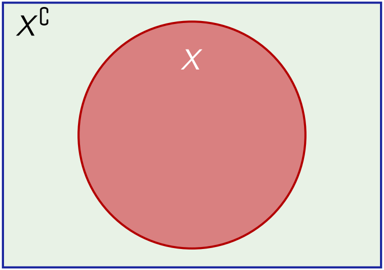

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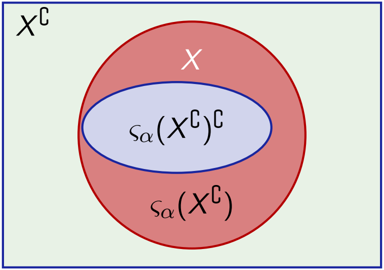

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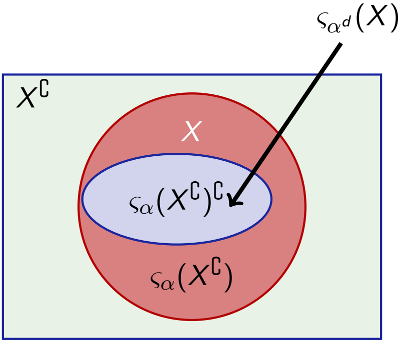
$$\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement$$



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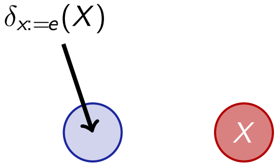

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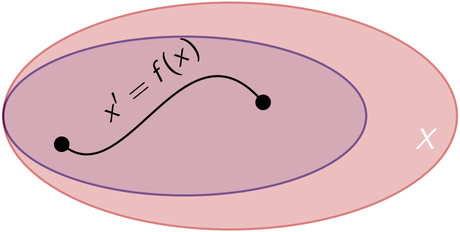

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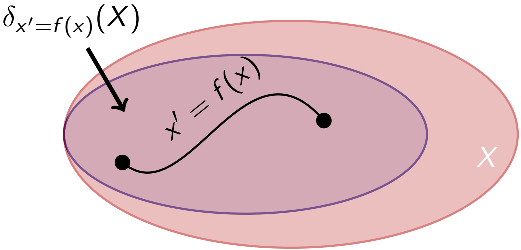

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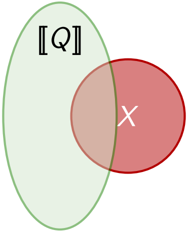

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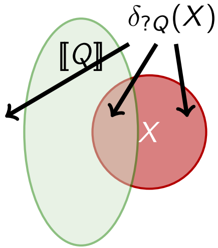
$$\delta_{?Q}(X) = [\![Q]\!]^\complement \cup X$$



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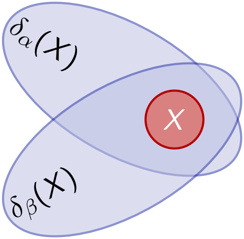

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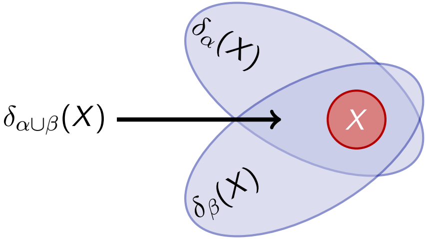

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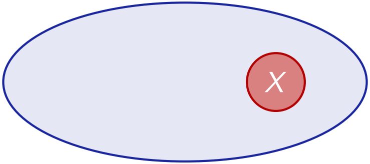

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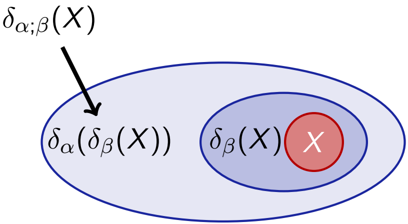

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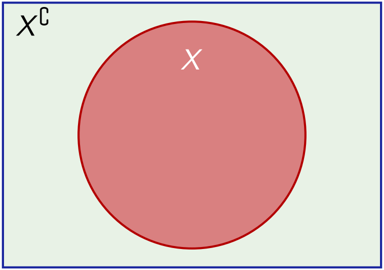

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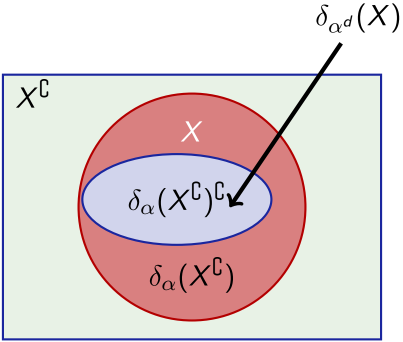

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \text{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
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\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

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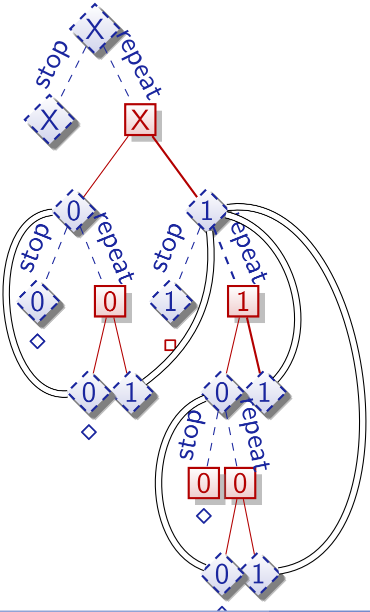
Filibusters & The Significance of Finitude

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Semantics of Repetition

Definition (Hybrid game α)

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Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$

$[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!]$ where $\alpha^{n+1} \equiv \alpha^n; \alpha \quad \alpha^0 \equiv ?true$ for HP α

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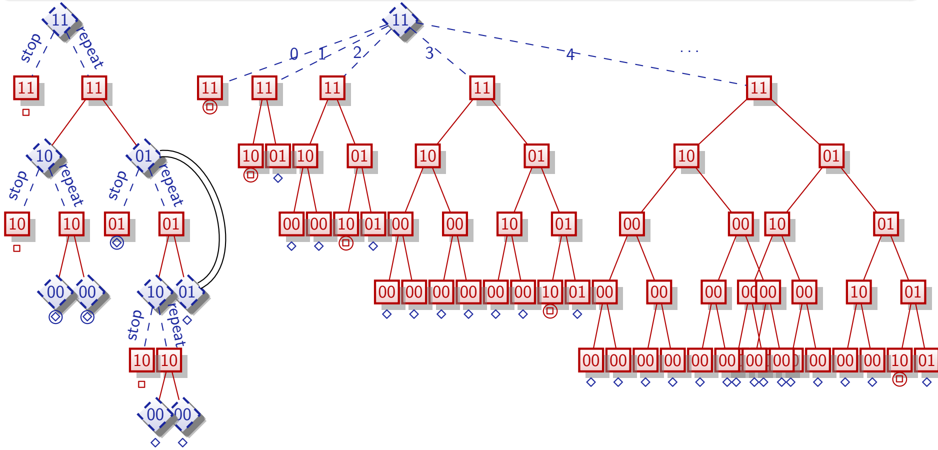
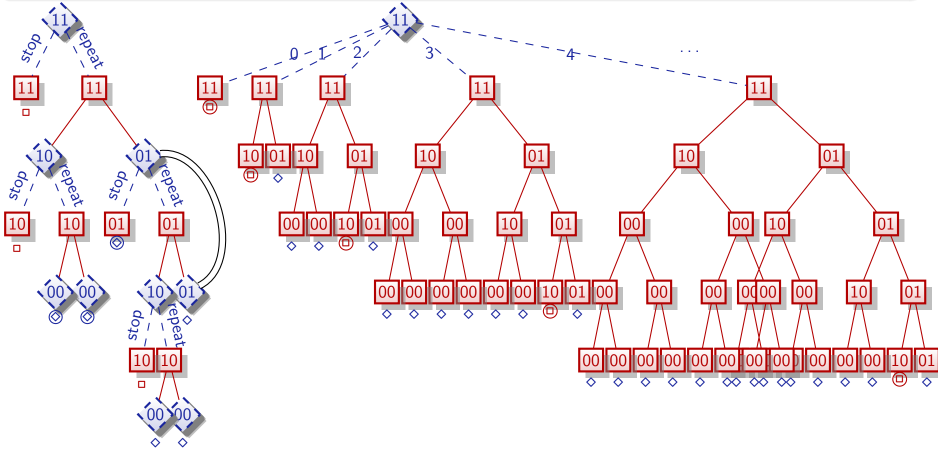


Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ advance notice semantics

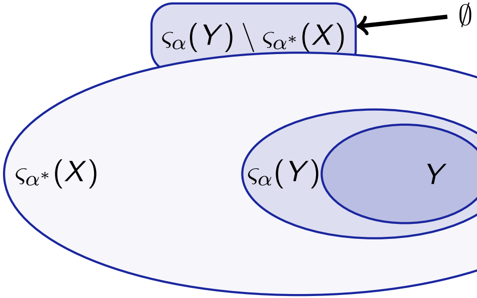


+1 Argument

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one round away from Y.



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Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle \, (0 \leq x < 1)$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$ ω-semantics

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$$\varsigma_\alpha^\lambda(X) \stackrel{\text{def}}{=} \bigcup_{\kappa<\lambda} \varsigma_\alpha^\kappa(X) \qquad \lambda \neq 0 \text{ a limit ordinal}$$

Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle\, (0 \le x < 1)$ $\varsigma_\alpha^n([0,1)) = [0,n) \neq \mathbb{R}$

$\varsigma_\alpha^\omega([0,1)) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n([0,1)) = [0,\infty) \neq \mathbb{R}$

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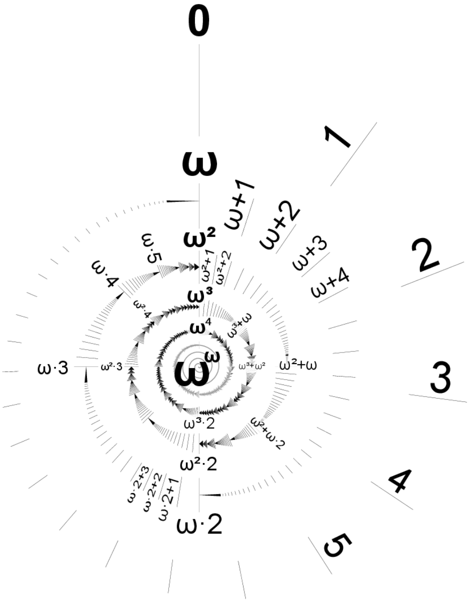
$\varsigma_\alpha^{\omega+1}([0,1)) = \varsigma_\alpha([0,\infty)) = \mathbb{R} \qquad \varsigma_\alpha^\omega([0,1)) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n([0,1)) = [0,\infty) \neq \mathbb{R}$

Strategic Closure Ordinal

$\geq \omega_1^{CK}$



Theorem

Hybrid game closure ordinal $>\omega^\omega$

$$\iota + 0 = \iota$$

$$\iota + (\kappa{+}1) = (\iota + \kappa) + 1 \quad \text{successor } \kappa{+}1$$

$$\iota + \lambda = \bigsqcup_{\kappa<\lambda} \iota + \kappa \quad \text{limit } \lambda$$

$$\iota \cdot 0 = 0$$

$$\iota \cdot (\kappa{+}1) = (\iota \cdot \kappa) + \iota \quad \text{successor } \kappa{+}1$$

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$$\iota^0 = 1$$

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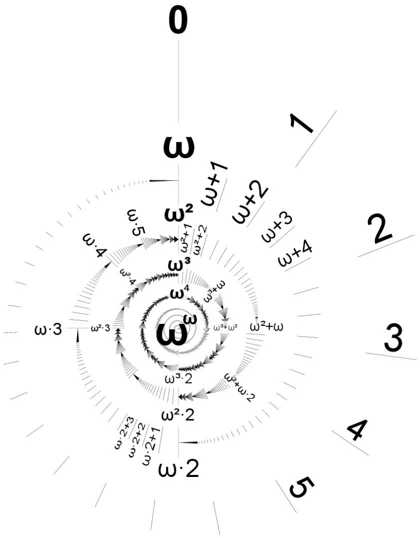
$$2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4$$



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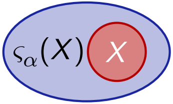

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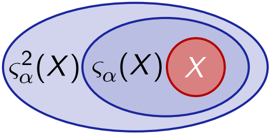

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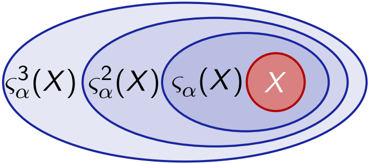

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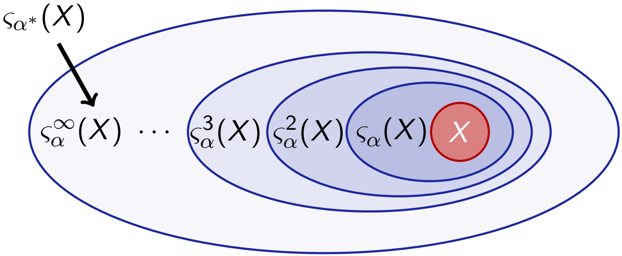


The Power of Implicit Definitions

Implicit Definitions

The advantages of implicit definition over construction are roughly those of theft over honest toil.

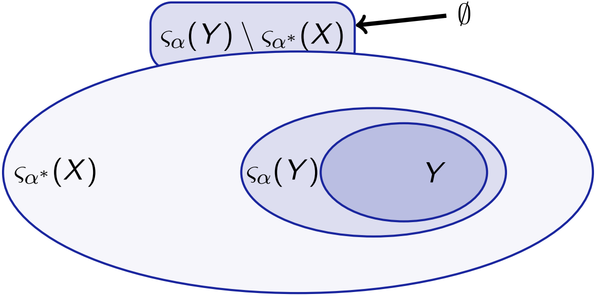
— Bertrand Russell

+1 Argument

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_{\alpha}(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_{\alpha}(Y)$ is just one round away from Y.



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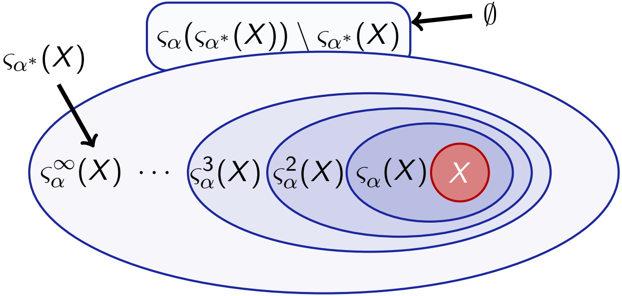
- Still too small: $X \subseteq Z$ since Angel may decide not to repeat

Fixpoints and Pre-Fixpoints

Definition (Pre-fixpoint)

$X \cup \varsigma_\alpha(Z) \subseteq Z$ for the winning region $Z \stackrel{\text{def}}{=} \varsigma_{\alpha^*}(X)$



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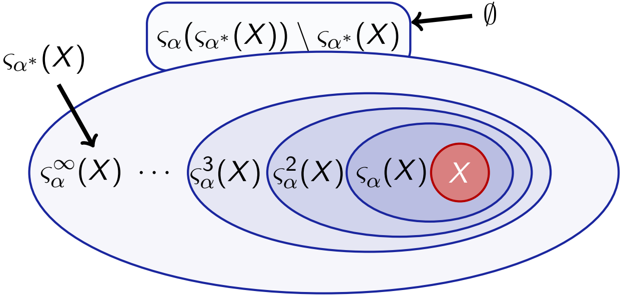

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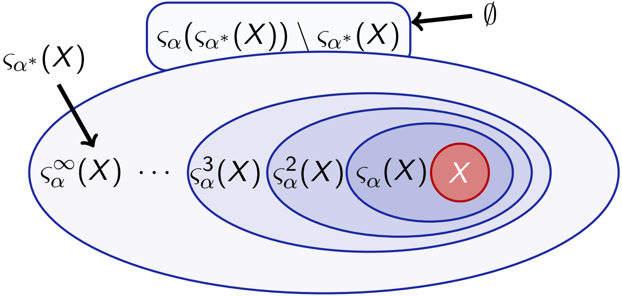

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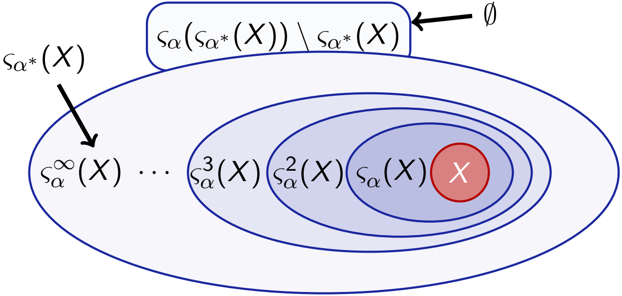

- Which Z is the right one?
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- Does such a Z exist?
- Existence: $Z = \mathcal{S}$ but that's too big and independent of α

Comparing (Pre-)Fixpoints

Lemma ( )

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
$$X \cup \varsigma_\alpha(Z) \subseteq Z$$

are pre-fixpoints, then

Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
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are pre-fixpoints, then $Y \cap Z$ *is a smaller pre-fixpoint.*

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Proof.

$$X \cup \varsigma_\alpha(Y \cap Z) \overset{\text{mon}}{\subseteq} X \cup (\varsigma_\alpha(Y) \cap \varsigma_\alpha(Z)) \overset{\text{above}}{\subseteq} Y \cap Z$$

□

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Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!

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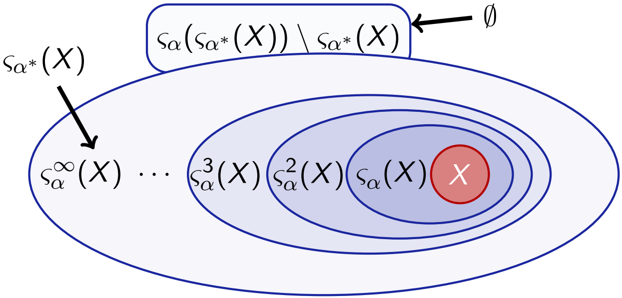
Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!
So: repetition semantics is the smallest pre-fixpoint (well-founded)

Semantics of Repetition

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \;:\; X \cup \varsigma_\alpha(Z) \subseteq Z\}$$



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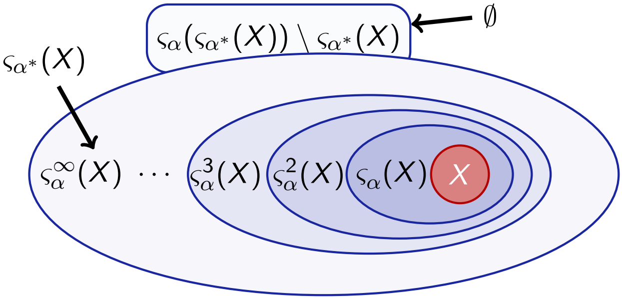

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$\varsigma_{\alpha^*}(X)$ intersection of solution

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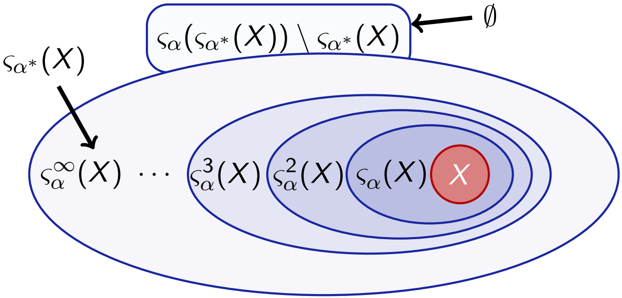

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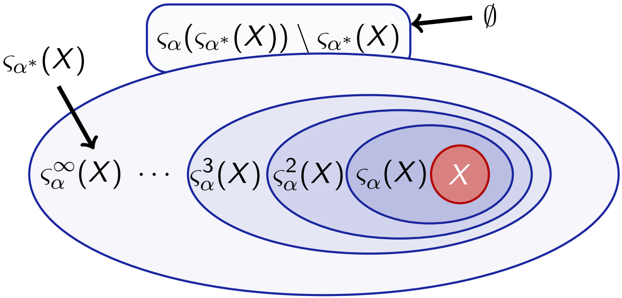

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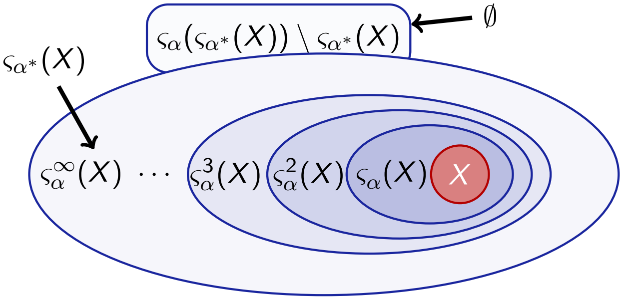

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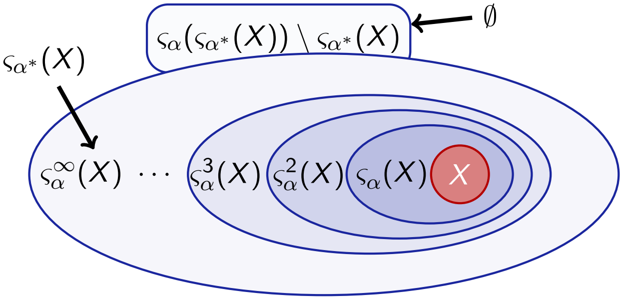

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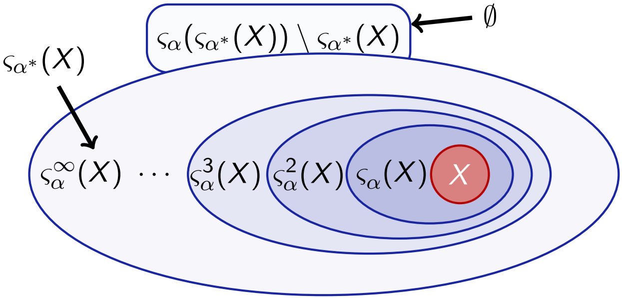

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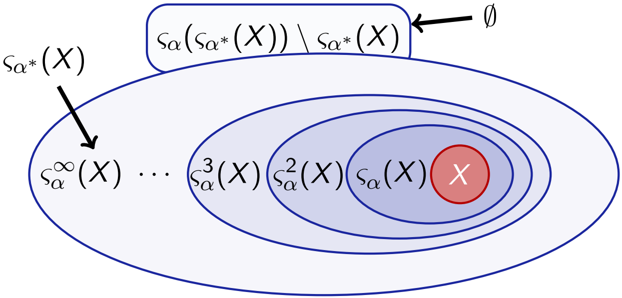
$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \,:\, X \cup \varsigma_\alpha(Z) = Z\} = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ by Knaster-Tarski



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Outline

1. Learning Objectives
2. Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics
3. Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly
4. Summary

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \text{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} \;:\; \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} \;:\; \varphi(r) \in X, \frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(z) = \varphi(z)[\![f(x)]\!] \text{ for all } z\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X) \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}$$

Definition (dGL Formula P) $[\![\cdot]\!] : \text{Fml} \to \wp(\mathcal{S})$

$$\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} \;:\; \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
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Differential Game Logic: Denotational Semantics

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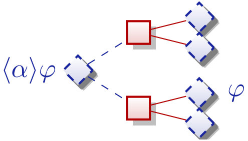
\end{aligned}
$$

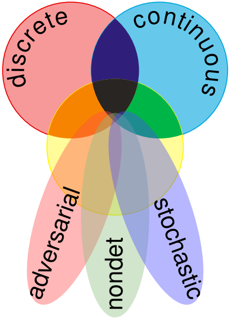
Summary

differential game logic

$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG} = \mathsf{dL} + {}^{d}$



- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint



Next lecture

1. Axiomatics
2. How to win and prove hybrid games

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