02: Differential Equations & Domains
Logical Foundations of Cyber-Physical Systems

André Platzer
Carnegie Mellon University
Computer Science Department
Outline

1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
   - Terms
   - First-Order Formulas
   - Continuous Programs
6. Summary
Outline

1 Learning Objectives

2 Introduction

3 Differential Equations

4 Examples of Differential Equations

5 Domains of Differential Equations
   • Terms
   • First-Order Formulas
   • Continuous Programs

6 Summary
Learning Objectives

Differential Equations & Domains

- semantics of differential equations
- descriptive power of differential equations
- syntax versus semantics

CT

M&C

CPS

- continuous dynamics
- differential equations
- evolution domains
- first-order logic

continuous operational effects
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
  y'(t) &= f(t, y) \\
  y(t_0) &= y_0
\end{align*}
\]

Intuition:

1. At each point in space, plot the value of RHS $f(t, y)$ as a vector.
2. Start at initial state $y_0$ at initial time $t_0$.
3. Follow the direction of the vector. The diagram should really show infinitely many vectors.

Your car's ODE:

\[
\begin{align*}
  x' &= v \\
  v' &= a
\end{align*}
\]

Well it's a wee bit more complicated.

André Platzer (CMU)
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
    y'(t) &= f(t, y) \\
    y(t_0) &= y_0
\end{align*}
\]

Intuition:

1. At each point in space, plot the value of RHS \( f(t, y) \) as a vector
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
    y'(t) &= f(t, y) \\
    y(t_0) &= y_0
\end{align*}
\]

Intuition:
1. At each point in space, plot the value of RHS \( f(t, y) \) as a vector
2. Start at initial state \( y_0 \) at initial time \( t_0 \)
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
    y'(t) &= f(t, y) \\
    y(t_0) &= y_0
\end{align*}
\]

Intuition:
1. At each point in space, plot the value of RHS \( f(t, y) \) as a vector
2. Start at initial state \( y_0 \) at initial time \( t_0 \)
3. Follow the direction of the vector
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
  y'(t) &= f(t, y) \\
  y(t_0) &= y_0
\end{align*}
\]

Intuition:

1. At each point in space, plot the value of RHS \( f(t, y) \) as a vector
2. Start at initial state \( y_0 \) at initial time \( t_0 \)
3. Follow the direction of the vector
4. The diagram should really show infinitely many vectors ...
Example (Vector field and one solution of a differential equation)

\[
\begin{align*}
\frac{dy(t)}{dt} &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]

Intuition:

1. At each point in space, plot the value of RHS $f(t, y)$ as a vector
2. Start at initial state $y_0$ at initial time $t_0$
3. Follow the direction of the vector
4. The diagram should really show infinitely many vectors . . .

Your car’s ODE: $x' = v, v' = a$
Example (Vector field and one solution of a differential equation)

\[ \begin{align*}
    y'(t) &= f(t, y) \\
    y(t_0) &= y_0
\end{align*} \]

Intuition:

1. At each point in space, plot the value of RHS $f(t, y)$ as a vector.
2. Start at initial state $y_0$ at initial time $t_0$.
3. Follow the direction of the vector.
4. The diagram should really show infinitely many vectors . . .

Your car’s ODE: $x' = v, v' = a$

Well it’s a wee bit more complicated.
\( x' = \frac{1}{4} x \)

\( x(0) = 1 \)
\begin{align*}
\left( x'(t) = \frac{1}{4}x(t) \right) & \quad \sim \quad \left( x(t + \Delta) := x(t) + \frac{1}{4}x(t)\Delta \right) \\
\left( x(0) = 1 \right) & \quad \quad \quad \left( x(0) := 1 \right)
\end{align*}
Intuition for Differential Equations

\[
\begin{align*}
\left( x'(t) &= \frac{1}{4} x(t) \\
    x(0) &= 1 \right) \quad &\sim \quad \left( x(t + \Delta) := x(t) + \frac{1}{4} x(t) \Delta \\
    x(0) &= 1 \right)
\end{align*}
\]
\[
\begin{align*}
(x'(t) &= \frac{1}{4}x(t) \\
x(0) &= 1
\end{align*}
\begin{align*}
\xrightarrow{4}
(x(t + \Delta) &:= x(t) + \frac{1}{4}x(t)\Delta \\
x(0) &:= 1
\end{align*}
\]
\[
\begin{aligned}
(x'(t) &= \frac{1}{4}x(t) \\
x(0) &= 1
\end{aligned}
\implies
\begin{aligned}
x(t + \Delta) &:= x(t) + \frac{1}{4}x(t)\Delta \\
x(0) &:= 1
\end{aligned}
\]
\[ \left( \frac{dx}{dt} = \frac{1}{4}x(t) \right) \quad \text{and} \quad x(0) = 1 \]

\[ \rightarrow \left( x(t + \Delta) := x(t) + \frac{1}{4}x(t)\Delta \right) \quad \text{and} \quad x(0) := 1 \]
\begin{align*}
\frac{dx(t)}{dt} &= \frac{1}{4} x(t) \\
\int_{0}^{t} \frac{dx(t)}{dt} &= \int_{0}^{t} \frac{1}{4} x(t) dt \\
\Rightarrow x(t) &= \frac{1}{4} \int_{0}^{t} x(t) dt + x(0) \\
\Rightarrow x(t) &= \frac{1}{4} \left( \int_{0}^{t} x(t) dt + x(0) \right)
\end{align*}
Outline

1 Learning Objectives

2 Introduction

3 Differential Equations

4 Examples of Differential Equations

5 Domains of Differential Equations
   • Terms
   • First-Order Formulas
   • Continuous Programs

6 Summary
The Meaning of Differential Equations

1. What exactly is a vector field?
2. What does it mean to describe directions of evolution at every point in space?
3. Could these directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure. We immediately want to get into the habit of studying the behavior and exact meaning of all relevant aspects of CPS.
Definition (Ordinary Differential Equation, ODE)

\( f : D \rightarrow \mathbb{R}^n \) on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) (i.e., open connected set). Then \( Y : I \rightarrow \mathbb{R}^n \) is solution of initial value problem (IVP)

\[
\begin{align*}
(y'(t)) &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]

on the interval \( I \subseteq \mathbb{R} \), iff, for all times \( t \in I \),

If \( f \in C(D, \mathbb{R}^n) \), then \( Y \in C^1(I, \mathbb{R}^n) \).

If \( f \) continuous, then \( Y \) continuously differentiable.
Definition (Ordinary Differential Equation, ODE)

\( f : D \to \mathbb{R}^n \) on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) (i.e., open connected set). Then \( Y : I \to \mathbb{R}^n \) is solution of initial value problem (IVP)

\[
\begin{align*}
(y'(t) &= f(t, y)) \\
y(t_0) &= y_0
\end{align*}
\]

on the interval \( I \subseteq \mathbb{R} \), iff, for all times \( t \in I \),

1. defined \( (t, Y(t)) \in D \)
Definition (Ordinary Differential Equation, ODE)

\( f : D \rightarrow \mathbb{R}^n \) on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) (i.e., open connected set). Then \( Y : I \rightarrow \mathbb{R}^n \) is solution of initial value problem (IVP)

\[
\begin{align*}
(y'(t) &= f(t, y)) \\
y(t_0) &= y_0
\end{align*}
\]

on the interval \( I \subseteq \mathbb{R} \), iff, for all times \( t \in I \),

1. defined \( (t, Y(t)) \in D \)
2. time-derivative \( Y'(t) \) exists and satisfies \( Y'(t) = f(t, Y(t)) \).
Definition (Ordinary Differential Equation, ODE)

Let $f : D \to \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected set). Then $Y : I \to \mathbb{R}^n$ is solution of initial value problem (IVP)

\[
\begin{align*}
(y'(t) &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]

on the interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

1. defined $(t, Y(t)) \in D$
2. time-derivative $Y'(t)$ exists and satisfies $Y'(t) = f(t, Y(t))$.
3. initial value $Y(t_0) = y_0$
Definition (Ordinary Differential Equation, ODE)

\[ f : D \to \mathbb{R}^n \text{ on domain } D \subseteq \mathbb{R} \times \mathbb{R}^n \text{ (i.e., open connected set). Then } \]

\( Y : I \to \mathbb{R}^n \text{ is solution of initial value problem (IVP) } \]

\[
\begin{align*}
  (y'(t) &= f(t, y)) \\
  y(t_0) &= y_0 
\end{align*}
\]

on the interval \( I \subseteq \mathbb{R} \), iff, for all times \( t \in I \),

1. defined \( (t, Y(t)) \in D \)
2. time-derivative \( Y'(t) \) exists and satisfies \( Y'(t) = f(t, Y(t)) \).
3. initial value \( Y(t_0) = y_0 \)

If \( f \in C(D, \mathbb{R}^n) \), then \( Y \in C^1(I, \mathbb{R}^n) \).

If \( f \) continuous, then \( Y \) continuously differentiable.
Outline

1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
   - Terms
   - First-Order Formulas
   - Continuous Programs
6. Summary
Example (Initial value problem)

\[
\begin{align*}
  x'(t) &= \frac{1}{2} \\
  x(0) &= -1
\end{align*}
\]

has a solution
Example: A Constant Differential Equation

Example (Initial value problem)

\[
\begin{pmatrix}
x'(t) &= \frac{1}{2} \\ x(0) &= -1
\end{pmatrix}
\] has a solution \( x(t) = \frac{1}{2} t - 1 \)

Check by inserting solution into ODE+IVP.

\[
\left( x'(t) = \frac{1}{2} \right)
\]

\[
\left( x(0) = -1 \right)
\]

André Platzer (CMU)
Example (Initial value problem)

\[
\begin{pmatrix}
  x'(t) = \frac{1}{2} \\
  x(0) = -1
\end{pmatrix}
\]
has a solution \( x(t) = \frac{1}{2} t - 1 \)

\[\text{Check by inserting solution into ODE+IVP.}\]
Example (Initial value problem)

\[
\begin{align*}
  x'(t) &= \frac{1}{2} \\
  x(0) &= -1
\end{align*}
\]

has a solution \( x(t) = \frac{1}{2} t - 1 \)

Check by inserting solution into ODE+IVP.

\[
\begin{align*}
  x'(t) &= \frac{1}{2} t - 1 \\
  x(0) &= \frac{1}{2} \cdot 0 - 1 = -1
\end{align*}
\]
Example: A Constant Differential Equation

Example (Initial value problem)

\[
\begin{cases}
x'(t) = \frac{1}{2} \\
x(0) = -1
\end{cases}
\]
has a solution \( x(t) = \frac{1}{2} t - 1 \)
Example: A Constant Differential Equation

Example (Initial value problem)

\[
\begin{align*}
  x'(t) &= \frac{1}{2} \\
  x(0) &= -1
\end{align*}
\]

has a solution \( x(t) = \frac{1}{2} t - 1 \)

Check by inserting solution into ODE+IVP.

\[
\begin{align*}
  ((x(t))') &= (\frac{1}{2} t - 1)' = \frac{1}{2} \\
  x(0) &= \frac{1}{2} \cdot 0 - 1 = -1
\end{align*}
\]
Example (Initial value problem)

\[
\begin{pmatrix}
  x'(t) = \frac{1}{4}x(t) \\
  x(0) = 1
\end{pmatrix}
\]

has a solution

\[
x(t) = e^{t/4}
\]
Example (Initial value problem)

\[
\begin{align*}
\left( x'(t) &= \frac{1}{4}x(t) \\ x(0) &= 1 \right)
\end{align*}
\]

has a solution \( x(t) = e^{\frac{t}{4}} \)

Check by inserting solution into ODE+IVP.

\[
\begin{align*}
\left( x(t) \right)' &= \left( e^{\frac{t}{4}} \right)' \\
&= e^{\frac{t}{4}} \left( \frac{1}{4} \right) \\
&= \frac{1}{4} x(t) \\
x(0) &= e^{\frac{0}{4}} = 1
\end{align*}
\]
Example (Initial value problem)

\[
\begin{cases}
    x'(t) = \frac{1}{4}x(t) \\
    x(0) = 1
\end{cases}
\]

has a solution \( x(t) = e^{t/4} \).
Example (Initial value problem)

\[
\begin{align*}
(x'(t) &= \frac{1}{4}x(t)) \\
(x(0) &= 1)
\end{align*}
\]

has a solution \( x(t) = e^{\frac{t}{4}} \)

Check by inserting solution into ODE+IVP.

\[
\begin{align*}
(x(t))' &= (e^{\frac{t}{4}})' \\
&= e^{\frac{t}{4}} \cdot \frac{1}{4} \\
&= \frac{1}{4}x(t)
\end{align*}
\]

\[
x(0) = e^{0} \cdot \frac{1}{4} = 1
\]
Example (Initial value problem)

\[
\begin{align*}
\begin{cases}
 x'(t) &= \frac{1}{4}x(t) \\
 x(0) &= 1
\end{cases}
\end{align*}
\]

has a solution \( x(t) = e^{t/4} \).

\[\text{Check by inserting solution into ODE+IVP.}\]

\[
\begin{align*}
(x(t))' &= (e^{t/4})' = e^{t/4} \cdot \frac{1}{4} = \frac{1}{4}x(t) = 1/4 \cdot x(t) \\
x(0) &= e^{0/4} = 1
\end{align*}
\]
Example: A Linear Differential Equation from before

Example (Initial value problem)

\[
\begin{align*}
(x'(t) &= \frac{1}{4}x(t) \\
x(0) &= 1
\end{align*}
\]
has a solution \( x(t) = e^{\frac{t}{4}} \)

Check by inserting solution into ODE+IVP.

\[
\begin{align*}
(x(t))' &= (e^{\frac{t}{4}})' = e^{\frac{t}{4}} (\frac{t}{4})' = e^{\frac{t}{4}} \frac{1}{4} = \frac{1}{4} x(t) \\
x(0) &= e^{\frac{0}{4}} = 1
\end{align*}
\]
Example (Initial value problem)

\[
\begin{align*}
\frac{d}{dt} v(t) &= w(t) \\
\frac{d}{dt} w(t) &= -v(t) \\
v(0) &= 0 \\
w(0) &= 1
\end{align*}
\]

has solution

\[
\begin{align*}
v(t) &= \sin(t) \\
w(t) &= \cos(t)
\end{align*}
\]
Example (Initial value problem)

\[
\begin{pmatrix}
  v'(t) = w(t) \\
w'(t) = -v(t) \\
v(0) = 0 \\
w(0) = 1
\end{pmatrix}
\]

has solution

\[
\begin{pmatrix}
v(t) = \sin(t) \\
w(t) = \cos(t)
\end{pmatrix}
\]
Example: Rotational Dynamics

Example (Initial value problem)

\[
\begin{align*}
  v'(t) &= \omega w(t) \\
  w'(t) &= -\omega v(t) \\
  v(0) &= 0 \\
  w(0) &= 1
\end{align*}
\]

has solution

\[
\begin{align*}
  v(t) &= \sin(t) \\
  w(t) &= \cos(t)
\end{align*}
\]
Example: Rotational Dynamics

Example (Initial value problem)

\[
\begin{align*}
\begin{bmatrix}
  v'(t) &= \omega w(t) \\
  w'(t) &= -\omega v(t) \\
  v(0) &= 0 \\
  w(0) &= 1
\end{bmatrix}
\end{align*}
\]

has solution

\[
\begin{align*}
  v(t) &= \sin(\omega t) \\
  w(t) &= \cos(\omega t)
\end{align*}
\]
Example (Initial value problem)

\[
\begin{align*}
    x'(t) &= v(t) \\
    y'(t) &= w(t) \\
    v'(t) &= \omega w(t) \\
    w'(t) &= -\omega v(t) \\
    x(0) &= x_0 \\
    y(0) &= y_0 \\
    v(0) &= v_0 \\
    w(0) &= w_0
\end{align*}
\]
Example: Planar Motion Dynamics

Example (Initial value problem)

\[
\begin{aligned}
&x'(t) = v(t) \\
&y'(t) = w(t) \\
&v'(t) = \omega w(t) \\
&w'(t) = -\omega v(t) \\
&x(0) = x_0 \\
&y(0) = y_0 \\
&v(0) = v_0 \\
&w(0) = w_0
\end{aligned}
\]
<table>
<thead>
<tr>
<th>ODE</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x' = 1, x(0) = x_0$</td>
<td>$x(t) = x_0 + t$</td>
</tr>
<tr>
<td>$x' = 5, x(0) = x_0$</td>
<td>$x(t) = x_0 + 5t$</td>
</tr>
<tr>
<td>$x' = x, x(0) = x_0$</td>
<td>$x(t) = x_0 e^t$</td>
</tr>
<tr>
<td>$x' = x^2, x(0) = x_0$</td>
<td>$x(t) = \frac{x_0}{1 - tx_0}$</td>
</tr>
<tr>
<td>$x' = \frac{1}{x}, x(0) = 1$</td>
<td>$x(t) = \sqrt{1 + 2t}$ ...</td>
</tr>
<tr>
<td>$y'(x) = -2xy, y(0) = 1$</td>
<td>$y(x) = e^{-x^2}$</td>
</tr>
<tr>
<td>$x'(t) = tx, x(0) = x_0$</td>
<td>$x(t) = x_0 e^{t^2}$</td>
</tr>
<tr>
<td>$x' = \sqrt{x}, x(0) = x_0$</td>
<td>$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0 + x_0}$</td>
</tr>
<tr>
<td>$x' = y, y' = -x, x(0) = 0, y(0) = 1$</td>
<td>$x(t) = \sin t, y(t) = \cos t$</td>
</tr>
<tr>
<td>$x' = 1 + x^2, x(0) = 0$</td>
<td>$x(t) = \tan t$</td>
</tr>
<tr>
<td>$x'(t) = \frac{2}{t^3} x(t)$</td>
<td>$x(t) = e^{-\frac{1}{t^2}}$ non-analytic</td>
</tr>
<tr>
<td>$x' = x^2 + x^4$</td>
<td></td>
</tr>
<tr>
<td>$x'(t) = e^{t^2}$</td>
<td>non-elementary</td>
</tr>
<tr>
<td>ODE</td>
<td>Solution</td>
</tr>
<tr>
<td>-----</td>
<td>----------</td>
</tr>
<tr>
<td>$x' = 1, x(0) = x_0$</td>
<td>$x(t) = x_0 + t$</td>
</tr>
<tr>
<td>$x' = 5, x(0) = x_0$</td>
<td>$x(t) = x_0 + 5t$</td>
</tr>
<tr>
<td>$x' = x, x(0) = x_0$</td>
<td>$x(t) = x_0 e^t$</td>
</tr>
<tr>
<td>$x' = x^2, x(0) = x_0$</td>
<td>$x(t) = \frac{x_0}{1 - t x_0}$</td>
</tr>
<tr>
<td>$x' = \frac{1}{x}, x(0) = 1$</td>
<td>$x(t) = \sqrt{1 + 2t} \ldots$</td>
</tr>
<tr>
<td>$y'(x) = -2xy, y(0) = 1$</td>
<td>$y(x) = e^{-x^2}$</td>
</tr>
<tr>
<td>$x'(t) = tx, x(0) = x_0$</td>
<td>$x(t) = x_0 e^{\frac{t^2}{2}}$</td>
</tr>
<tr>
<td>$x' = \sqrt{x}, x(0) = x_0$</td>
<td>$x(t) = \frac{t^2}{4} \pm t \sqrt{x_0 + x_0}$</td>
</tr>
<tr>
<td>$x' = y, y' = -x, x(0) = 0, y(0) = 1$</td>
<td>$x(t) = \sin t, y(t) = \cos t$</td>
</tr>
<tr>
<td>$x' = 1 + x^2, x(0) = 0$</td>
<td>$x(t) = \tan t$</td>
</tr>
<tr>
<td>$x'(t) = \frac{2}{t^3}x(t)$</td>
<td>$x(t) = e^{-\frac{1}{t^2}}$ non-analytic</td>
</tr>
<tr>
<td>$x' = x^2 + x^4$</td>
<td>non-elementary</td>
</tr>
<tr>
<td>$x'(t) = e^{t^2}$</td>
<td></td>
</tr>
</tbody>
</table>
Solutions of differential equations can be much more involved than the differential equations themselves.

Representational and descriptive power of differential equations!

Simple differential equations can describe quite complicated physical processes.

Local description as the direction into which the system evolves.
Outline

1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
   - Terms
   - First-Order Formulas
   - Continuous Programs
6. Summary
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain $Q$ is denoted by

$$x' = f(x) \land Q$$

conjunctive notation ($\land$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain $Q$. 

\[ x' = f(x) \land Q \]
Enable Cyber to interact with Physics

**Definition (Evolution domain constraints)**

A *differential equation* \( x' = f(x) \) *with evolution domain* \( Q \) is denoted by

\[
x' = f(x) & Q
\]

conjunctive notation (&) signifies that the system obeys the differential equation \( x' = f(x) \) and the evolution domain \( Q \).
Evolution Domain Constraints

Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain $Q$ is denoted by

$$x' = f(x) \& Q$$

conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain $Q$. 

\[ x' = f(x) \& Q \]
Enable Cyber to interact with Physics

**Definition (Evolution domain constraints)**

A differential equation $x' = f(x)$ with evolution domain $Q$ is denoted by

$$x' = f(x) \& Q$$

 Conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain $Q$.

- $x' = v, v' = a, t' = 1 \& t \leq \varepsilon$ stops at clock $\varepsilon$ at the latest
- $x' = v, v' = a, t' = 1 \& v \geq 0$ stops before velocity negative
- $x' = y, y' = x + y^2 \& true$ no constraint
Evolution Domain Constraints

**Define: Terms**

Cyber to interact with Physics

**Define: Formulas**

Definition (Evolution domain constraints)

A *differential equation* \( x' = f(x) \) *with evolution domain* \( Q \) is denoted by

\[
\begin{align*}
x' &= f(x) & \text{conjunctive notation (&) signifies that the system obeys the differential equation} \\
&\quad \text{and the evolution domain} \ Q.
\end{align*}
\]

\[
\begin{align*}
x' &= v, \; v' = a, \; t' = 1 & \text{stops at clock} \ \varepsilon \ \text{at the latest} \\
x' &= v, \; v' = a, \; t' = 1 & \text{stops before velocity negative} \\
x' &= y, \; y' = x + y^2 & \text{no constraint}
\end{align*}
\]
A term $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

where $e, \tilde{e}$ are terms, $x \in \mathcal{V}$ is a variable, $c \in \mathbb{Q}$ a rational number constant.
Definition (Syntax of terms)

A *term* $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Variable  Constant  Add  Multiply
Definition (Syntax of terms)

A *term* $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms) \((\llbracket \cdot \rrbracket : \text{Trm} \to (\mathcal{S} \to \mathbb{R}))\)

The *value of term* $e$ in state $\omega : \mathcal{V} \to \mathbb{R}$ is a real number denoted $\omega \llbracket e \rrbracket$ and is defined by induction on the structure of $e$:

- $\omega \llbracket x \rrbracket = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega \llbracket c \rrbracket = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega \llbracket e + \tilde{e} \rrbracket = \omega \llbracket e \rrbracket + \omega \llbracket \tilde{e} \rrbracket$ addition of reals
- $\omega \llbracket e \cdot \tilde{e} \rrbracket = \omega \llbracket e \rrbracket \cdot \omega \llbracket \tilde{e} \rrbracket$ multiplication of reals
Terms: Syntax & Semantics

Definition (Syntax of terms)

A term $e$ is a polynomial term defined by the grammar:

$$ e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e} $$

Definition (Semantics of terms) \hspace{1cm} \left( \llbracket \cdot \rrbracket : \text{Trm} \rightarrow (\mathcal{S} \rightarrow \mathbb{R}) \right) \\

The value of term $e$ in state $\omega : \mathcal{V} \rightarrow \mathbb{R}$ is a real number denoted $\omega[e]$ and is defined by induction on the structure of $e$:

- $\omega[x] = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega[c] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[e + \tilde{e}] = \omega[e] + \omega[\tilde{e}]$ addition of reals
- $\omega[e \cdot \tilde{e}] = \omega[e] \cdot \omega[\tilde{e}]$ multiplication of reals

$$ \omega\llbracket 4 + x \cdot 2 \rrbracket = \begin{cases} 
\omega(x) & \text{if } \omega(x) = 5 
\end{cases} $$
Definition (Syntax of terms)

A *term* $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms)

The *value* of *term* $e$ in state $\omega : \mathcal{V} \rightarrow \mathbb{R}$ is a real number denoted $\omega[e]$ and is defined by induction on the structure of $e$:

- $\omega[x] = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega[c] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[e + \tilde{e}] = \omega[e] + \omega[\tilde{e}]$ addition of reals
- $\omega[e \cdot \tilde{e}] = \omega[e] \cdot \omega[\tilde{e}]$ multiplication of reals

For example:

$$\omega[4 + x \cdot 2] = \omega[4] + \omega[x] \cdot \omega[2] = 4 + \omega(x) \cdot 2 = 14$$ if $\omega(x) = 5$
A term $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x | c | e + \tilde{e} | e \cdot \tilde{e}$$

The value of term $e$ in state $\omega : \mathcal{V} \rightarrow \mathbb{R}$ is a real number denoted $\omega[e]$ and is defined by induction on the structure of $e$:

- $\omega[x] = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega[c] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[e + \tilde{e}] = \omega[e] + \omega[\tilde{e}]$ addition of reals
- $\omega[e \cdot \tilde{e}] = \omega[e] \cdot \omega[\tilde{e}]$ multiplication of reals

What about $x - y$?
A term \( e \) is a polynomial term defined by the grammar:

\[
e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}
\]

The value of term \( e \) in state \( \omega : \mathcal{V} \rightarrow \mathbb{R} \) is a real number denoted \( \omega[e] \) and is defined by induction on the structure of \( e \):

\[
\begin{align*}
\omega[x] &= \omega(x) & \text{if } x \in \mathcal{V} \text{ is a variable} \\
\omega[c] &= c & \text{if } c \in \mathbb{Q} \text{ is a rational constant} \\
\omega[e + \tilde{e}] &= \omega[e] + \omega[\tilde{e}] & \text{addition of reals} \\
\omega[e \cdot \tilde{e}] &= \omega[e] \cdot \omega[\tilde{e}] & \text{multiplication of reals}
\end{align*}
\]

What about \( x - y \)? Defined as \( x + (-1) \cdot y \)
Definition (Syntax of terms)

A term $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms)

The value of term $e$ in state $\omega : \mathcal{V} \rightarrow \mathbb{R}$ is a real number denoted $\omega[e]$ and is defined by induction on the structure of $e$:

- $\omega[x] = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega[c] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[e + \tilde{e}] = \omega[e] + \omega[\tilde{e}]$ addition of reals
- $\omega[e \cdot \tilde{e}] = \omega[e] \cdot \omega[\tilde{e}]$ multiplication of reals

What about $x^4$?
Terms: Syntax & Semantics

Definition (Syntax of terms)

A term $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

Definition (Semantics of terms) ($[[\cdot]] : \text{Trm} \rightarrow (S \rightarrow \mathbb{R})$)

The value of term $e$ in state $\omega : V \rightarrow \mathbb{R}$ is a real number denoted $\omega[[e]]$ and is defined by induction on the structure of $e$:

- $\omega[[x]] = \omega(x)$ if $x \in V$ is a variable
- $\omega[[c]] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[[e + \tilde{e}]] = \omega[[e]] + \omega[[\tilde{e}]]$ addition of reals
- $\omega[[e \cdot \tilde{e}]] = \omega[[e]] \cdot \omega[[\tilde{e}]]$ multiplication of reals

What about $x^4$? Defined as $x \cdot x \cdot x \cdot x$
A *term* $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x | c | e + \tilde{e} | e \cdot \tilde{e}$$

The *value* of term $e$ in state $\omega : \mathcal{V} \to \mathbb{R}$ is a real number denoted $\omega[e]$ and is defined by induction on the structure of $e$:

- $\omega[x] = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega[c] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[e + \tilde{e}] = \omega[e] + \omega[\tilde{e}]$ addition of reals
- $\omega[e \cdot \tilde{e}] = \omega[e] \cdot \omega[\tilde{e}]$ multiplication of reals

What about $x^n$?
Terms: Syntax & Semantics

**Definition (Syntax of terms)**

A *term* $e$ is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

**Definition (Semantics of terms)**

The *value* of *term* $e$ in state $\omega : \mathcal{V} \to \mathbb{R}$ is a real number denoted $\omega[e]$ and is defined by induction on the structure of $e$:

- $\omega[x] = \omega(x)$ if $x \in \mathcal{V}$ is a variable
- $\omega[c] = c$ if $c \in \mathbb{Q}$ is a rational constant
- $\omega[e + \tilde{e}] = \omega[e] + \omega[\tilde{e}]$ addition of reals
- $\omega[e \cdot \tilde{e}] = \omega[e] \cdot \omega[\tilde{e}]$ multiplication of reals

What about $x^n$? Defined as $x \cdot x \cdot x \cdot x \cdot x \cdot \ldots$, wait when do we stop???
Definition (Syntax of first-order logic formulas)

The formulas of FOL of real arithmetic are defined by the grammar:

\[ P, Q ::= e \geq \bar{e} \mid e = \bar{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x \ P \mid \exists x \ P \]
Definition (Syntax of first-order logic formulas)

The formulas of FOL of real arithmetic are defined by the grammar:

\[ P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x P \mid \exists x P \]

Greater-or-equal  Not  And  Or  Imply  Equiv  All  Exists
Definition (Syntax of first-order logic formulas)

The formulas of FOL of real arithmetic are defined by the grammar:

\[ P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x \ P \mid \exists x \ P \]

Definition (Semantics of first-order logic formulas)

First-order formula \( P \) is true in state \( \omega \), written \( \omega \models P \), defined inductively:

\[ \begin{align*}
\omega \models e = \tilde{e} & \iff \omega[e] = \omega[\tilde{e}] \\
\omega \models e \geq \tilde{e} & \iff \omega[e] \geq \omega[\tilde{e}] \\
\omega \models \neg P & \iff \omega \not\models P, \text{ i.e., if it is not the case that } \omega \models P \\
\omega \models P \land Q & \iff \omega \models P \text{ and } \omega \models Q \\
\omega \models P \lor Q & \iff \omega \models P \text{ or } \omega \models Q \\
\omega \models P \rightarrow Q & \iff \omega \not\models P \text{ or } \omega \models Q \\
\omega \models \forall x \ P & \iff \omega^d_x \models P \text{ for all } d \in \mathbb{R} \\
\omega \models \exists x \ P & \iff \omega^d_x \models P \text{ for some } d \in \mathbb{R} \\
\end{align*} \]

\[ \omega^d_x(y) = \begin{cases} 
  d & \text{ if } y = x \\
  \omega(y) & \text{ if } y \neq x 
\end{cases} \]
First-order logic formulas: Syntax & Semantics

\( \omega \models P \) formula \( P \) is true in state \( \omega \)

\( \models P \) formula \( P \) is valid, i.e., true in all states \( \omega \), i.e., \( \omega \models P \) for all \( \omega \)

\( \models [P] = \{ \omega : \omega \models P \} \) set of all states in which \( P \) is true

**Definition (Semantics of first-order logic formulas)**

First-order formula \( P \) is true in state \( \omega \), written \( \omega \models P \), defined inductively:

\( \omega \models e = \tilde{e} \) iff \( \omega[e] = \omega[\tilde{e}] \)

\( \omega \models e \geq \tilde{e} \) iff \( \omega[e] \geq \omega[\tilde{e}] \)

\( \omega \models \neg P \) iff \( \omega \not\models P \), i.e., if it is not the case that \( \omega \models P \)

\( \omega \models P \land Q \) iff \( \omega \models P \) and \( \omega \models Q \)

\( \omega \models P \lor Q \) iff \( \omega \not\models P \) or \( \omega \models Q \)

\( \omega \models P \rightarrow Q \) iff \( \omega \not\models P \) or \( \omega \models Q \)

\( \omega \models \forall x \ P \) iff \( \omega_x^d \models P \) for all \( d \in \mathbb{R} \)

\( \omega \models \exists x \ P \) iff \( \omega_x^d \models P \) for some \( d \in \mathbb{R} \)

\[ \omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases} \]
First-Order Logic Formulas: Syntax & Semantics

\( \omega \models P \) formula \( P \) is true in state \( \omega \)

\( \models P \) formula \( P \) is valid, i.e., true in all states \( \omega \), i.e., \( \omega \models P \) for all \( \omega \)

\( \llbracket P \rrbracket = \{ \omega : \omega \models P \} \) set of all states in which \( P \) is true

\[ \exists y \ (y^2 \leq x) \quad \text{for } \omega(x) = 5 \text{ and } \nu(x) = -5 \]

**Definition (Semantics of first-order logic formulas)**

First-order formula \( P \) is true in state \( \omega \), written \( \omega \models P \), defined inductively:

- \( \omega \models e = \tilde{e} \) iff \( \omega[e] = \omega[\tilde{e}] \)
- \( \omega \models e \geq \tilde{e} \) iff \( \omega[e] \geq \omega[\tilde{e}] \)
- \( \omega \models \neg P \) iff \( \omega \not\models P \), i.e., if it is not the case that \( \omega \models P \)
- \( \omega \models P \land Q \) iff \( \omega \models P \) and \( \omega \models Q \)
- \( \omega \models P \lor Q \) iff \( \omega \models P \) or \( \omega \models Q \)
- \( \omega \models P \rightarrow Q \) iff \( \omega \not\models P \) or \( \omega \models Q \)
- \( \omega \models \forall x \ P \) iff \( \omega^d_x \models P \) for all \( d \in \mathbb{R} \)
- \( \omega \models \exists x \ P \) iff \( \omega^d_x \models P \) for some \( d \in \mathbb{R} \)

\[ \omega^d_x(y) = \begin{cases} 
  d & \text{if } y = x \\
  \omega(y) & \text{if } y \neq x 
\end{cases} \]
First-Order Logic Formulas: Syntax & Semantics

\[ \omega \models P \text{ formula } P \text{ is true in state } \omega \]
\[ \models P \quad \text{formula } P \text{ is valid, i.e., true in all states } \omega, \text{ i.e., } \omega \models P \text{ for all } \omega \]
\[ \mathcal{P} = \{ \omega : \omega \models P \} \text{ set of all states in which } P \text{ is true} \]

\[ \omega \models \exists y \ (y^2 \leq x) \] but \[ \nu \not\models \exists y \ (y^2 \leq x) \] for \( \omega(x) = 5 \) and \( \nu(x) = -5 \)

**Definition (Semantics of first-order logic formulas)**

First-order formula \( P \) is true in state \( \omega \), written \( \omega \models P \), defined inductively:

\[ \begin{align*}
\omega \models e = \tilde{e} & \quad \text{iff } \omega[e] = \omega[\tilde{e}] \\
\omega \models e \geq \tilde{e} & \quad \text{iff } \omega[e] \geq \omega[\tilde{e}] \\
\omega \models \neg P & \quad \text{iff } \omega \not\models P, \text{ i.e., if it is not the case that } \omega \models P \\
\omega \models P \land Q & \quad \text{iff } \omega \models P \text{ and } \omega \models Q \\
\omega \models P \lor Q & \quad \text{iff } \omega \models P \lor \omega \models Q \\
\omega \models P \rightarrow Q & \quad \text{iff } \omega \not\models P \lor \omega \models Q \\
\omega \models \forall x \ P & \quad \text{iff } \omega_x^d \models P \text{ for all } d \in \mathbb{R} \\
\omega \models \exists x \ P & \quad \text{iff } \omega_x^d \models P \text{ for some } d \in \mathbb{R}
\end{align*} \]

\[ \omega_x^d(y) = \begin{cases} 
    d & \text{if } y = x \\
    \omega(y) & \text{if } y \neq x
\end{cases} \]
Definition (Semantics of differential equations)

A function $\varphi : [0, r] \to S$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \land Q$, written $\varphi |\!|= x' = f(x) \land Q$, iff:

1. $\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$ exists at all times $0 \leq z \leq r$

2. $\varphi(z) |\!|= x' = f(x) \land Q$ for all times $0 \leq z \leq r$

3. $\varphi(z) = \varphi(0)$ except at $x, x'$

André Platzer (CMU)
Definition (Semantics of differential equations)

A function $\varphi : [0, r] \to S$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \& Q$, written $\varphi \models x' = f(x) \wedge Q$, iff:

1. $\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x) \wedge Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at $x, x'$
Definition (Semantics of differential equations)

A function \( \varphi : [0, r] \to S \) of some duration \( r \geq 0 \) satisfies the differential equation \( x' = f(x) \) \& \( Q \), written \( \varphi \models x' = f(x) \land Q \), iff:

1. \( \varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z) \) exists at all times \( 0 \leq z \leq r \)
2. \( \varphi(z) \models x' = f(x) \land Q \) for all times \( 0 \leq z \leq r \)
3. \( \varphi(z) = \varphi(0) \) except at \( x, x' \)
Definition (Semantics of differential equations)

A function $\varphi : [0, r] \rightarrow S$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \land Q$, written $\varphi \models x' = f(x) \land Q$, iff:

1. $\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x) \land Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at $x, x'$
Outline

1. Learning Objectives
2. Introduction
3. Differential Equations
4. Examples of Differential Equations
5. Domains of Differential Equations
   - Terms
   - First-Order Formulas
   - Continuous Programs
6. Summary
Definition (Syntax of terms)

\[ e, \bar{e} ::= x \mid c \mid e + \bar{e} \mid e \cdot \bar{e} \]

Definition (Syntax of first-order logic formulas)

\[ P, Q ::= e \geq \bar{e} \mid e = \bar{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x \, P \mid \exists x \, P \]

Definition (Syntax of continuous programs)

A differential equation \( x' = f(x) \) with evolution domain \( Q \) is denoted by

\[ x' = f(x) \land Q \]
André Platzer.
*Logical Foundations of Cyber-Physical Systems.*
URL: http://www.springer.com/978-3-319-63587-3,
doi:10.1007/978-3-319-63588-0.

André Platzer.
*Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.*
doi:10.1007/978-3-642-14509-4.