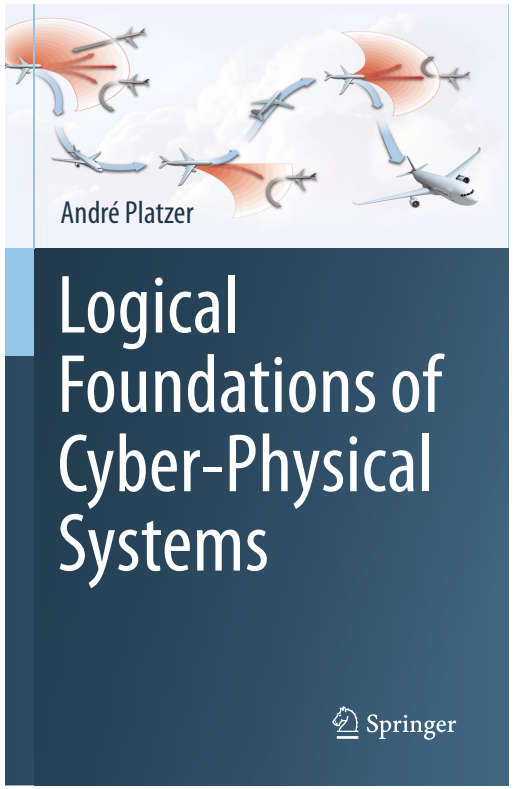
03: Choice & Control

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

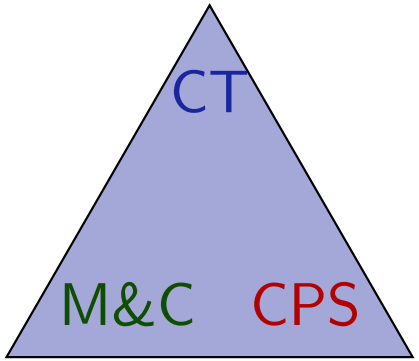
Outline

Outline

Learning Objectives

Choice & Control

nondeterminism
abstraction
programming languages for CPS
semantics
compositionality



models
core principles
discrete+
continuous

operational effect
operational precision

Outline

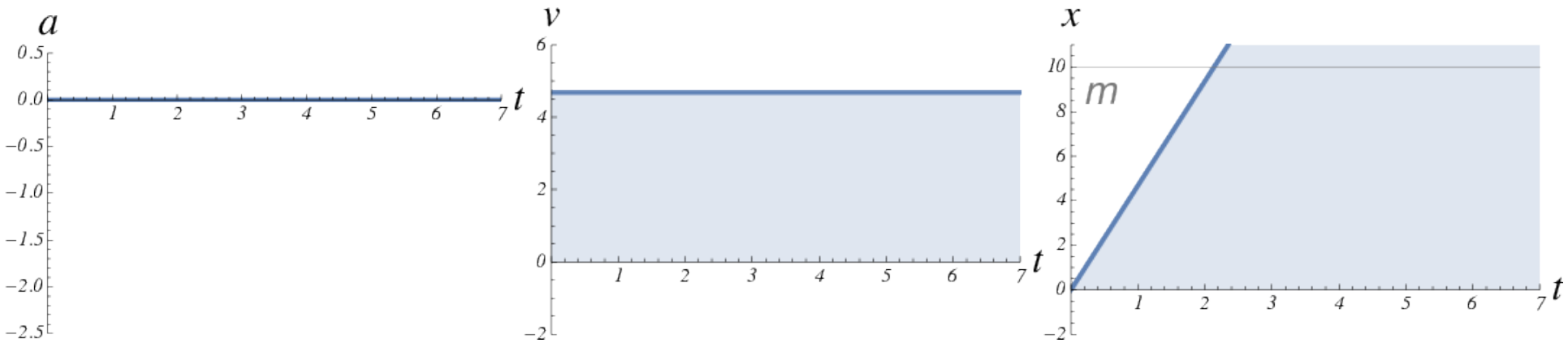
Playing with Acceleration and Braking

Example (Speedy the point)

$$\{x' = v, v' = a\}$$

Purely continuous dynamics

What about the cyber?



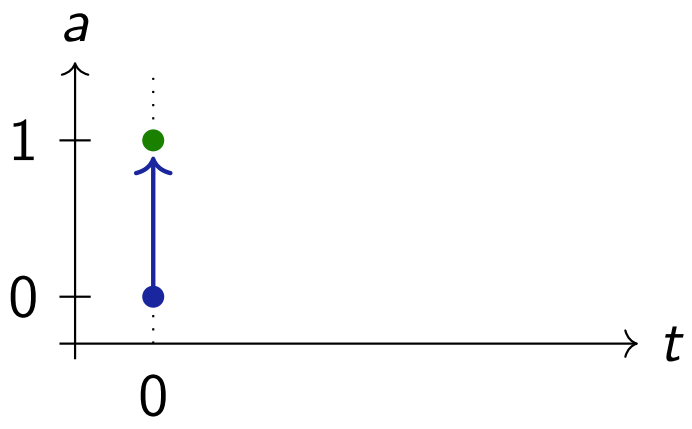
Playing with Acceleration and Braking

Example (Speedy the point)

$$a := a + 1$$

Purely discrete dynamics

How do both meet?

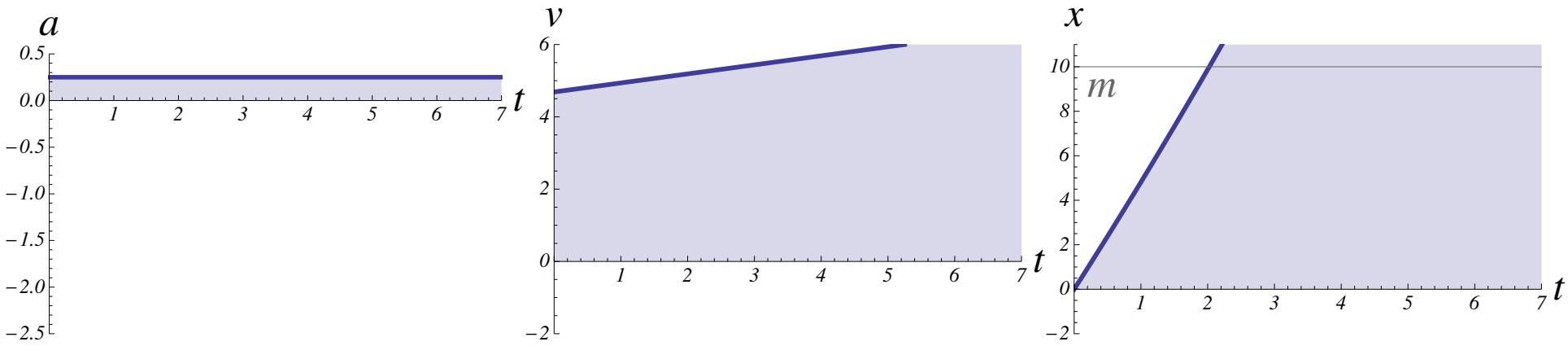


Playing with Acceleration and Braking

Example (Speedy the point)

$$a := a + 1;\ \{x' = v, v' = a\}$$

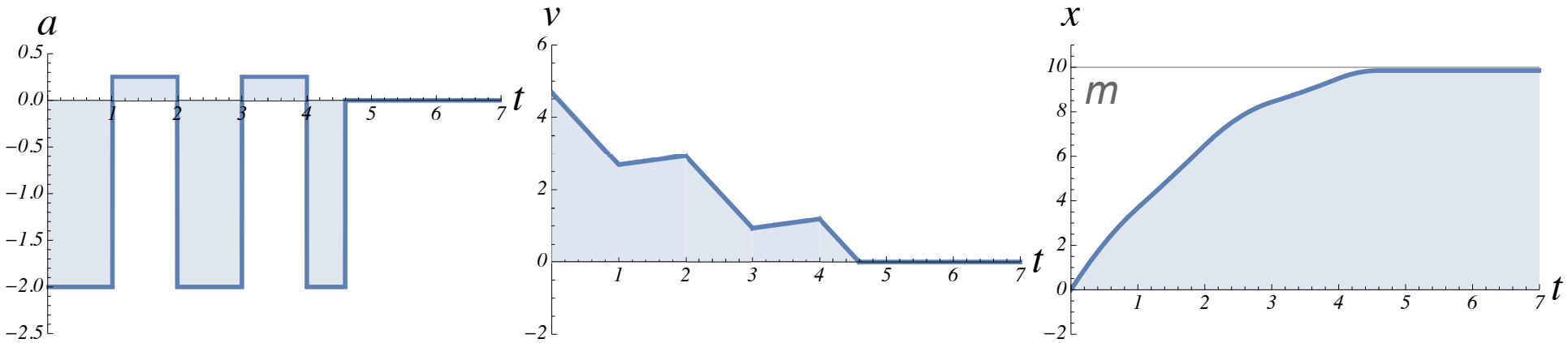
Hybrid dynamics, i.e., composition of continuous and discrete dynamics
Here: sequential composition first;second



Playing with Acceleration and Braking

Example (Speedy the point)

$$
\begin{aligned}
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\};\\
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\};\\
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\}
\end{aligned}
$$

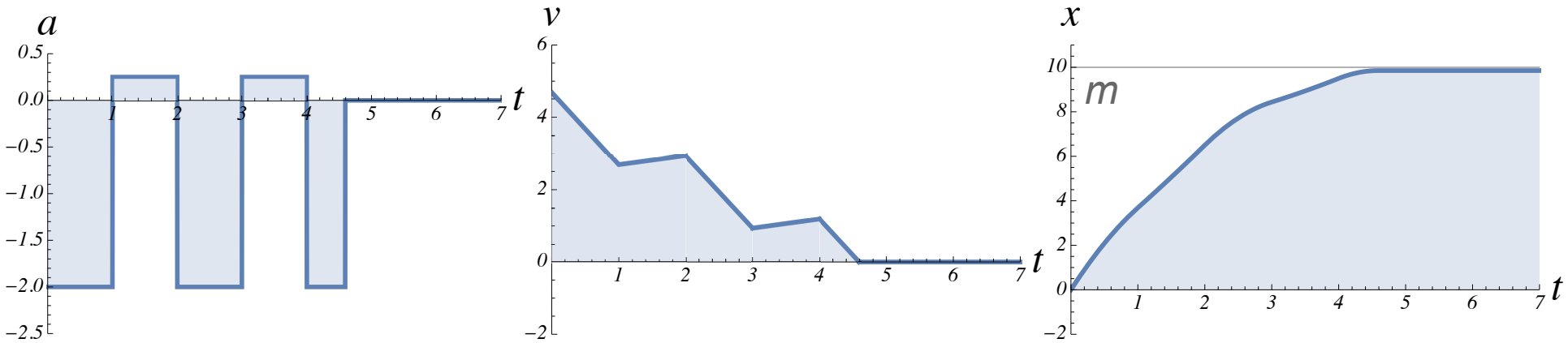


Playing with Acceleration and Braking

Example (Speedy the point)

$$
\begin{aligned}
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\};\\
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\};\\
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\}
\end{aligned}
$$

How long to follow an ODE?

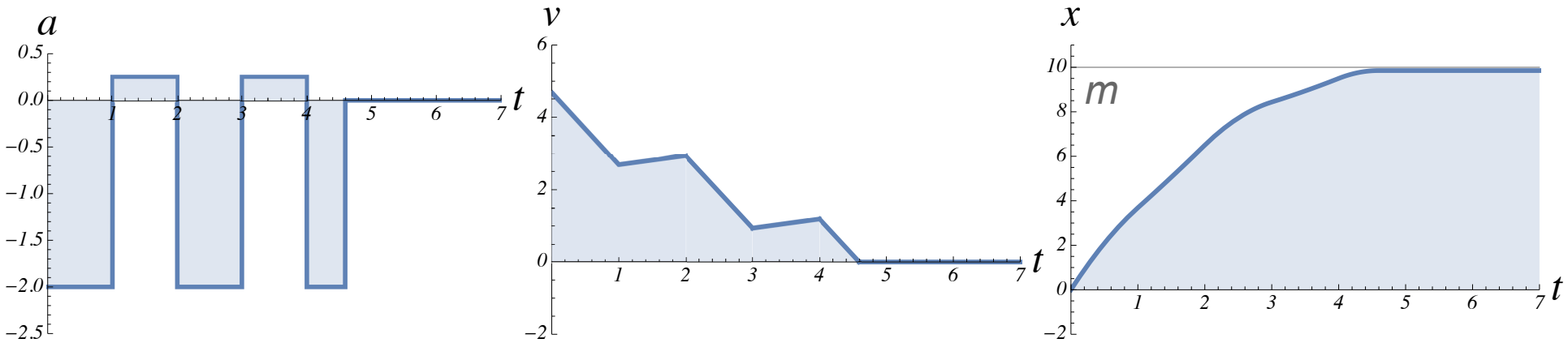


Playing with Acceleration and Braking

Example (Speedy the point)

$$
\begin{aligned}
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25;\{x' = v, v' = a\};\\
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25;\{x' = v, v' = a\};\\
&a := -2;\ \ \{x' = v, v' = a\};\\
&a := 0.25;\{x' = v, v' = a\}
\end{aligned}
$$

How to check conditions before actions?



Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(v < 4)\, a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Velocity-dependent control

Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(x - m > s)\, a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Distance-dependent control for obstacle m

Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(x - m > s \wedge v < 4)\, a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Velocity and distance-dependent control

Iterative Design

Start as simple as possible, then add challenges once basics are correct.

Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(x - m > s \wedge v < 4 \wedge \text{efficiency})\ a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Also only accelerate if it's efficient to do so

Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(x - m > s \wedge v < 4 \wedge \text{efficiency})\ a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Exact models are unnecessarily complex. Not all features are safety-critical.

Playing with Acceleration and Braking

Example (Speedy the point)

$$(a := a + 1 \ \cup \ a := -b);$$
$$\{x' = v, v' = a\}$$

Nondeterministic choice $\cup$ allows either side to be run, arbitrarily

Power of Abstraction

Only include relevant aspects, elide irrelevant detail.
The model and its analysis become simpler. And apply to more systems.

Playing with Acceleration and Braking

Example (Speedy the point)

$$(a := a + 1 \ \cup \ a := -b);$$
$$\{x' = v, v' = a\}$$

Nondeterministic choice $\cup$ allows either side to be run, arbitrarily

Oops, now it got too simple! Not every choice is always acceptable.

Playing with Acceleration and Braking

Example (Speedy the point)

$$(?v < 4; a := a + 1 \ \cup \ a := -b);$$
$$\{x' = v, v' = a\}$$

Test $?Q$ checks if formula Q is true in current state

Playing with Acceleration and Braking

Example (Speedy the point)

$$(?v < 4; a := a + 1 \ \cup \ a := -b);$$
$$\{x' = v, v' = a\}$$

Test $?Q$ checks if formula Q is true in current state, otherwise run fails.

Discarding failed runs and backtracking

System runs that fail tests are discarded and not considered further.

$?v < 4;\ v := v + 1$ only runs if
$v := v + 1;\ ?v < 4$ only runs if

Broader significance of nondeterminism

Nondeterminism is a tool for abstraction to focus on critical aspects.
Nondeterminism is essential to describe imperfectly known environment.

Playing with Acceleration and Braking

Example (Speedy the point)

$$(?v < 4; a := a + 1 \ \cup \ a := -b);$$
$$\{x' = v, v' = a\}$$

Test $?Q$ checks if formula Q is true in current state, otherwise run fails.

Discarding failed runs and backtracking

System runs that fail tests are discarded and not considered further.

$?v < 4;\ v := v + 1$	only runs if $v < 4$ initially true
$v := v + 1;\ ?v < 4$	only runs if $v < 3$ initially true

Broader significance of nondeterminism

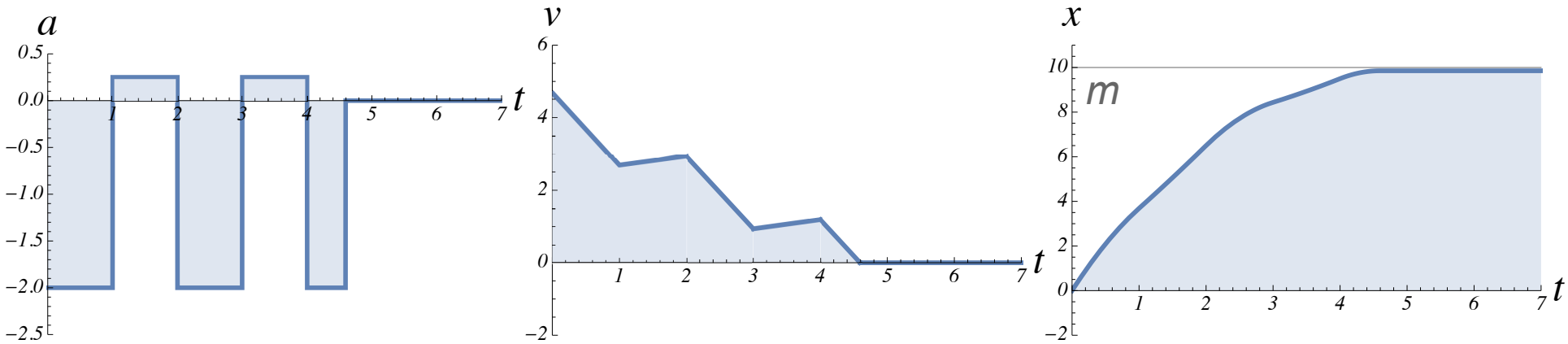
Nondeterminism is a tool for abstraction to focus on critical aspects.
Nondeterminism is essential to describe imperfectly known environment.

Playing with Acceleration and Braking

Example (Speedy the point)

$$
\begin{array}{l}
(?v < 4; a := a + 1 \cup a := -b); \\
\{x' = v, v' = a\}; \\
(?v < 4; a := a + 1 \cup a := -b); \\
\{x' = v, v' = a\}; \\
(?v < 4; a := a + 1 \cup a := -b); \\
\{x' = v, v' = a\}
\end{array}
$$

Repeated control needs longer programs, e.g., by copy&paste

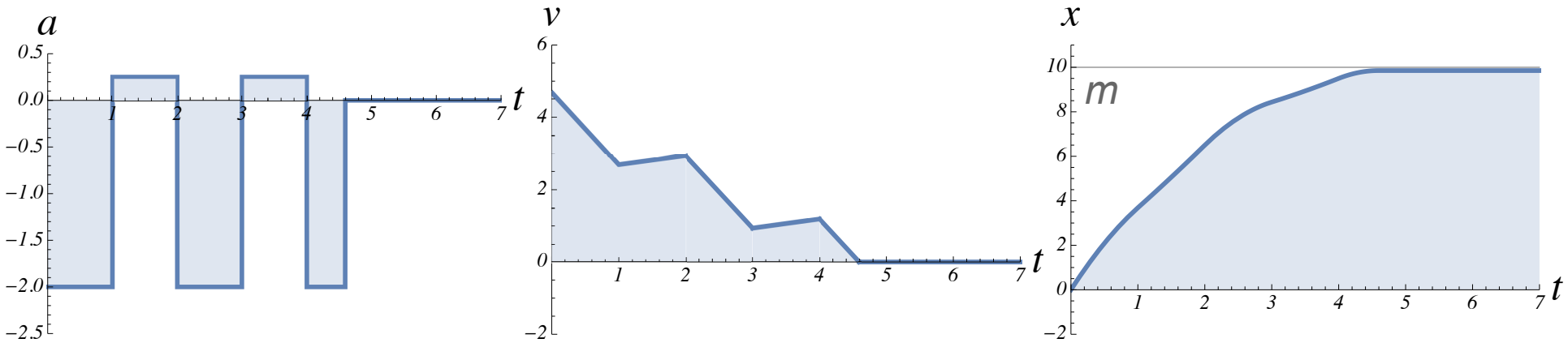


Playing with Acceleration and Braking

Example (Speedy the point)

$$\big((?v < 4; a := a + 1 \cup a := -b);\ \{x' = v, v' = a\}\big)^*$$

Nondeterministic repetition * repeats *any* arbitrary number of times



Outline

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

- $x := e$ — Discrete Assign
- $?Q$ — Test Condition
- $x' = f(x) \,\&\, Q$ — Differential Equation
- $\alpha \cup \beta$ — Nondet. Choice
- $\alpha; \beta$ — Seq. Compose
- α^* — Nondet. Repeat

Hybrid Programs: Syntax

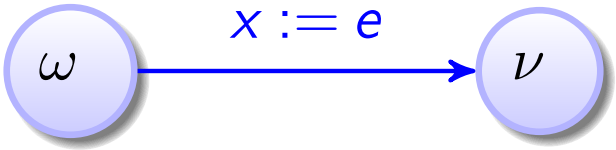
Definition (Syntax of hybrid program α)

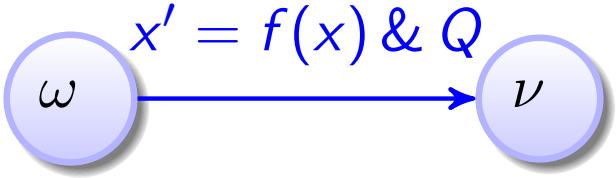
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

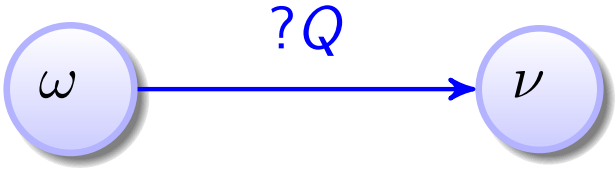
- $x := e$ — Discrete Assign
- $?Q$ — Test Condition
- $x' = f(x) \,\&\, Q$ — Differential Equation
- $\alpha \cup \beta$ — Nondet. Choice
- $\alpha; \beta$ — Seq. Compose
- α^* — Nondet. Repeat

Like regular expressions. Everything nondeterministic

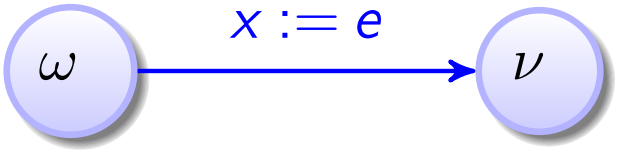
Hybrid Programs: Semantics

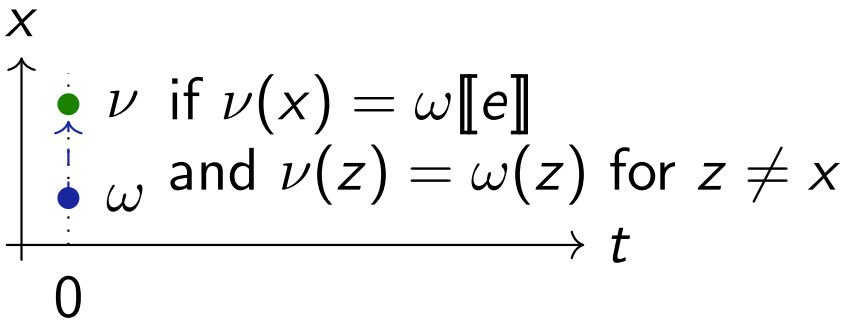


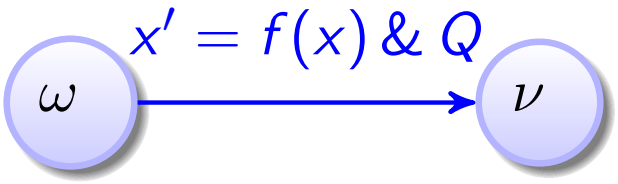


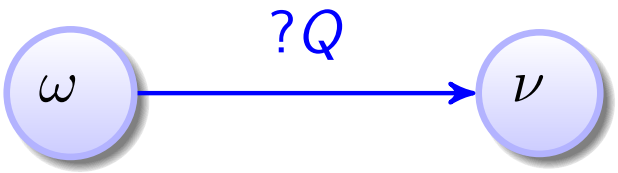


Hybrid Programs: Semantics

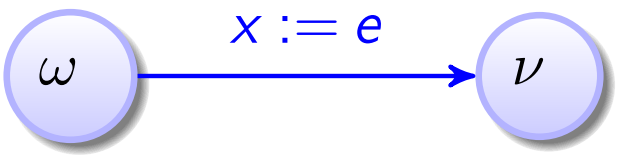


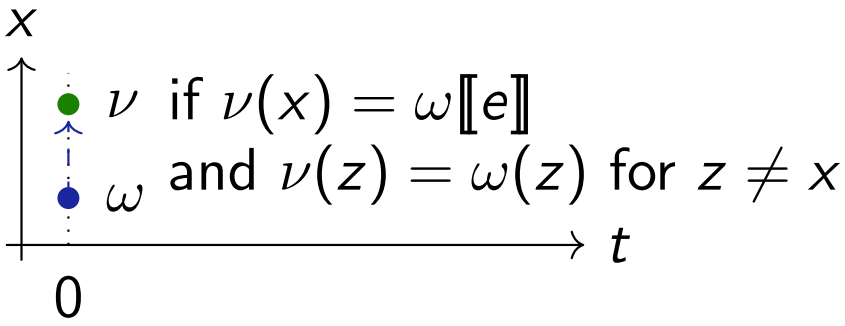


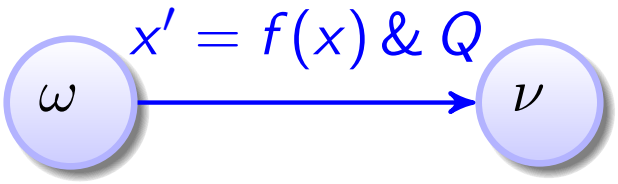


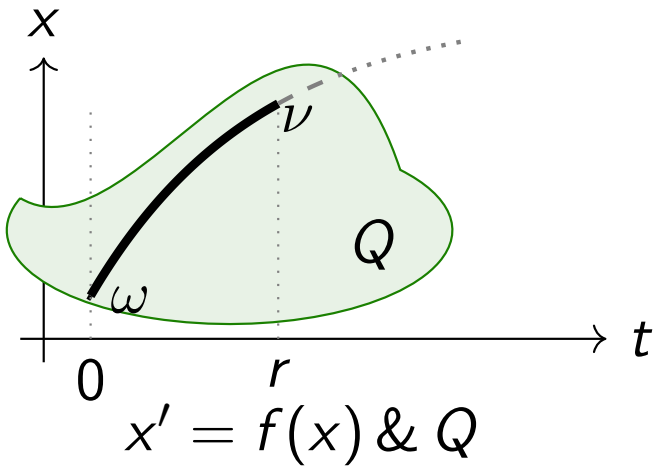


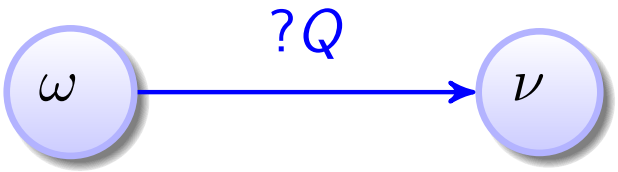
Hybrid Programs: Semantics



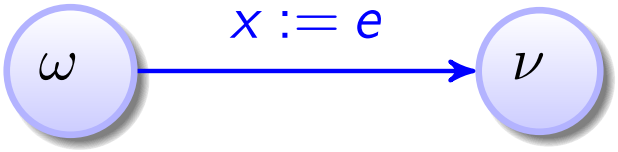


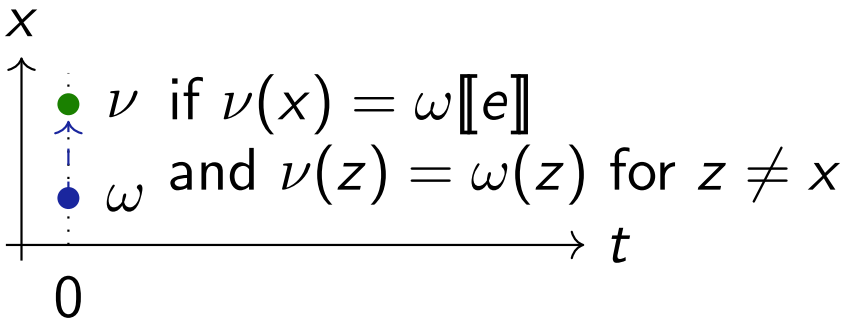


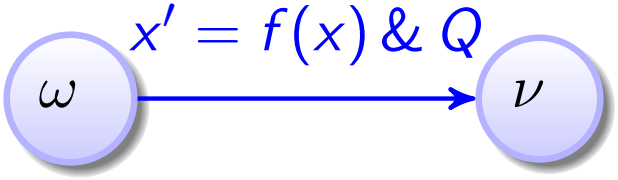


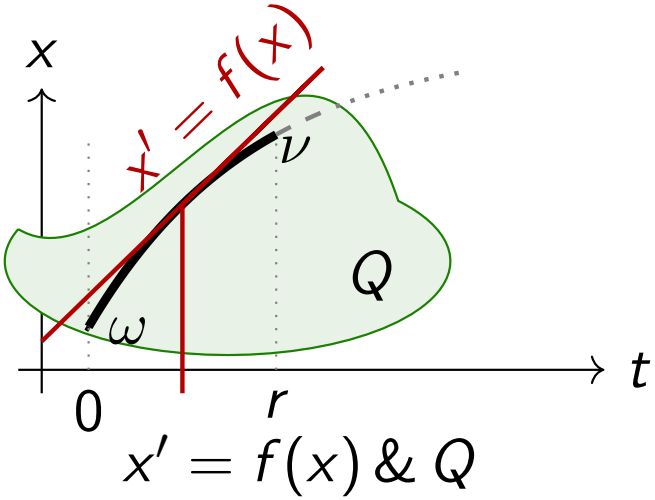


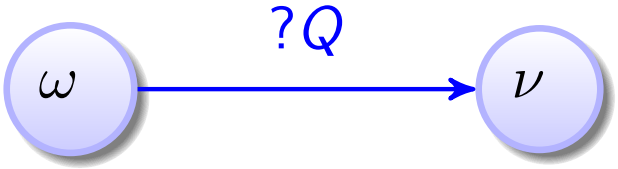
Hybrid Programs: Semantics



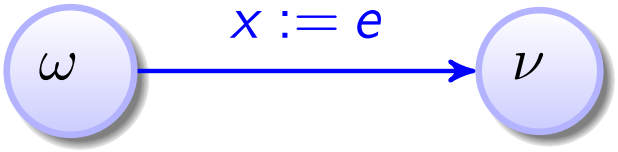


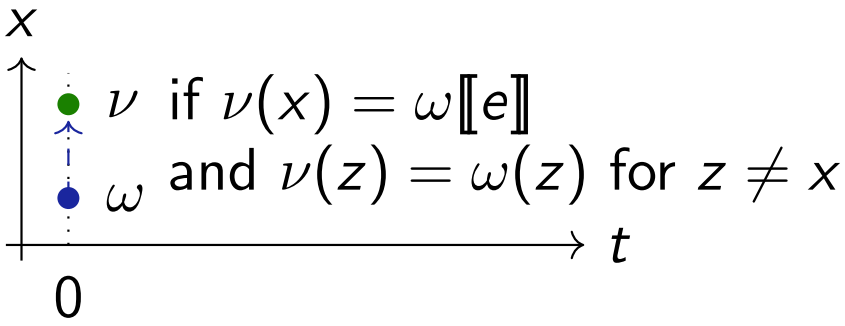


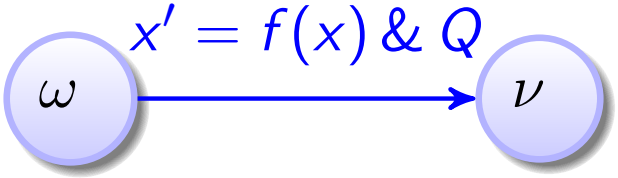


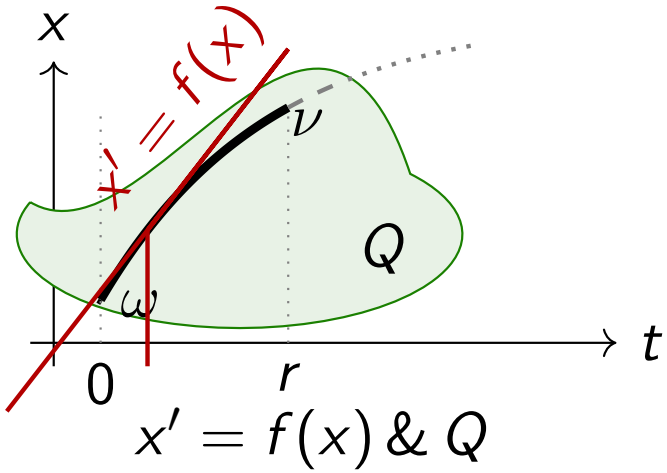


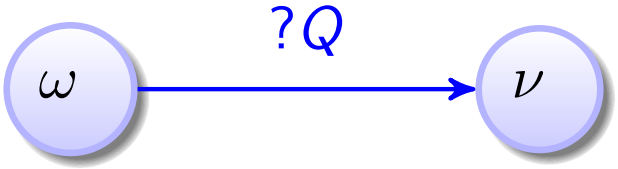
Hybrid Programs: Semantics



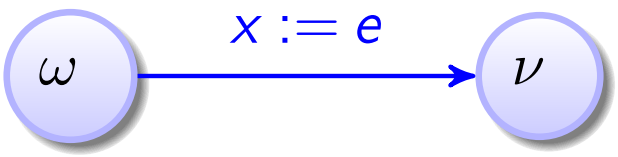


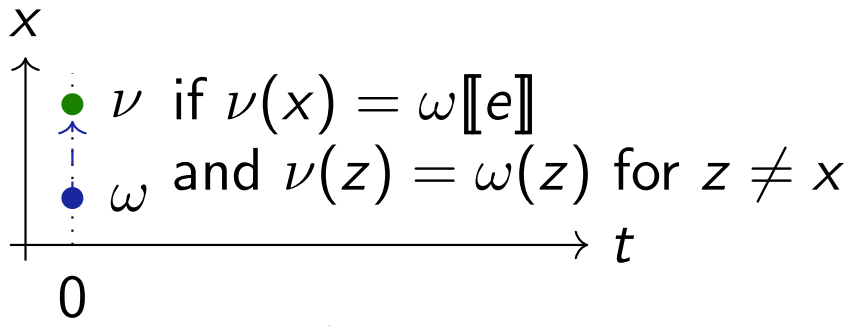


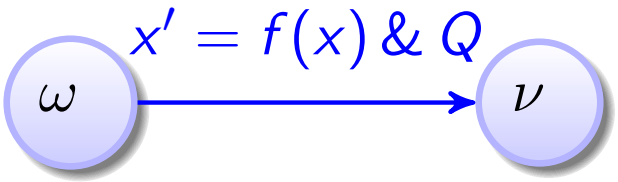


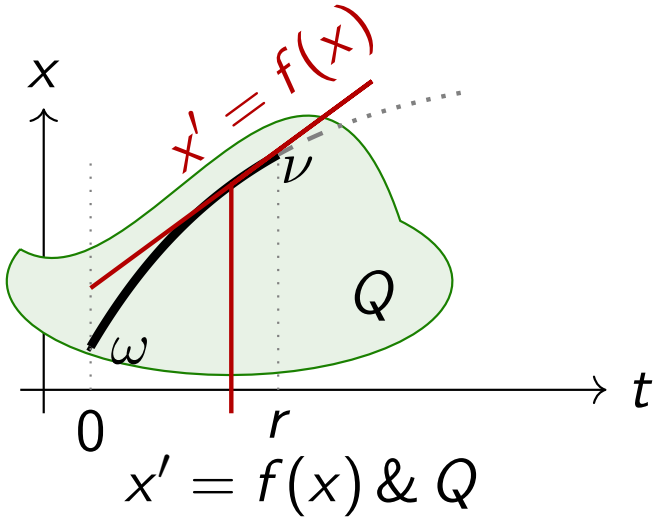


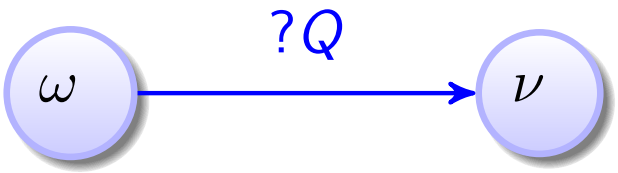
Hybrid Programs: Semantics



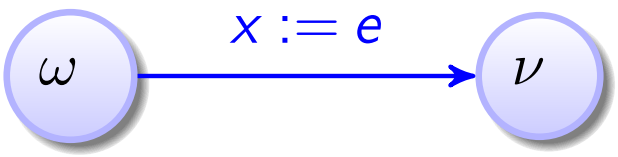


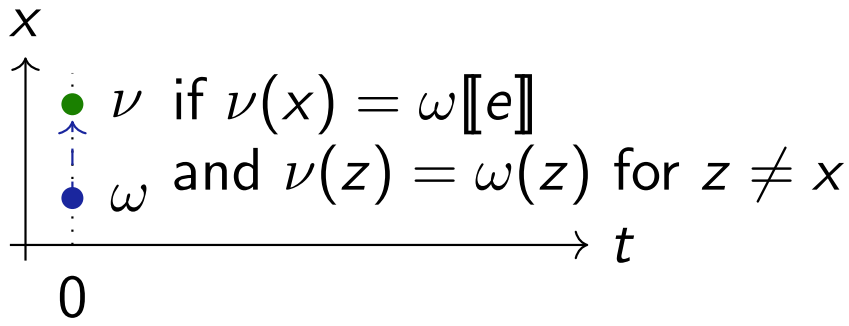


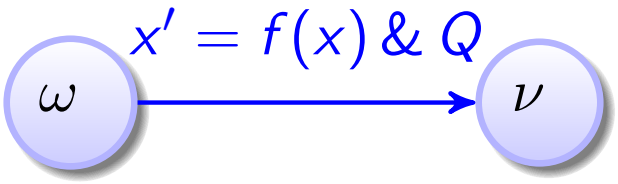


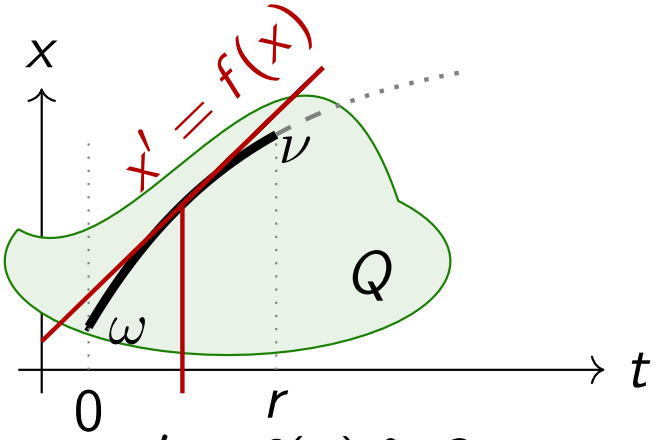


Hybrid Programs: Semantics

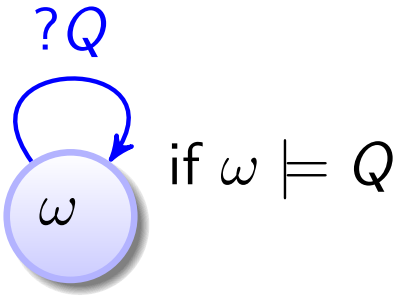


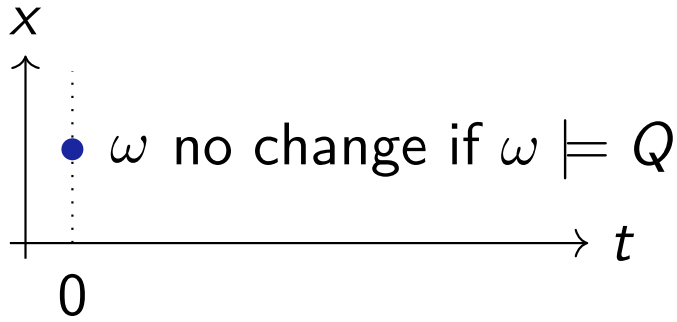




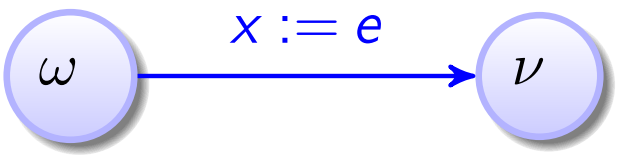


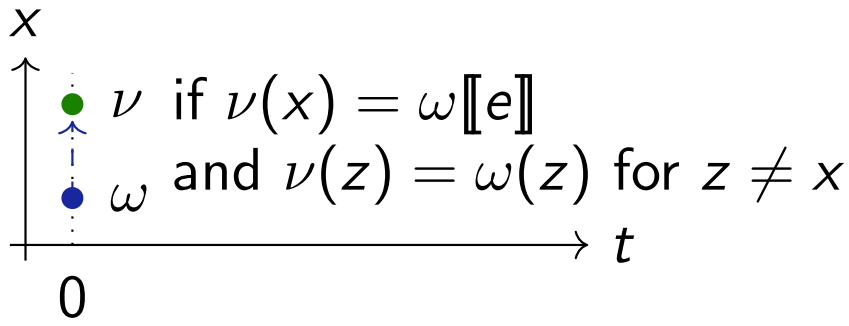
$x' = f(x) \,\&\, Q$

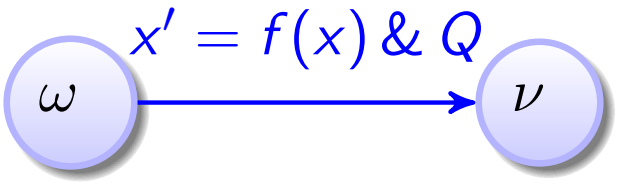


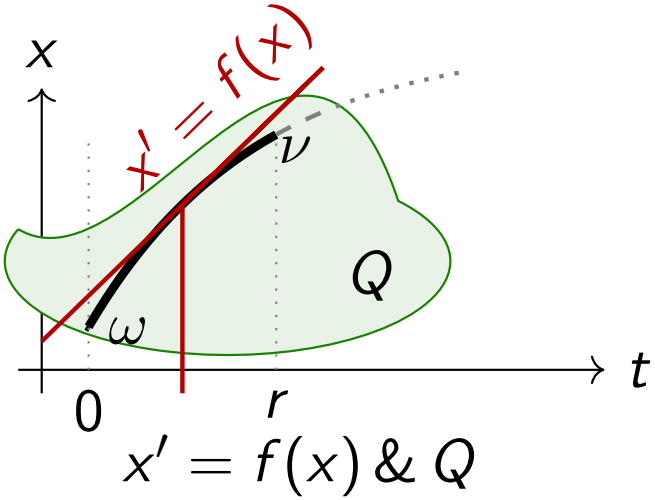


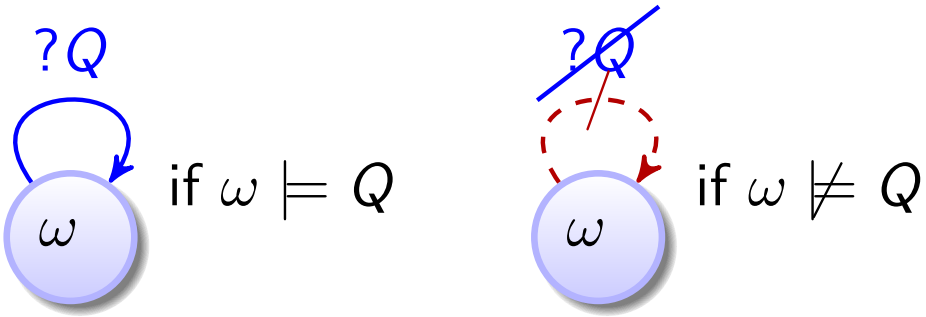
Hybrid Programs: Semantics

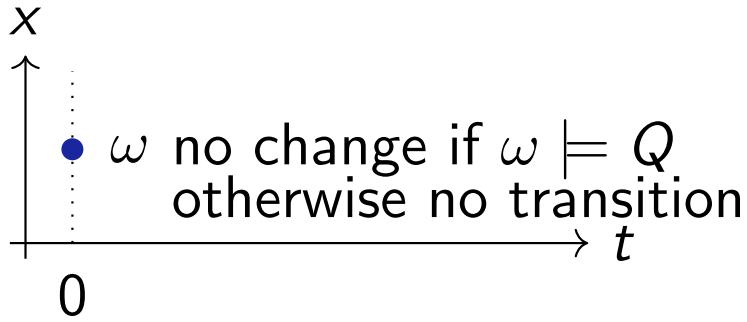




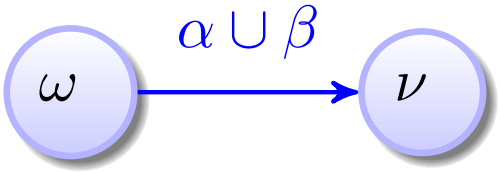


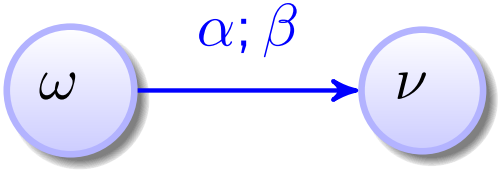


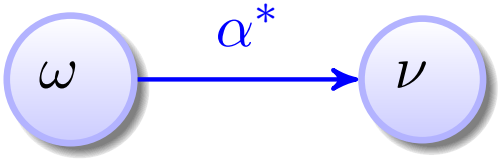




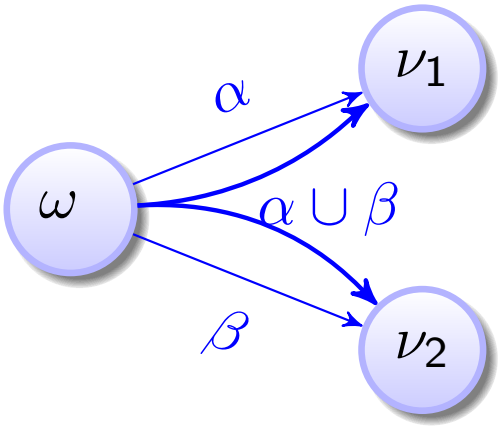
Hybrid Programs: Semantics

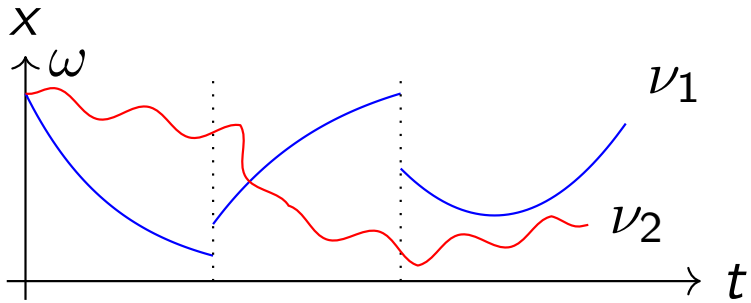


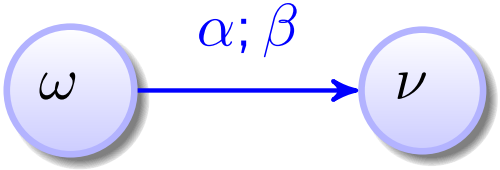


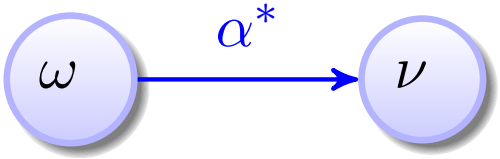


Hybrid Programs: Semantics

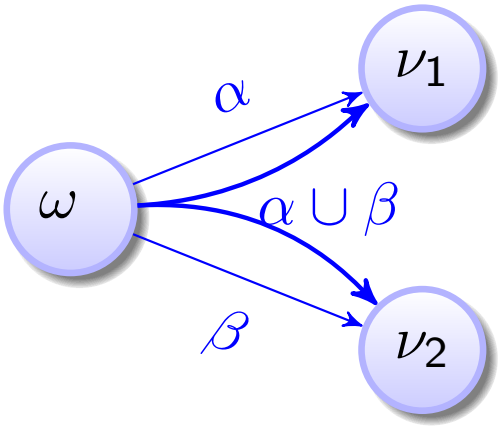


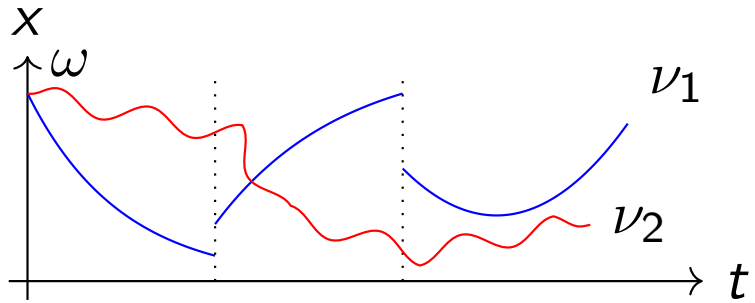


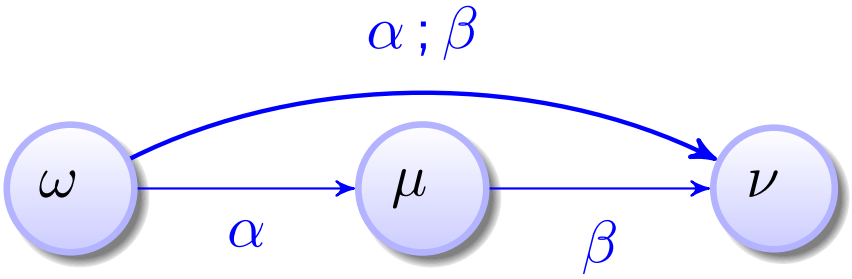


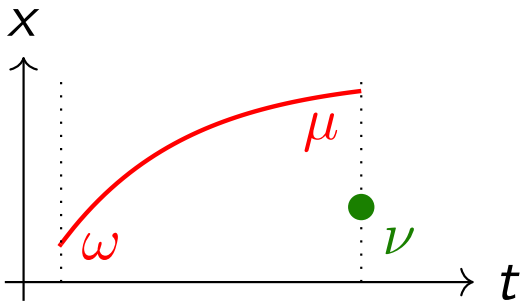


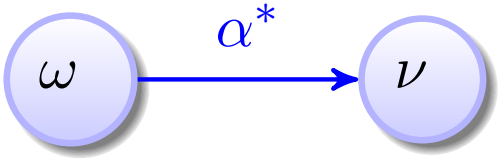
Hybrid Programs: Semantics



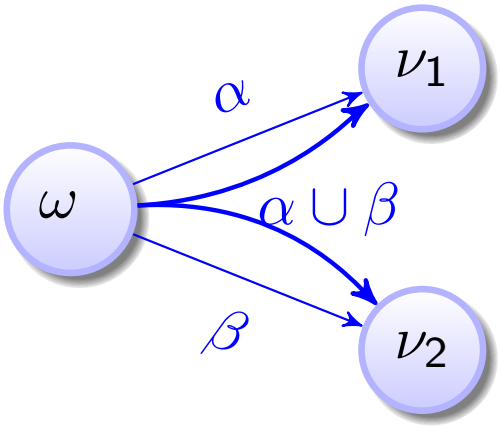


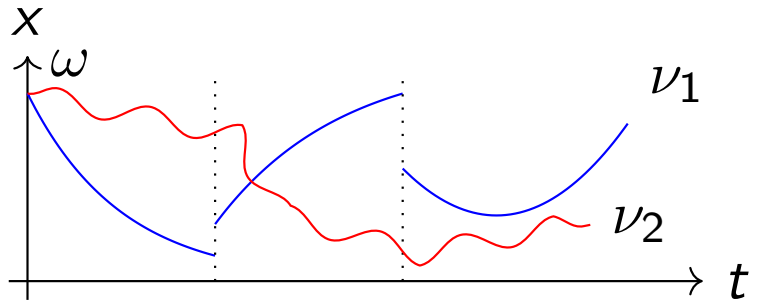


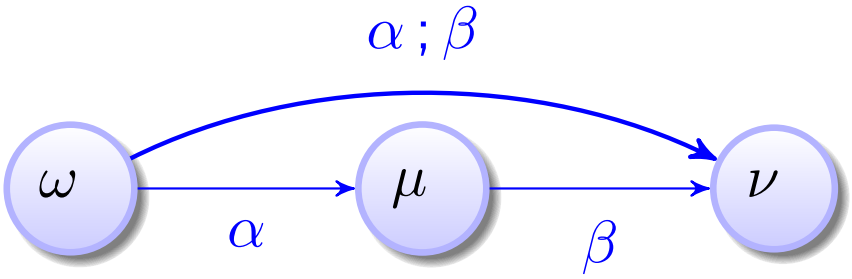


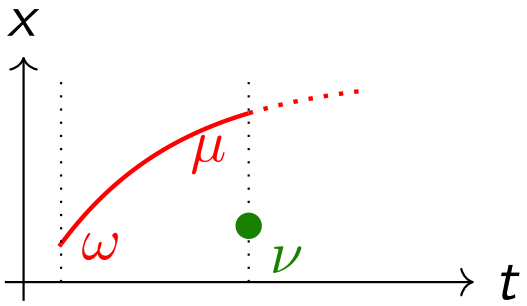


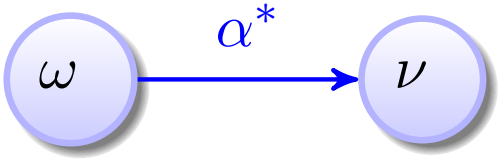
Hybrid Programs: Semantics



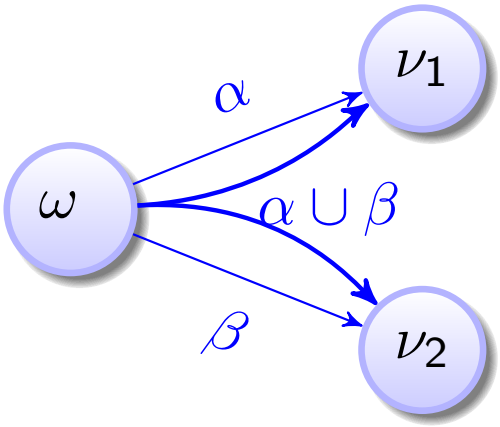


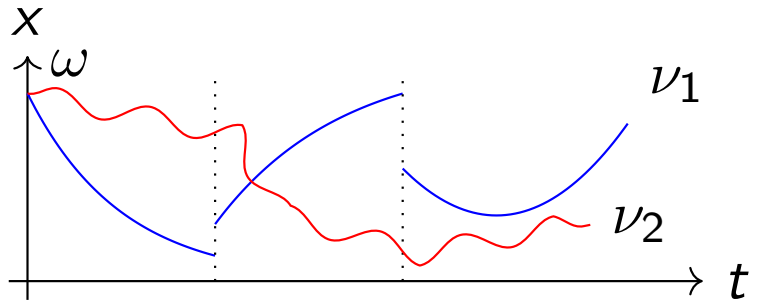


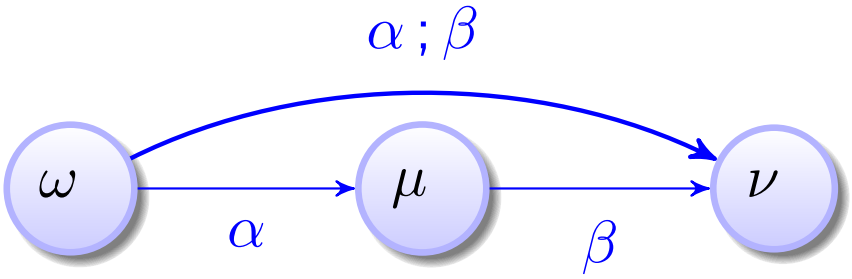


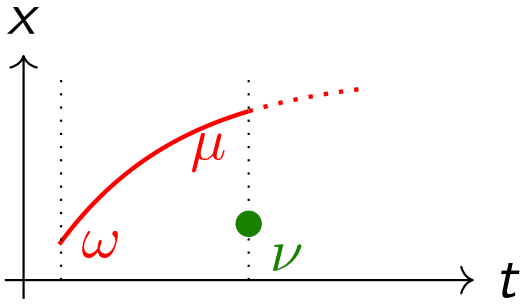


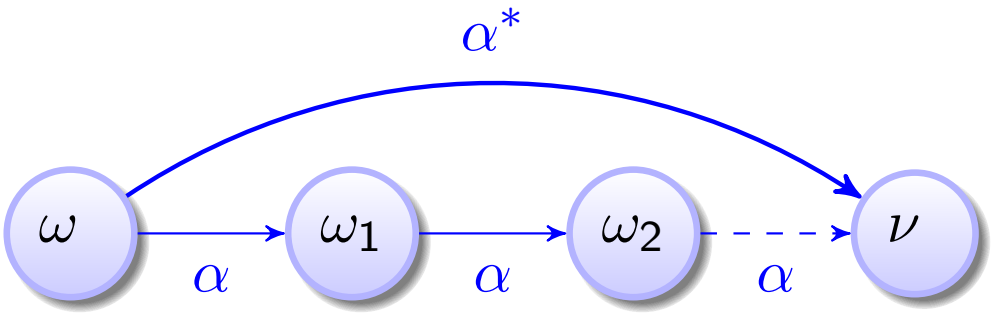
Hybrid Programs: Semantics

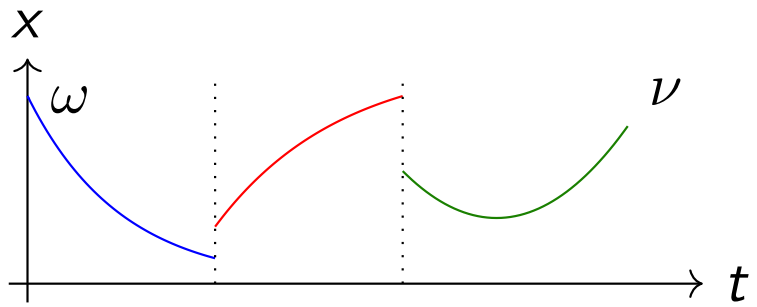




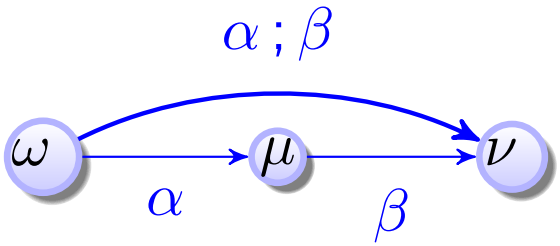


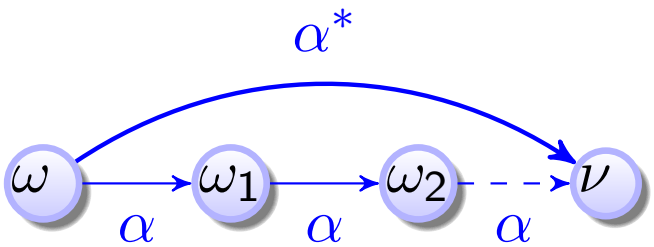


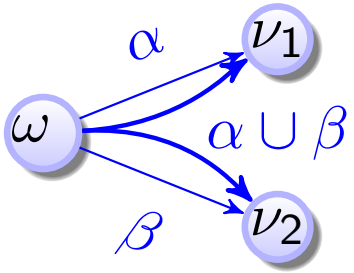




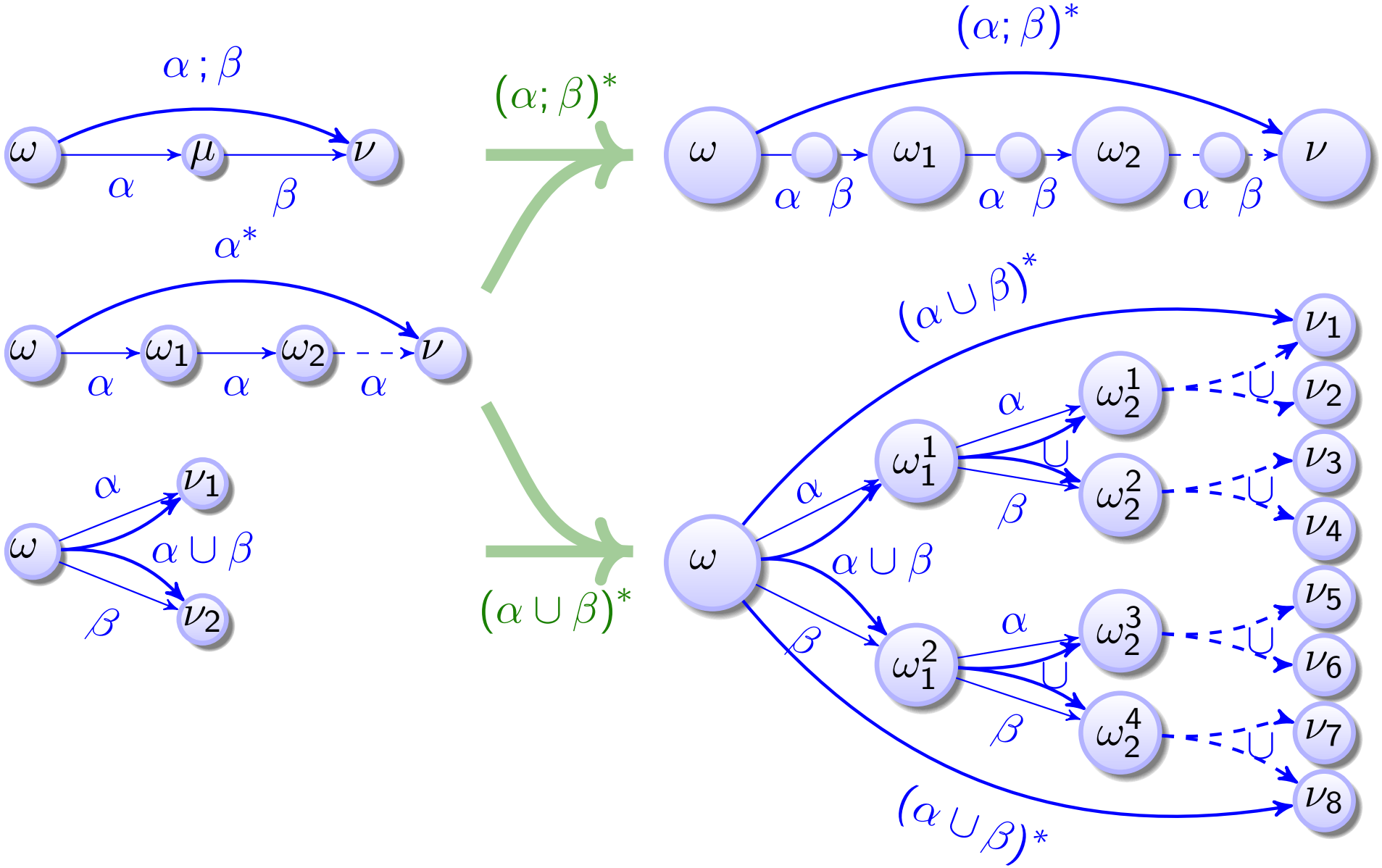
Plug-in for Semantics of Composed Hybrid Programs







Plug-in for Semantics of Composed Hybrid Programs



Hybrid Programs: Syntax & Semantics

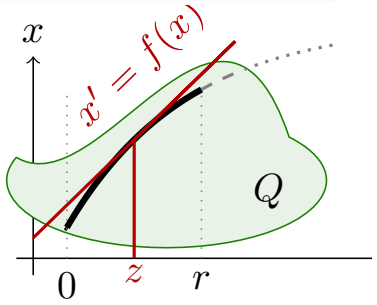
Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) $(\llbracket \cdot \rrbracket : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

$$\begin{aligned}
\llbracket x := e \rrbracket &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu\llbracket x \rrbracket = \omega\llbracket e \rrbracket\} \\
\llbracket ?Q \rrbracket &= \{(\omega, \omega) \ : \ \omega \models Q\} \\
\llbracket x' = f(x) \rrbracket &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
\llbracket \alpha \cup \beta \rrbracket &= \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket \\
\llbracket \alpha; \beta \rrbracket &= \llbracket \alpha \rrbracket \circ \llbracket \beta \rrbracket = \{(\omega, \nu) \ : \ (\omega, \mu) \in \llbracket \alpha \rrbracket \text{ and } (\mu, \nu) \in \llbracket \beta \rrbracket\} \\
\llbracket \alpha^* \rrbracket &= \llbracket \alpha \rrbracket^* = \bigcup_{n \in \mathbb{N}} \llbracket \alpha^n \rrbracket \qquad \alpha^n \equiv \underbrace{\alpha; \alpha; \alpha; \ldots; \alpha}_{n \text{ times}}
\end{aligned}$$

compositional



Hybrid Programs: Syntax & Semantics

Definition (Syntax of hybrid program α)

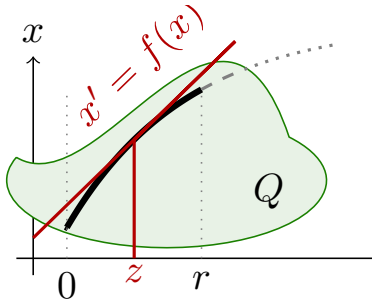
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) $(\llbracket \cdot \rrbracket : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

$$\begin{aligned}
\llbracket x := e \rrbracket &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu\llbracket x \rrbracket = \omega\llbracket e \rrbracket\} \\
\llbracket ?Q \rrbracket &= \{(\omega, \omega) \ : \ \omega \models Q\} \\
\llbracket x' = f(x) \rrbracket &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
\llbracket \alpha \cup \beta \rrbracket &= \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket \\
\llbracket \alpha; \beta \rrbracket &= \llbracket \alpha \rrbracket \circ \llbracket \beta \rrbracket \\
\llbracket \alpha^* \rrbracket &= \llbracket \alpha \rrbracket^* = \bigcup_{n \in \mathbb{N}} \llbracket \alpha^n \rrbracket
\end{aligned}$$

compositional

1. $\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x) \wedge Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'



Notational Conventions: Names

Example (Naming Conventions)

Letters	Convention
x, y, z	variables
$e, \tilde{e}$	terms
P, Q	formulas
α, β	programs
c	constant symbols
f, g, h	function symbols
p, q, r	predicate symbols

In CPS applications, all bets are off because names follow application: x position v velocity and a acceleration variables

Notational Conventions: Precedence

Convention (Operator Precedence)

1. Unary operators (including *, $\neg$ and $\forall x, \exists x$) bind stronger than binary.
2. $\wedge$ binds stronger than $\vee$, which binds stronger than $\rightarrow, \leftrightarrow$
3. ; binds stronger than $\cup$
4. Arithmetic operators $+, -, \cdot$ associate to the left
5. Logical and program operators associate to the right

Example (Operator Precedence)

$\forall x\, P \wedge Q \equiv (\forall x\, P) \wedge Q$ $\qquad$ $\forall x\, P \rightarrow Q \equiv (\forall x\, P) \rightarrow Q$.

$\alpha; \beta \cup \gamma \equiv (\alpha; \beta) \cup \gamma$ $\qquad$ $\alpha \cup \beta; \gamma \equiv \alpha \cup (\beta; \gamma)$ $\qquad$ $\alpha; \beta^* \equiv \alpha; (\beta^*)$

$P \rightarrow Q \rightarrow R \equiv P \rightarrow (Q \rightarrow R)$.

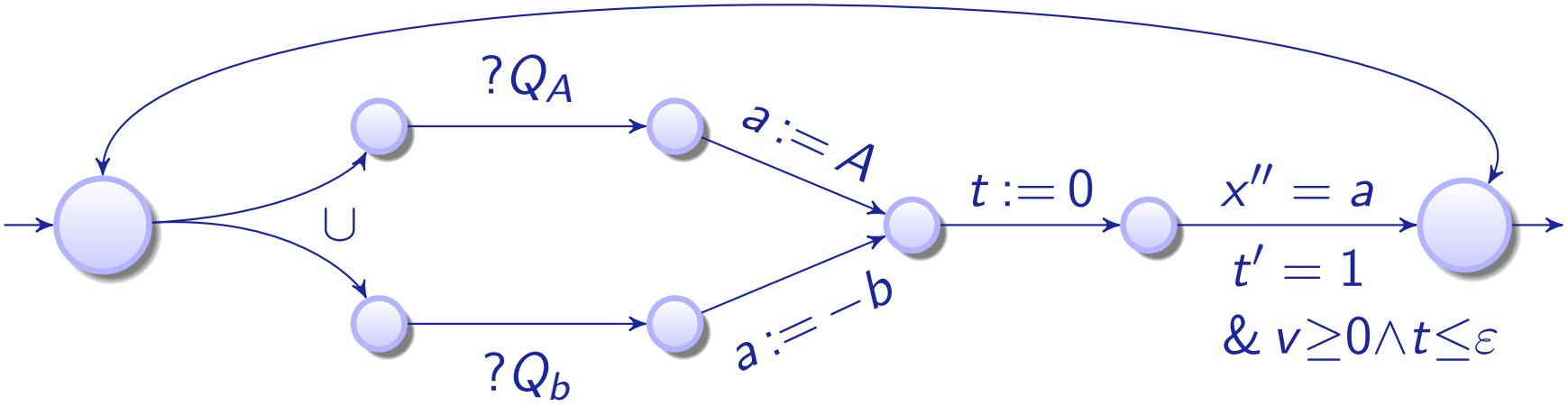
But $\rightarrow, \leftrightarrow$ expect explicit parentheses. Illegal: $P \rightarrow Q \leftrightarrow R$ $\quad$ $P \leftrightarrow Q \rightarrow R$

Outline

Branching Transition Structure in Hybrid Programs

$$
\begin{aligned}
\text{Robot} &\equiv (\text{ctrl}\,;\text{drive})^* \\
\text{ctrl} &\equiv (?Q_A; a := A) \\
&\quad \cup (?Q_b; a := -b) \\
\text{drive} &\equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}
\end{aligned}
$$

Branching Transition Structure in Hybrid Programs

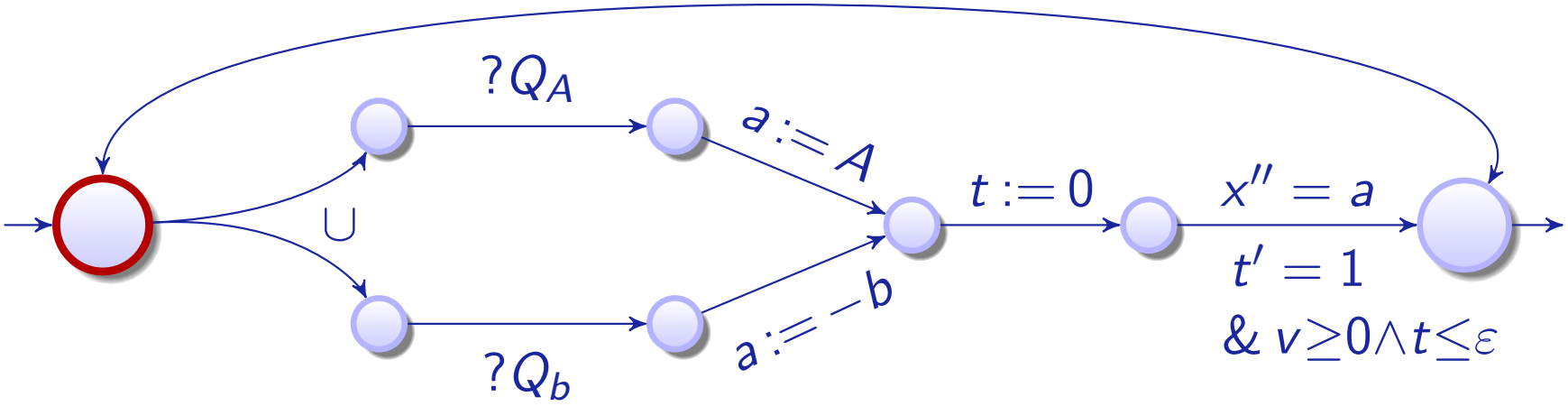


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

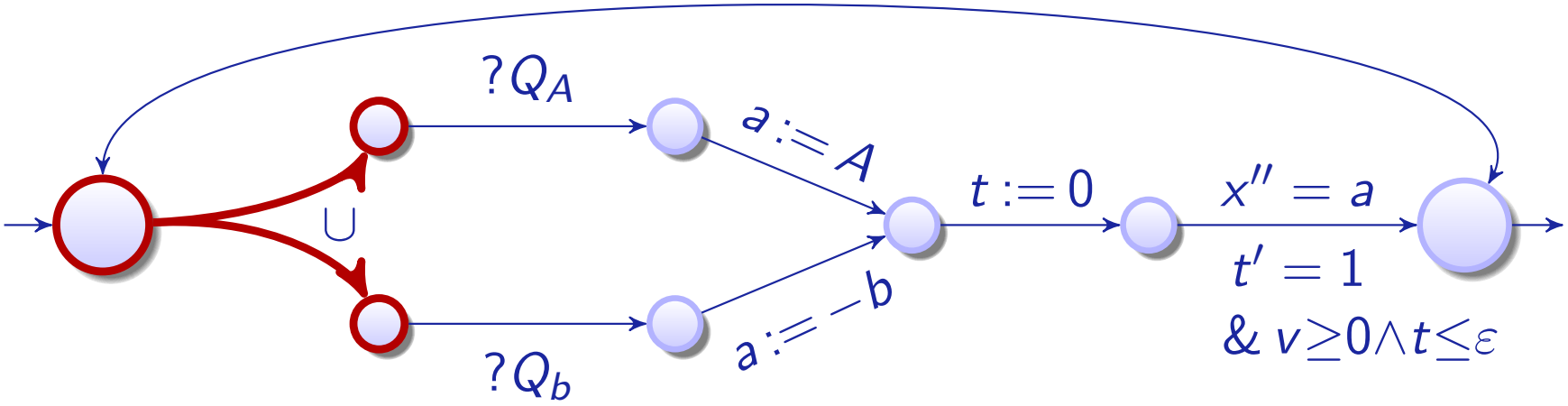


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

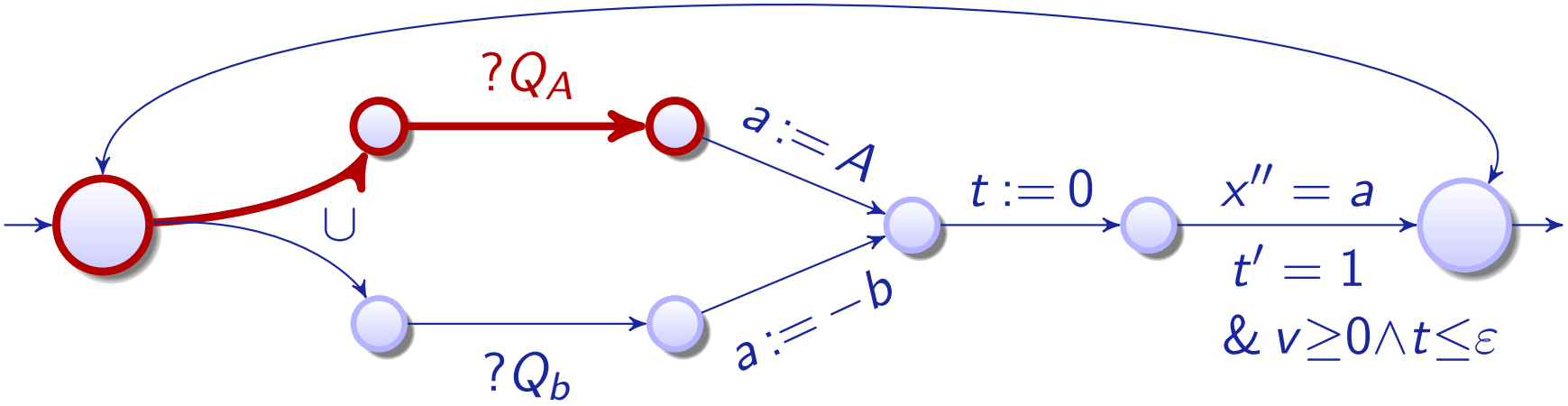


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

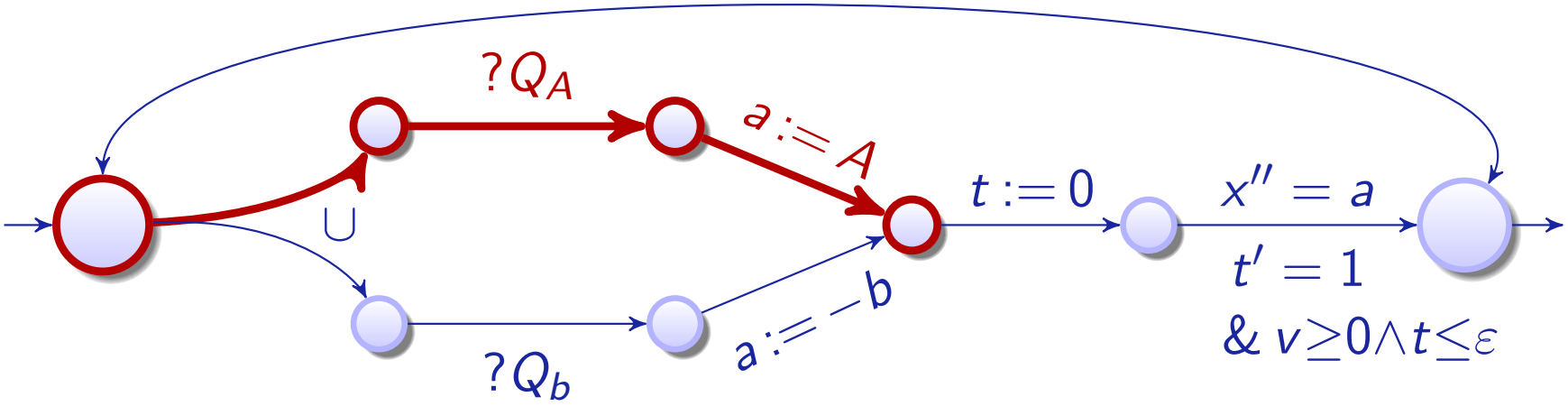


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

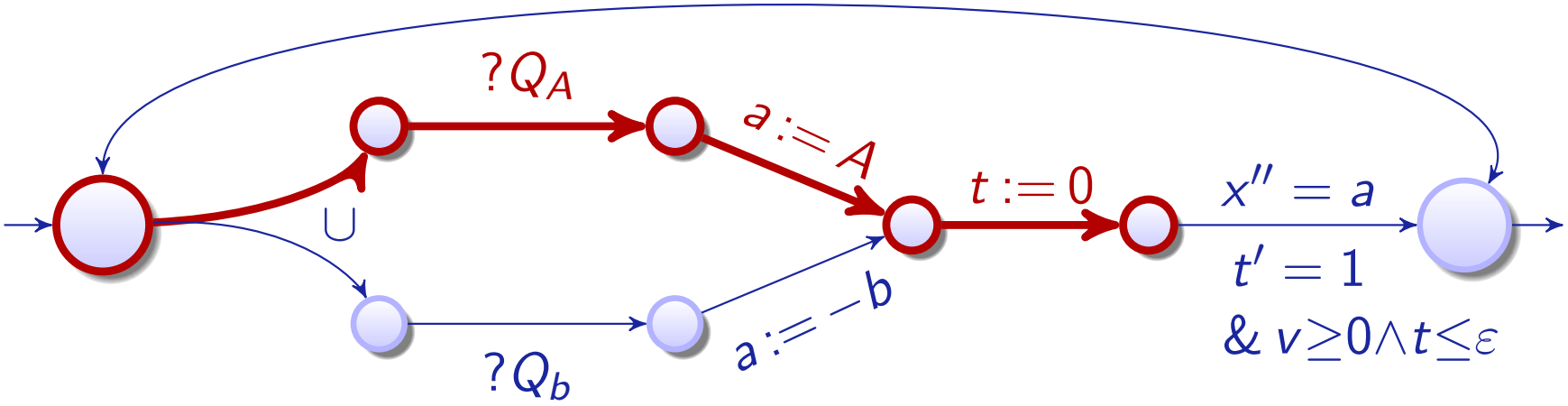


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

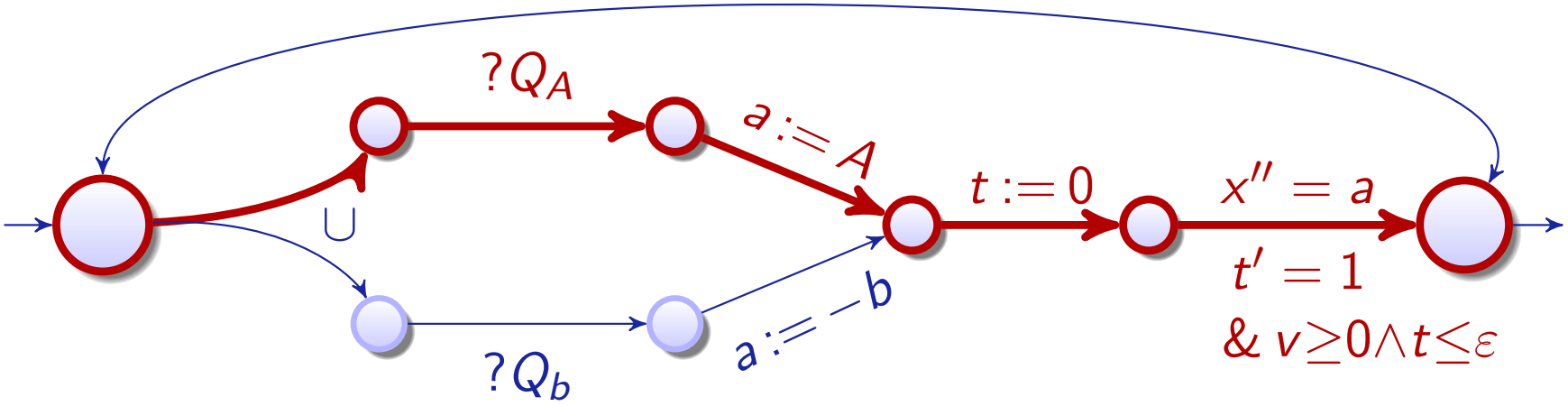


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

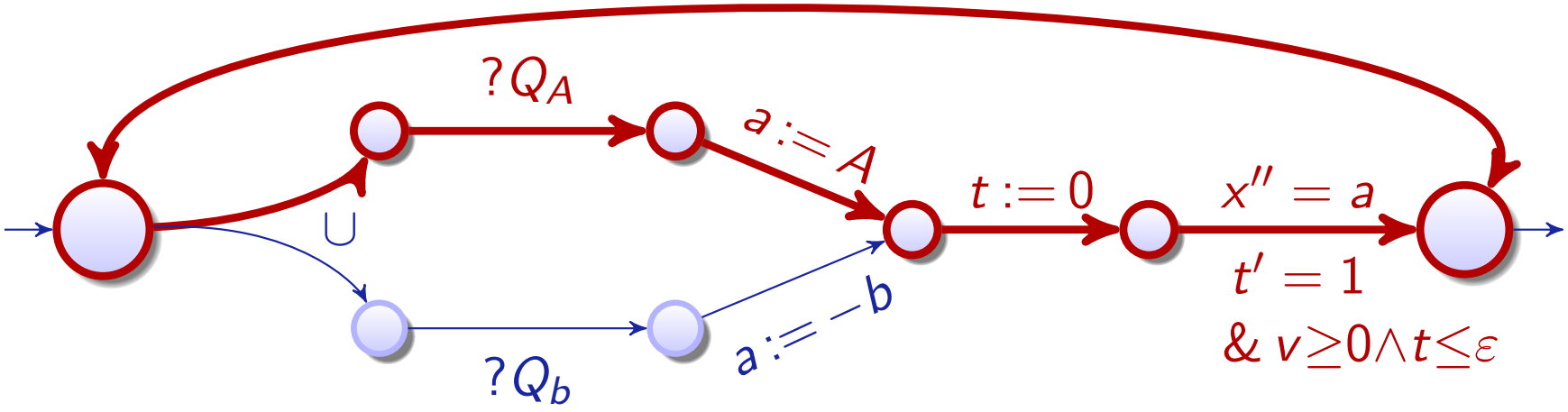


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

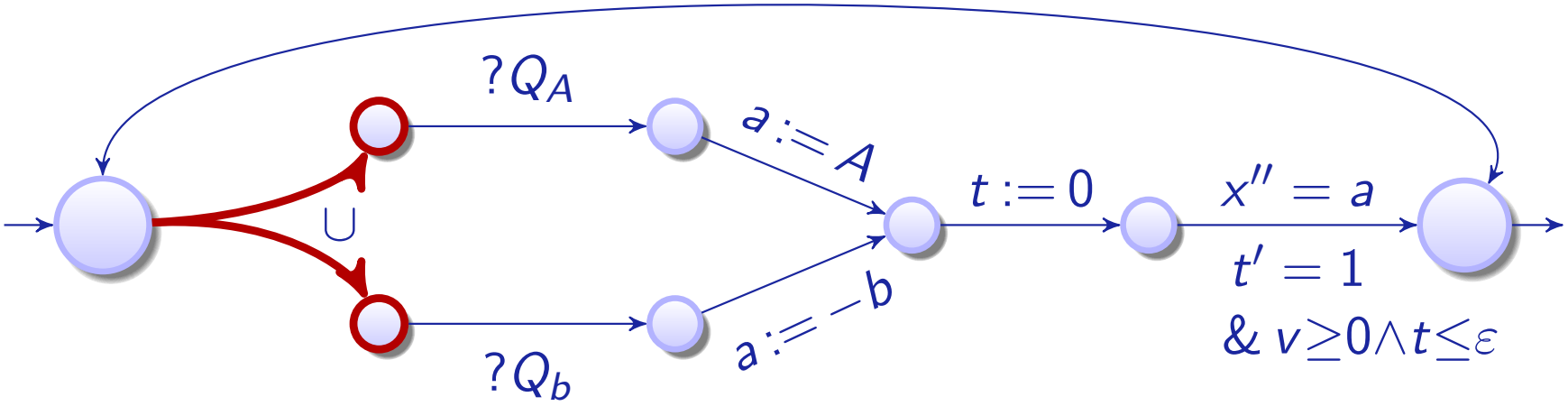


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

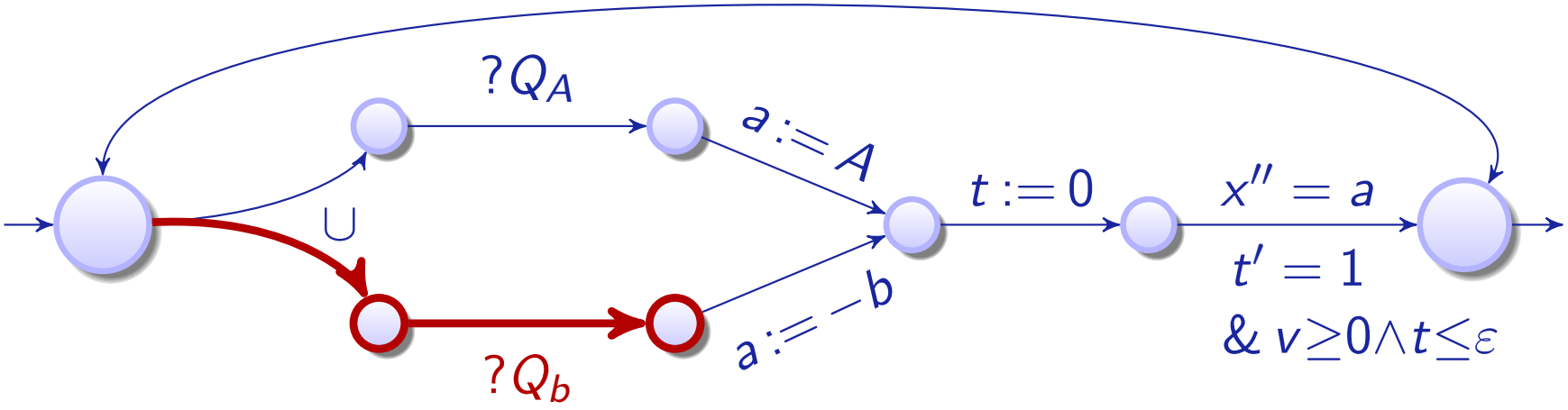


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

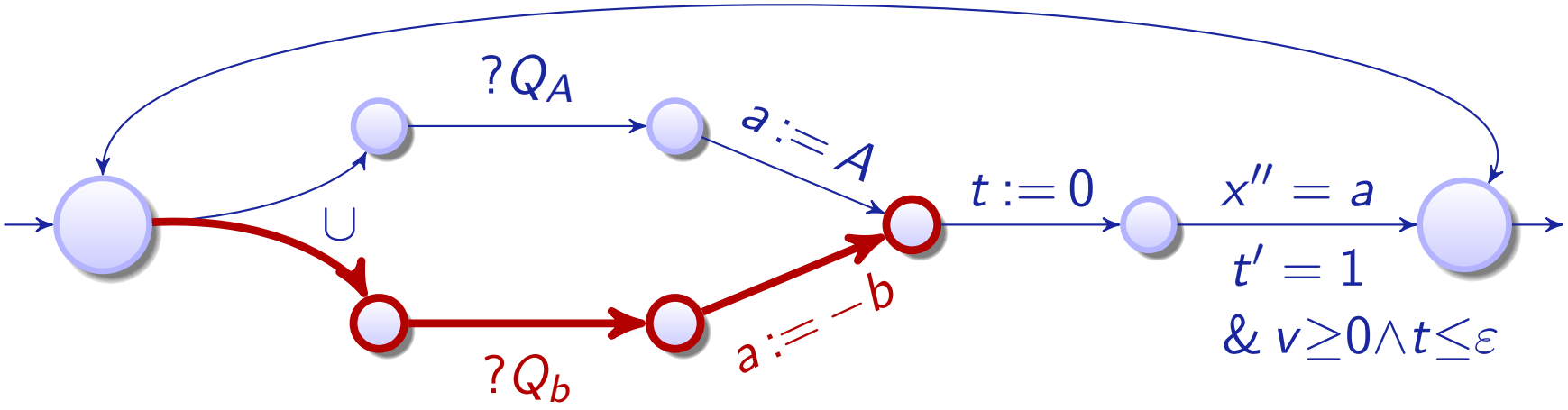


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

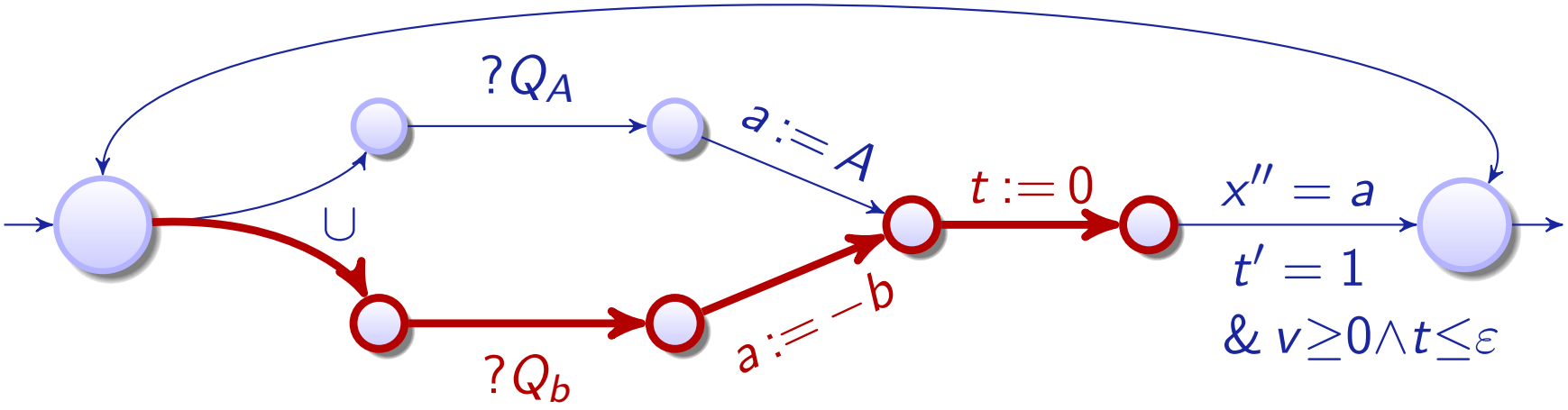


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

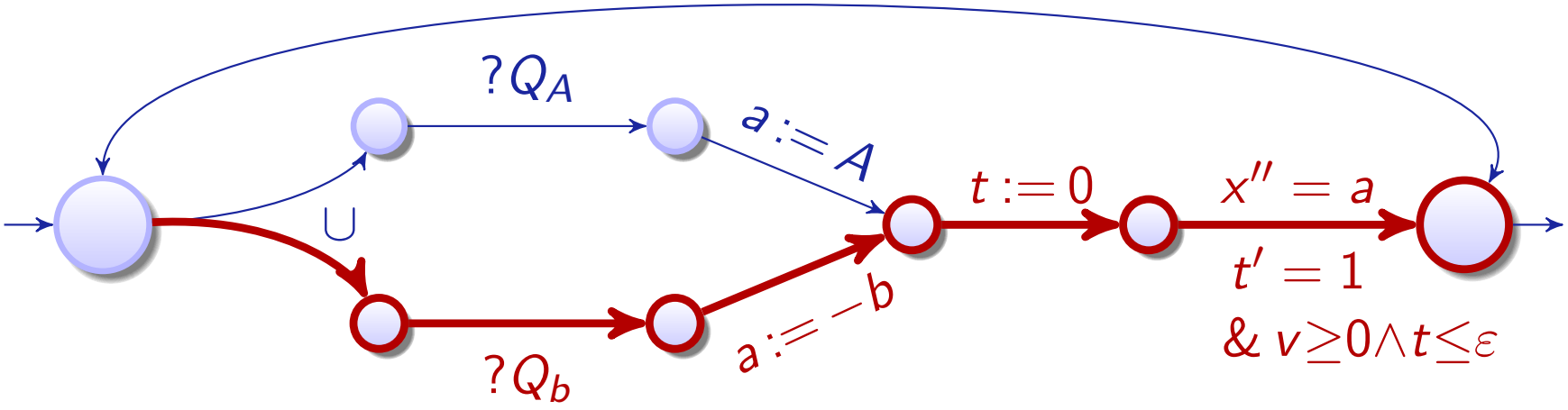


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

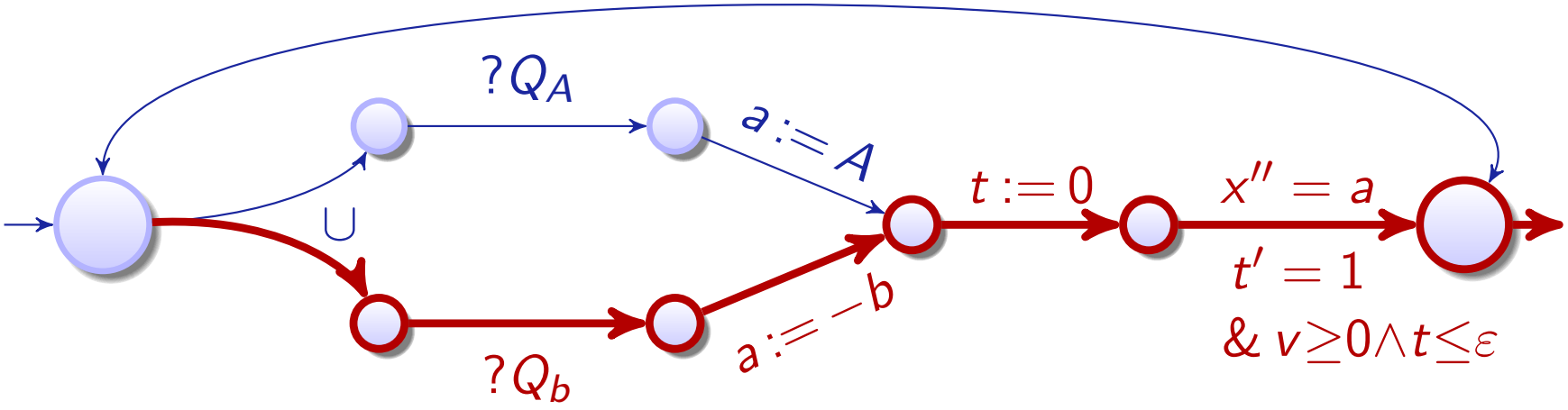


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs

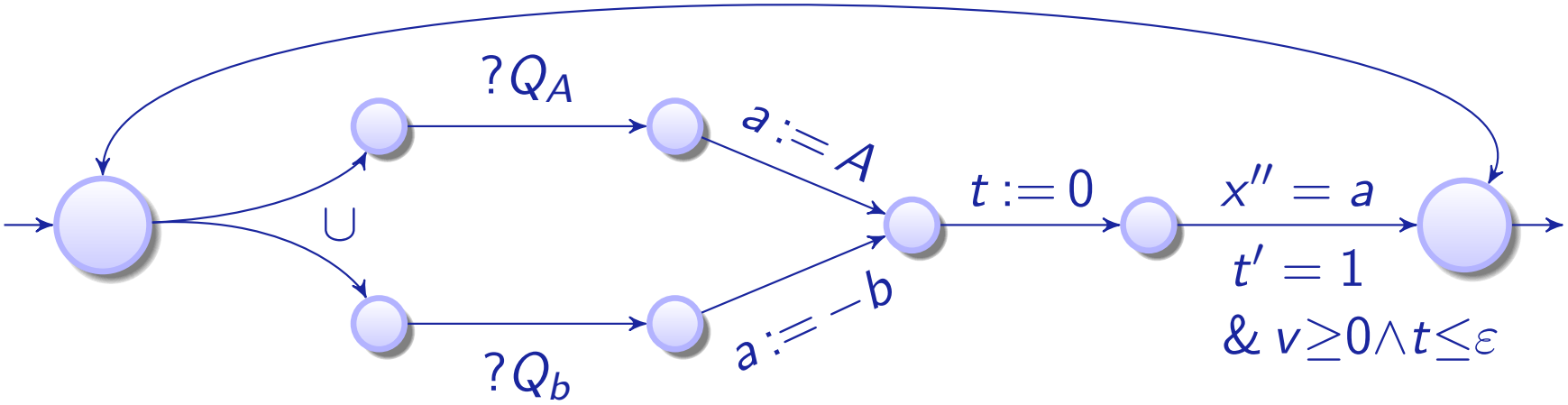


$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs



$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv$

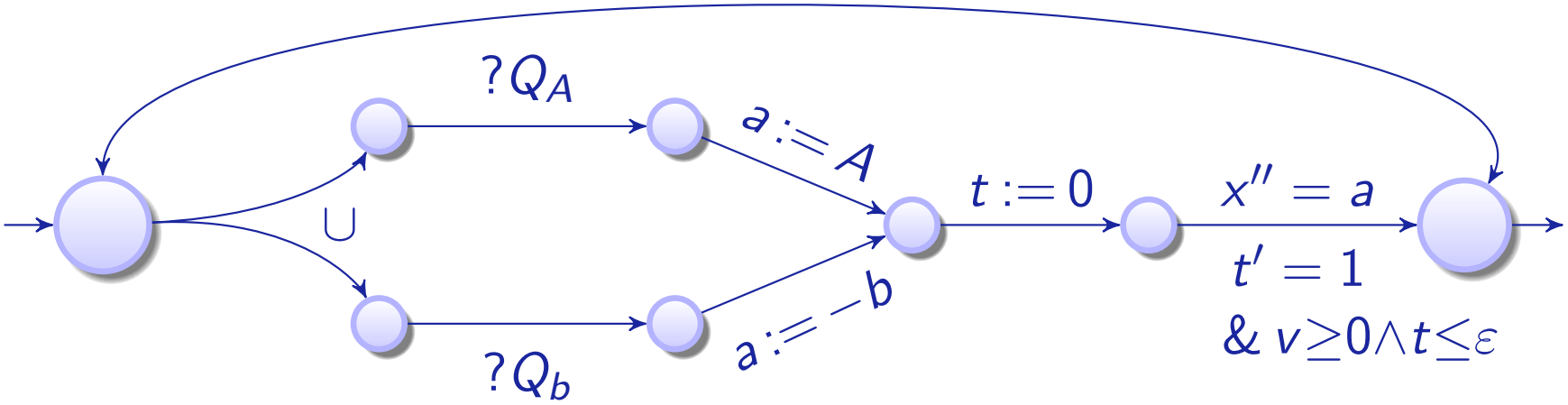
$\text{while}(Q)\,\alpha \equiv$

$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

Branching Transition Structure in Hybrid Programs



$$\text{if}(Q)\,\alpha \text{ else } \beta \equiv (?Q; \alpha) \cup (?\neg Q; \beta)$$

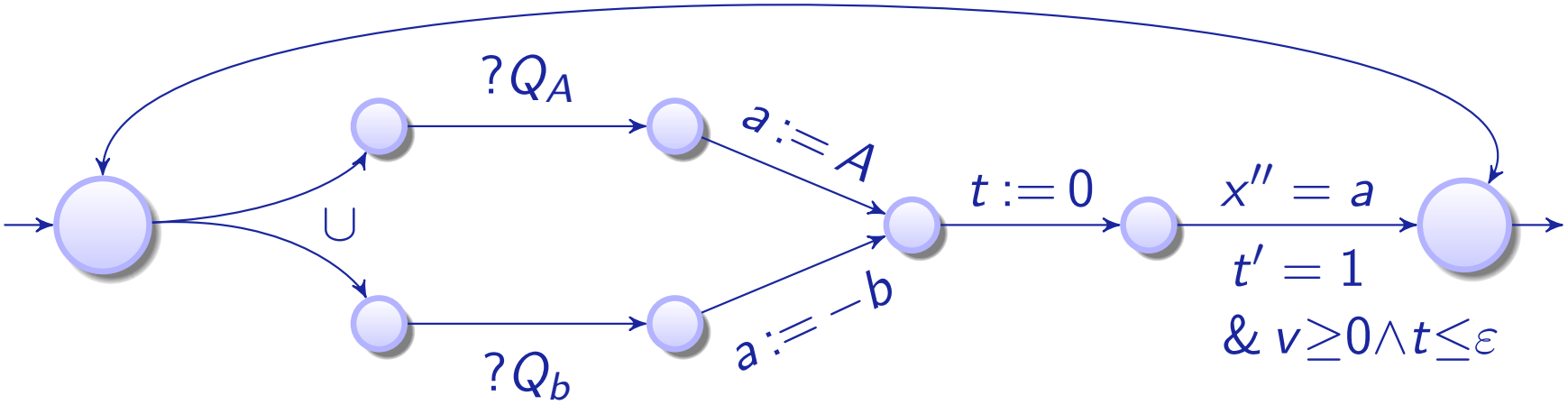
$$\text{while}(Q)\,\alpha \equiv$$

$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A) \cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}$$

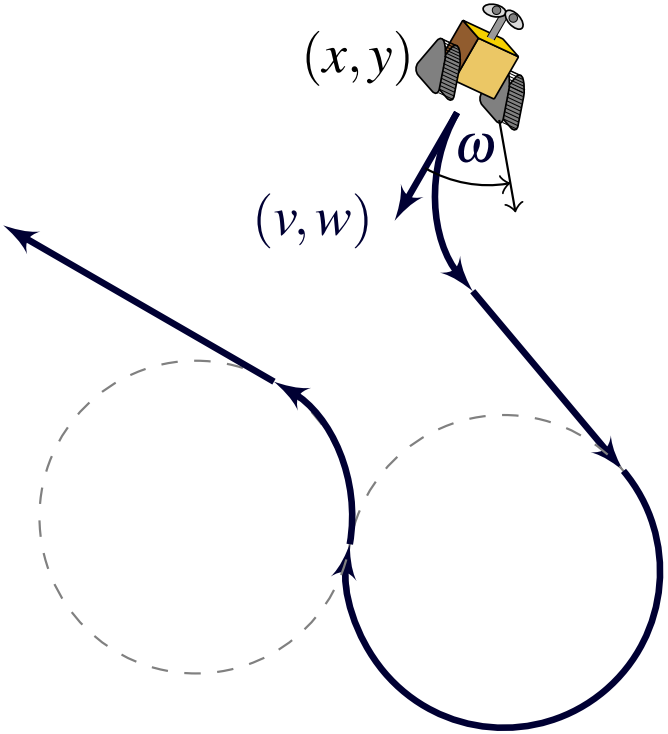
Branching Transition Structure in Hybrid Programs



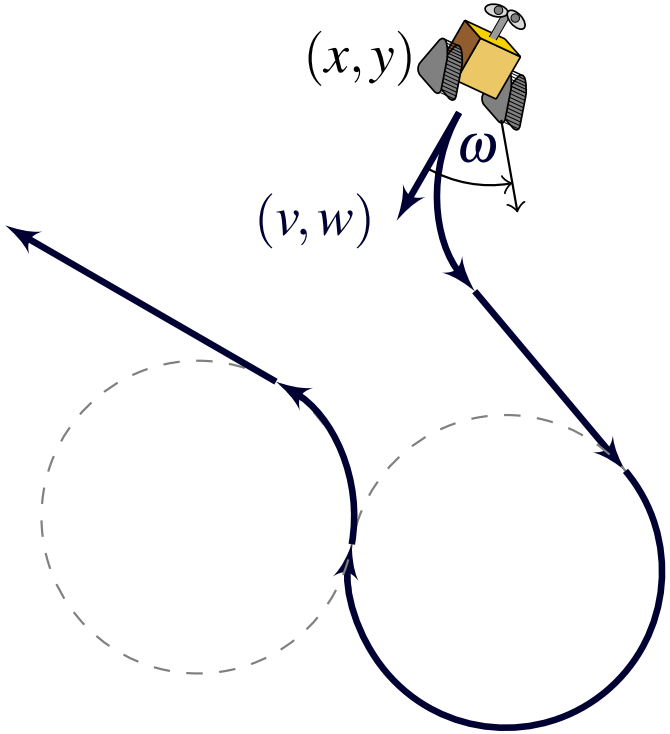
$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$
$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\begin{aligned}
\text{Robot} &\equiv (\text{ctrl}\,;\text{drive})^* \\
\text{ctrl} &\equiv (?Q_A; a := A) \\
&\quad \cup (?Q_b; a := -b) \\
\text{drive} &\equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}
\end{aligned}$$

Runaround Robot with Dubins Paths



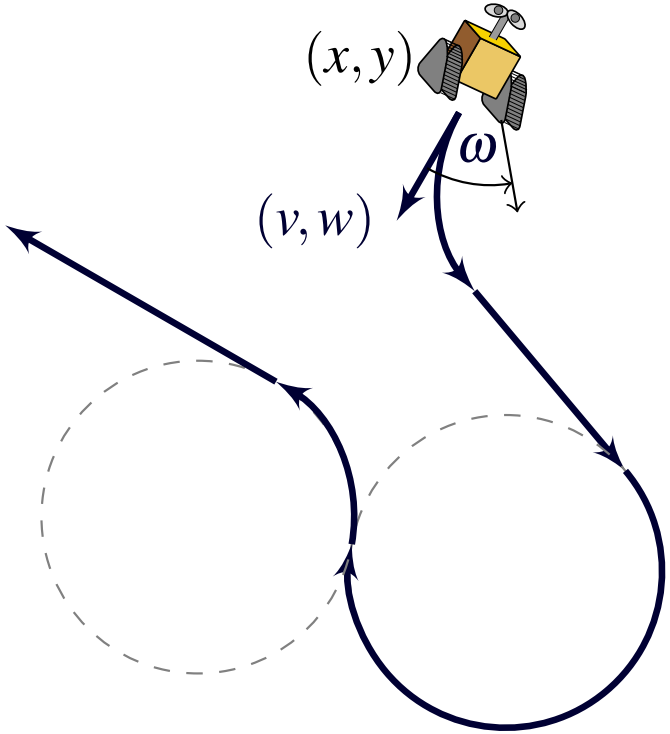
Runaround Robot with Dubins Paths



Example (Runaround Robot)

$$\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$

Runaround Robot with Dubins Paths



Example (Runaround Robot)

$$\big((?Q_{-1}; \omega := -1 \cup ?Q_1; \omega := 1 \cup ?Q_0; \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$

A Matter of Choice: Recall Speedy

Example (Speedy the point)

$$
\begin{aligned}
&(?v < 4; a := a + 1 \cup a := -b);\\
&\{x' = v, v' = a\};\\
&(?v < 4; a := a + 1 \cup a := -b);\\
&\{x' = v, v' = a\};\\
&(?v < 4; a := a + 1 \cup a := -b);\\
&\{x' = v, v' = a\}
\end{aligned}
$$

A Matter of Choice: Recall Speedy, Remove Stuff

Example (Speedy the point)

$$
\begin{aligned}
&?v < 4; a := a + 1\,; \\
&\{x' = v, v' = a\}; \\
&?v < 4; a := a + 1\,; \\
&\{x' = v, v' = a\}; \\
&?v < 4; a := a + 1\,; \\
&\{x' = v, v' = a\}
\end{aligned}
$$

A Matter of Choice: Recall Speedy, Remove Stuff

Example (Speedy the point)

$$
\begin{aligned}
&?v < 4; a := a + 1\,; \\
&\{x' = v, v' = a\}; \\
&?v < 4; a := a + 1\,; \\
&\{x' = v, v' = a\}; \\
&?v < 4; a := a + 1\,; \\
&\{x' = v, v' = a\}
\end{aligned}
$$

No wait, now it's a bad model! The HP assumes the test $v < 4$ passes after each ODE. No other choices are available.

Don't let your controller discard important cases!

Outline

Hybrid Programs: Syntax & Semantics

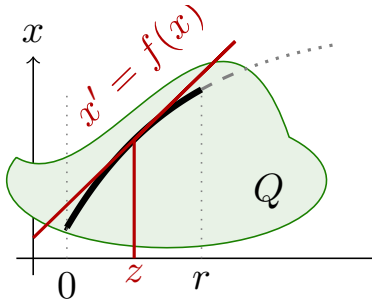
Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) $(\llbracket \cdot \rrbracket : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

$$\begin{aligned}
\llbracket x := e \rrbracket &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu\llbracket x \rrbracket = \omega\llbracket e \rrbracket\} \\
\llbracket ?Q \rrbracket &= \{(\omega, \omega) \ : \ \omega \models Q\} \\
\llbracket x' = f(x) \rrbracket &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
\llbracket \alpha \cup \beta \rrbracket &= \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket \\
\llbracket \alpha; \beta \rrbracket &= \llbracket \alpha \rrbracket \circ \llbracket \beta \rrbracket \\
\llbracket \alpha^* \rrbracket &= \llbracket \alpha \rrbracket^* = \bigcup_{n \in \mathbb{N}} \llbracket \alpha^n \rrbracket
\end{aligned}$$

compositional



André Platzer.
Logical Foundations of Cyber-Physical Systems.
Springer, Switzerland, 2018.
URL: http://www.springer.com/978-3-319-63587-3,
doi:10.1007/978-3-319-63588-0.

André Platzer.
Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.
Springer, Heidelberg, 2010.
doi:10.1007/978-3-642-14509-4.

André Platzer.
Logics of dynamical systems.
In *LICS*, pages 13–24, Los Alamitos, 2012. IEEE.
doi:10.1109/LICS.2012.13.

André Platzer.
Differential dynamic logic for hybrid systems.
J. Autom. Reas., 41(2):143–189, 2008.

doi:10.1007/s10817-008-9103-8.

André Platzer.
A complete uniform substitution calculus for differential dynamic logic.

J. Autom. Reas., 59(2):219–265, 2017.
doi:10.1007/s10817-016-9385-1.