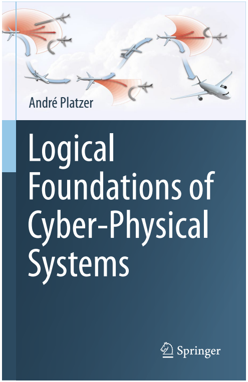
05: Dynamical Systems & Dynamic Axioms

Logical Foundations of Cyber-Physical Systems



André Platzer

Logical Foundations of Cyber-Physical Systems

Springer

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Carnegie Mellon University
Computer Science Department

Outline

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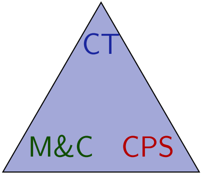
1. Learning Objectives
2. Approach & Reminder
3. Intermediate Conditions for CPS
4. Dynamic Axioms for Dynamical Systems
    - Nondeterministic Choices
    - Assignments
    - Differential Equations
    - Tests
    - Sequential Compositions
    - Loops
    - Soundness
    - Diamonds
5. First Bouncing Ball Proof
6. Summary

Learning Objectives

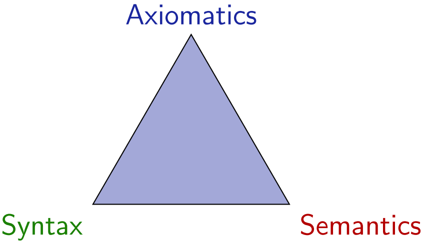
rigorous reasoning about CPS
dL as verification language



cyber+physics interaction
relate discrete+continuous

align semantics+reasoning
operational CPS effects

Logical Trinity



Syntax	defines the notation What problems are we allowed to write down?
Semantics	what carries meaning. What real or mathematical objects does the syntax stand for?
Axiomatics	internalizes semantic relations into universal syntactic transformations. How does the semantics of A relate to semantics of $A \land B$, syntactically? If A is true, is $A \land B$ true, too? Conversely?

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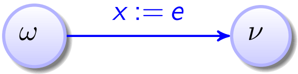
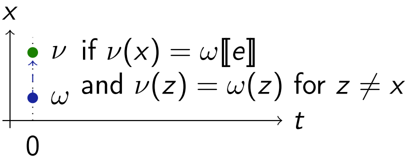
Approach

Logical guiding principle: Compositionality

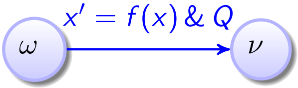
1. Every CPS is modeled by a hybrid program (or game . . .)
2. All hybrid programs are combinations of simpler hybrid programs (by a program operator such as $\cup$ and $;$ and *)
3. All CPS can be analyzed if only we identify one suitable analysis technique for each operator.
4. Analysis of a big CPS is an analysis chain for all individual parts.

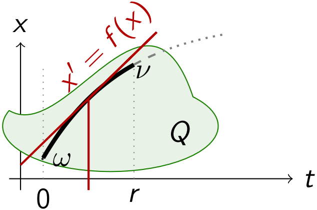
Hybrid Programs: Semantics



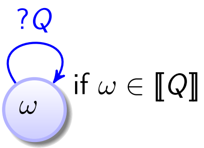


ν if $\nu(x) = \omega[\![e]\!]$ and $\nu(z) = \omega(z)$ for $z \neq x$

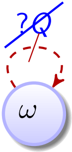




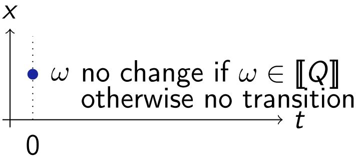
$x' = f(x) \,\&\, Q$



if $\omega \in [\![Q]\!]$

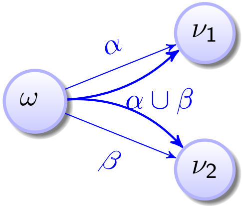


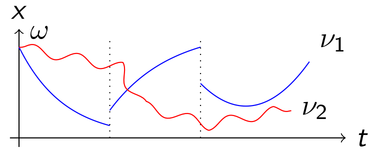
if $\omega \notin [\![Q]\!]$

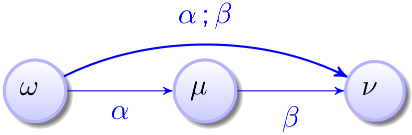


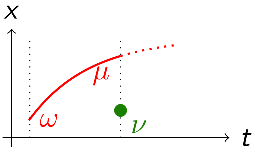
ω no change if $\omega \in [\![Q]\!]$
otherwise no transition

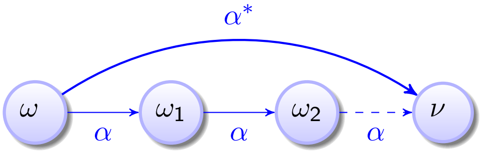
Hybrid Programs: Semantics

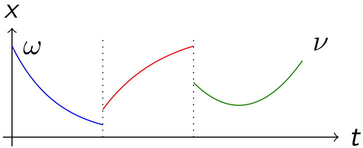




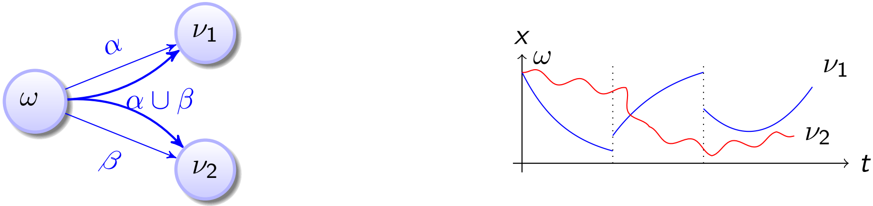


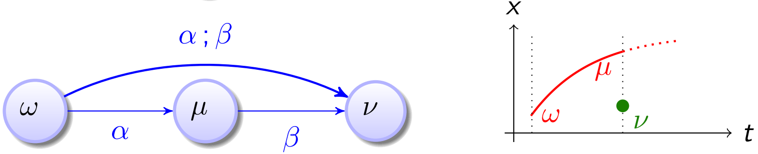


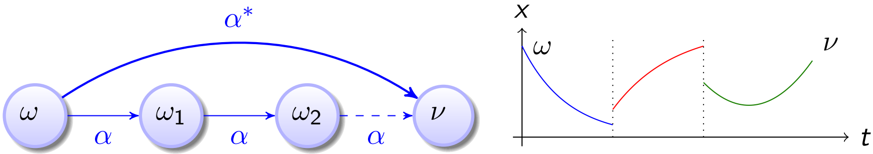




Hybrid Programs: Semantics

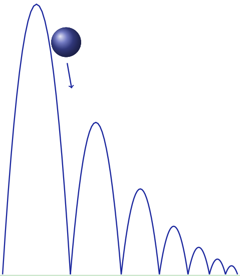






compositional semantics $\Rightarrow$ compositional proofs!

Conjecture: Quantum the Acrophobic Bouncing Ball



Example (Quantum the Bouncing Ball)

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow$$
$$\big[\big(\{x' = v, v' = -g \,\&\, x \ge 0\}; (?x{=}0; v := -cv \cup ?x{\ne}0)\big)^*\big]\,(0 \le x \land x \le H)$$

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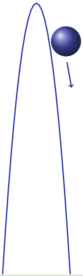

Example (Quantum the Bouncing Ball) (Single-hop)

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Removing the repetition grotesquely changes the behavior to a single hop

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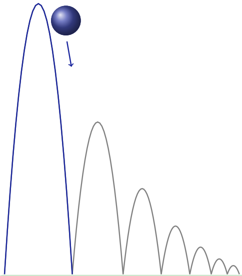

Example (Quantum the Bouncing Ball) **(Single-hop)**

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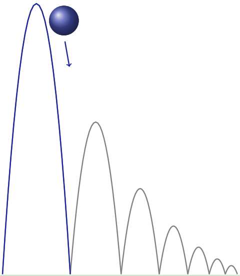

How to prove ;

Example (Quantum the Bouncing Ball) (Single-hop)

$$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \to$$
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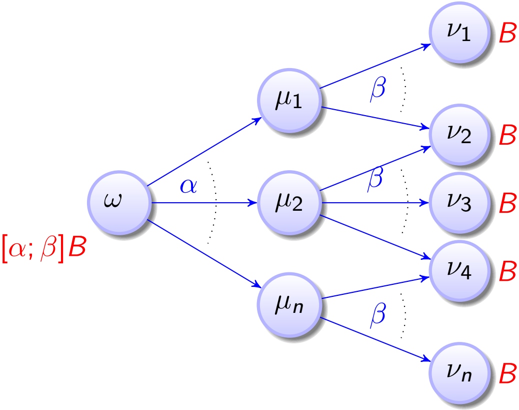
Intermediate Conditions for CPS

H; $\dfrac{\qquad\qquad\qquad}{A \rightarrow [\alpha;\beta]B}$



Intermediate Conditions for CPS

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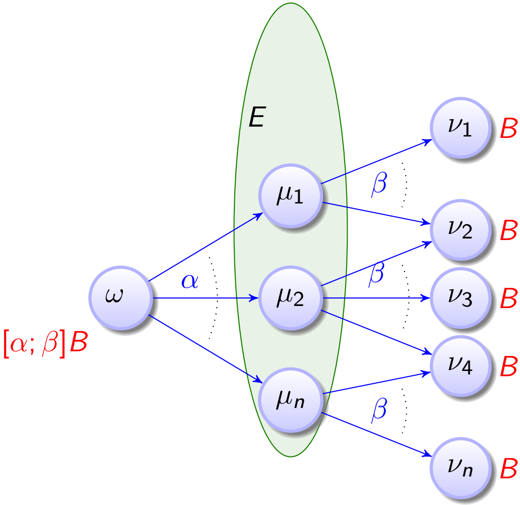
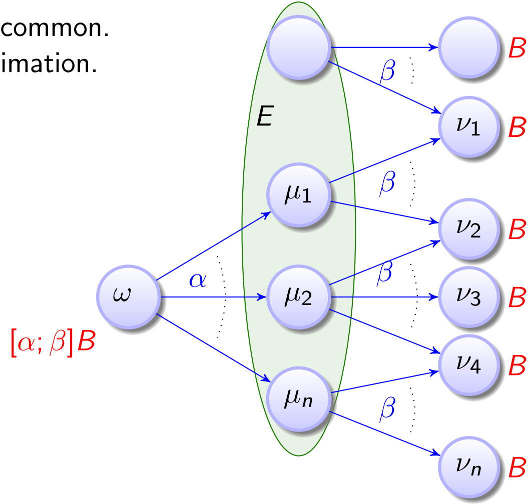


Intermediate Conditions for CPS

E summarizes what μ_i have in common.
E is often imprecise overapproximation.

$$\text{H;}\quad \frac{A \to [\alpha]E \quad E \to [\beta]B}{A \to [\alpha;\beta]B}$$



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Just need to find this E ...

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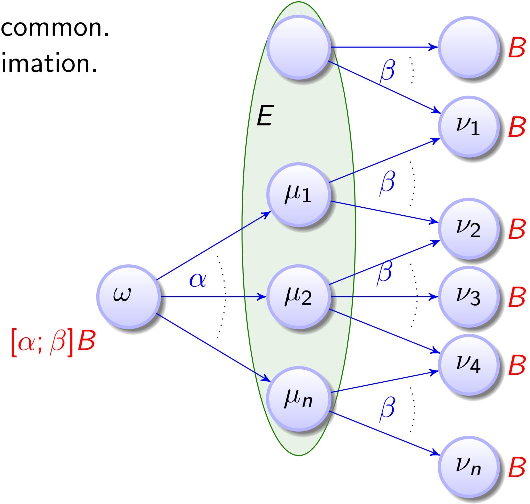

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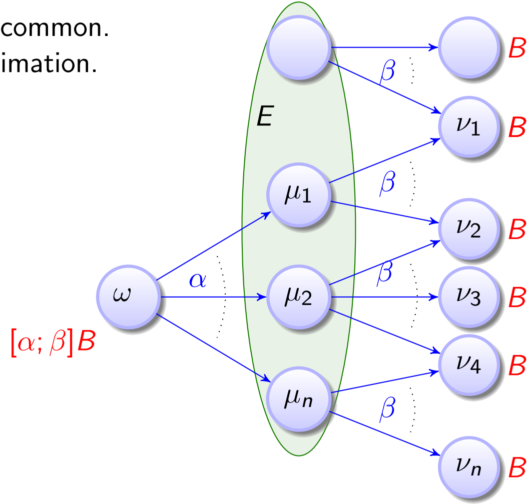

Example (Quantum the Bouncing Ball) (Single-hop)

$$0\le x=H\land v=0\land g>0\land 1\ge c\ge 0 \rightarrow \left[x' = v, v' = -g \,\&\, x \ge 0\right]E$$

$$E \rightarrow \left[?x=0; v := -cv \cup ?x\ne 0\right](0\le x \land x\le H)$$

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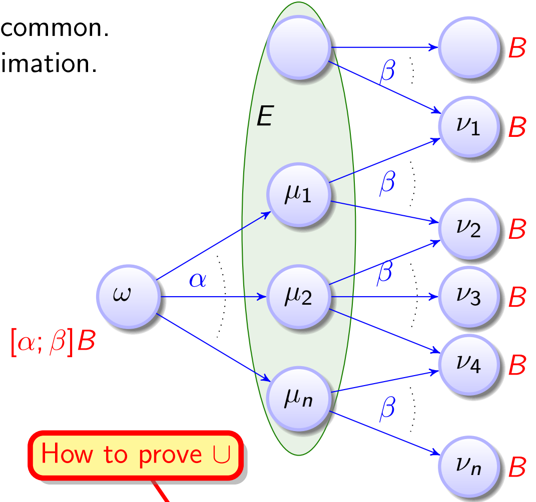

How to prove $\cup$

Example (Quantum the Bouncing Ball) (Single-hop)

$$0\le x=H \land v=0 \land g>0 \land 1\ge c\ge 0 \rightarrow \left[x'=v, v'=-g \,\&\, x\ge 0\right]E$$

$$E \rightarrow \left[?x=0; v:=-cv \cup ?x\ne 0\right](0\le x \land x\le H)$$

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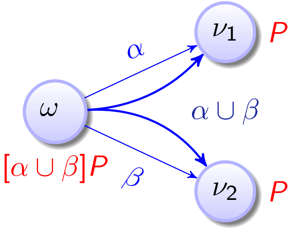
Motivating Proofs for Nondeterministic Choices

Semantics

$[\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$



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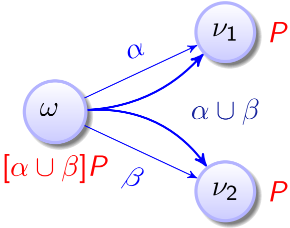

- $\omega \in [\![[\alpha \cup \beta]P]\!]$ iff $\nu \in [\![P]\!]$ for all ν with $(\omega, \nu) \in [\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$

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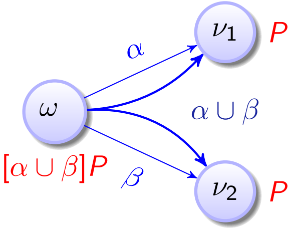

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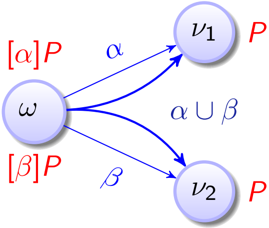

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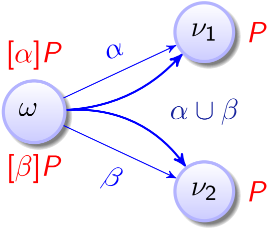

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- And vice versa.

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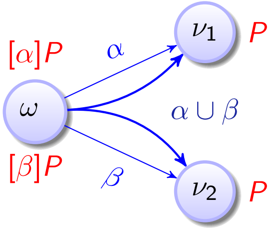

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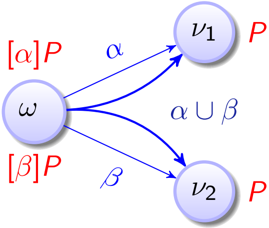

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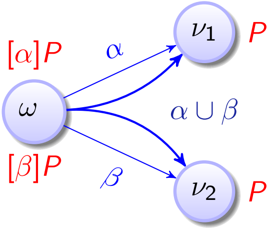

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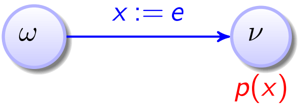
Lemma

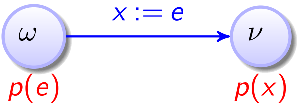
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$ *is a sound axiom, i.e., all its instances valid.*

Dynamic Axioms for Dynamical Systems



$[:=]\quad [x := e]p(x) \leftrightarrow$

Dynamic Axioms for Dynamical Systems



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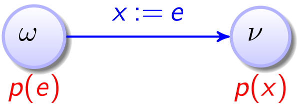
Dynamic Axioms for Dynamical Systems



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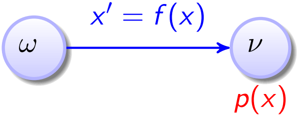
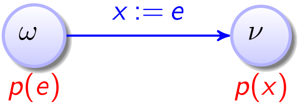
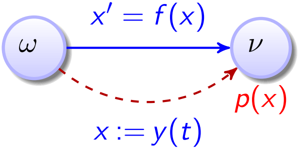


['] $[x' = f(x)]p(x) \leftrightarrow$

Dynamic Axioms for Dynamical Systems

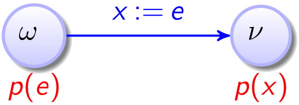


$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

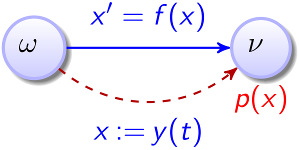


$$['] \quad [x' = f(x)]p(x) \leftrightarrow \qquad [x := y(t)]p(x)$$

Dynamic Axioms for Dynamical Systems



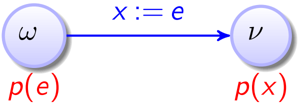
$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

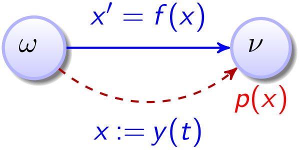


$$['] \quad [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0[x := y(t)]p(x)$$

Dynamic Axioms for Dynamical Systems

$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$

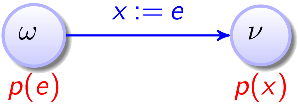




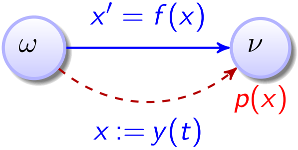
$[']$ $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0[x := y(t)]p(x)$

$[']$ $[x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0 \left([x := y(t)]p(x)\right)$

Dynamic Axioms for Dynamical Systems



$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$

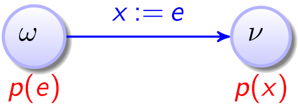


$[']$ $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0[x := y(t)]p(x)$

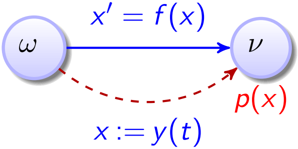
$[']$ $[x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0 \big(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

Dynamic Axioms for Dynamical Systems

[:=] $[x := e]p(x) \leftrightarrow p(e)$

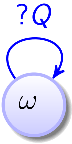


['] $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0[x := y(t)]p(x)$



['] $[x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0 \left(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\right)$

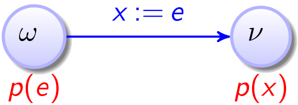
[?] $[?Q]P \leftrightarrow$



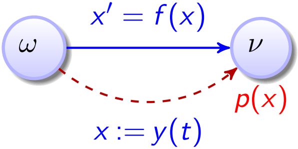
if $\omega \in [\![Q]\!]$

Dynamic Axioms for Dynamical Systems

[:=] $[x := e]p(x) \leftrightarrow p(e)$

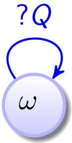


['] $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0[x := y(t)]p(x)$



['] $[x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0 \left(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\right)$

[?] $[?Q]P \leftrightarrow (Q \rightarrow P)$

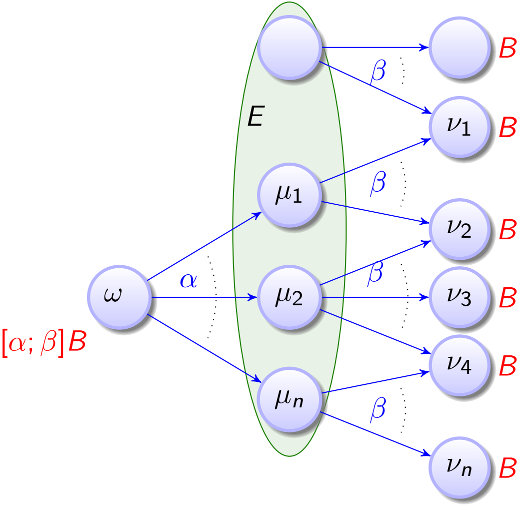


if $\omega \in [\![Q]\!]$

Sequential Compositions via Intermediate Conditions

What is the most precise E summary?

$$\text{H;}\quad \frac{A \to [\alpha]E \qquad E \to [\beta]B}{A \to [\alpha;\beta]B}$$



Sequential Compositions via Intermediate Conditions

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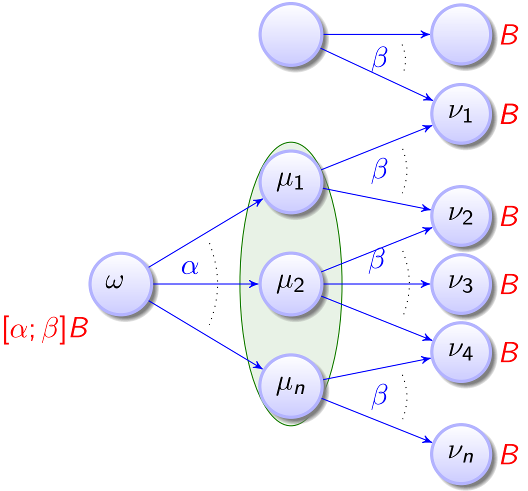
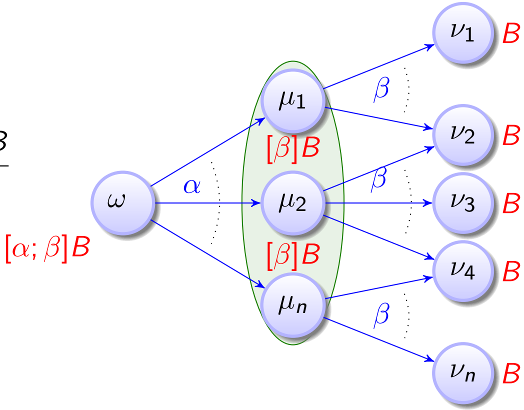


Sequential Compositions via Intermediate Conditions

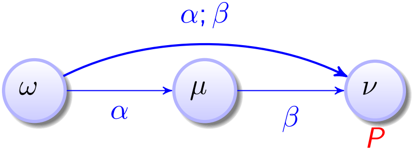
What is the most precise E summary? $[\beta]B$

$$\text{H;}\ \frac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha;\beta]B}$$

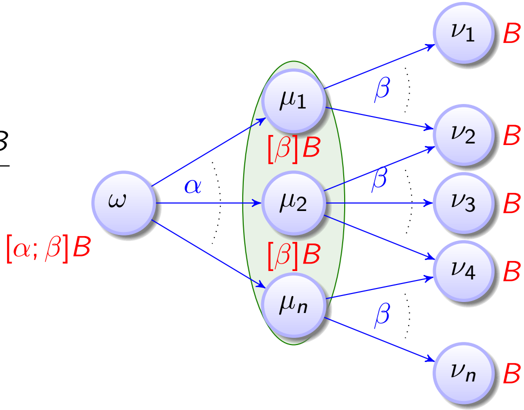


Sequential Compositions via Intermediate Conditions

[;] $[\alpha;\beta]P \leftrightarrow$

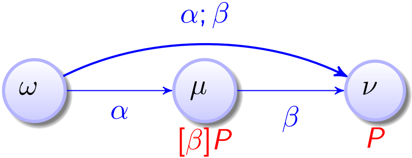


H; $$\frac{A \to [\alpha][\beta]B \quad [\beta]B \to [\beta]B}{A \to [\alpha;\beta]B}$$

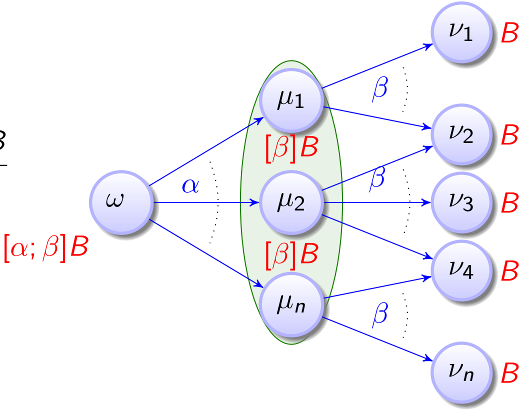


Sequential Compositions via Intermediate Conditions

[;] $[\alpha;\beta]P \leftrightarrow$

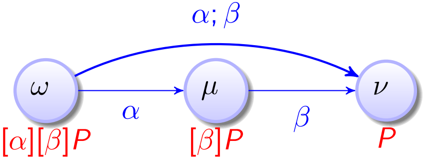


H; $$\frac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha;\beta]B}$$

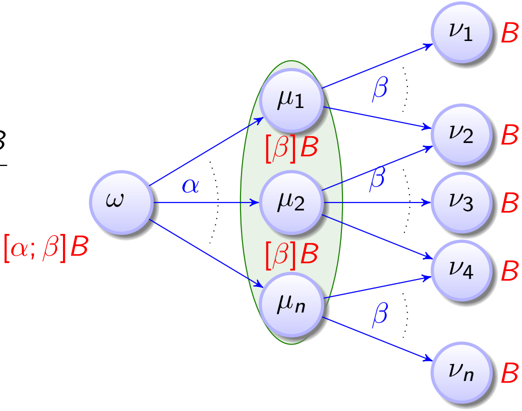


Sequential Compositions via Intermediate Conditions

[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



H; $\dfrac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha;\beta]B}$

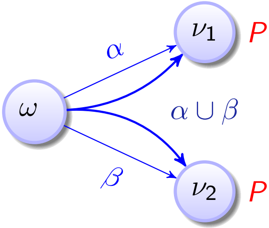


Dynamic Axioms for Dynamical Systems

compositional semantics $\Rightarrow$ compositional axioms!

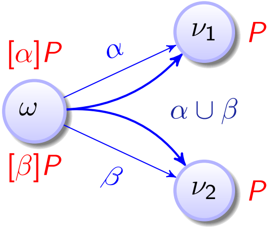
Dynamic Axioms for Dynamical Systems

$[\cup]\quad [\alpha \cup \beta]P \leftrightarrow$



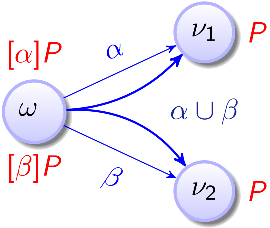
Dynamic Axioms for Dynamical Systems

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

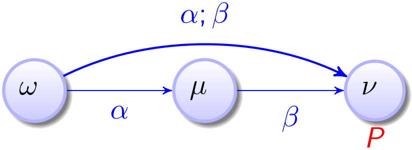


Dynamic Axioms for Dynamical Systems

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

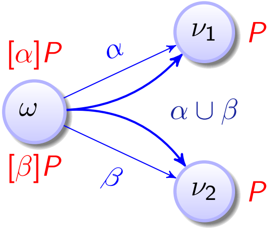


$[;]$ $[\alpha;\beta]P \leftrightarrow$

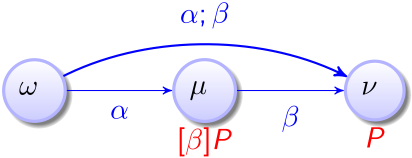


Dynamic Axioms for Dynamical Systems

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

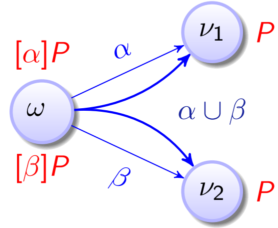


$[;]$ $[\alpha ; \beta]P \leftrightarrow$

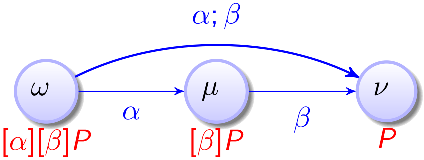


Dynamic Axioms for Dynamical Systems

[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

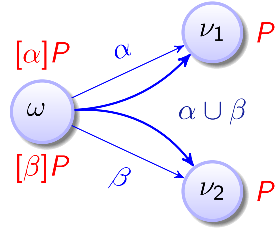


[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

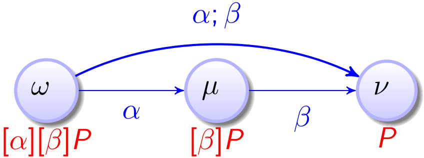


Dynamic Axioms for Dynamical Systems

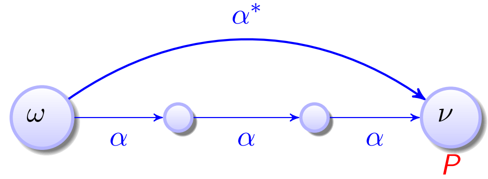
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

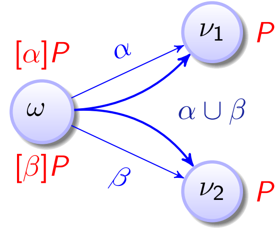


$[^*]$ $[\alpha^*]P \leftrightarrow$

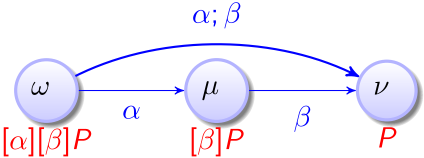


Dynamic Axioms for Dynamical Systems

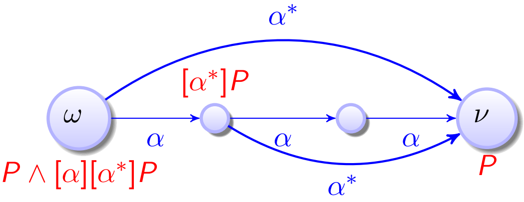
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



[*] $[\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$



Soundness of Dynamic Axioms

Lemma

$[\cup]$ $\quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$ *is sound.*

Lemma

$[;]$ $\quad [\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$ *is sound.*

Soundness of Dynamic Axioms

Lemma

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$ *is sound.*

Proof using $[\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$.

$(\omega, \nu) \in [\![\alpha \cup \beta]\!]$ iff $(\omega, \nu) \in [\![\alpha]\!]$ or $(\omega, \nu) \in [\![\beta]\!]$.
Thus, $\omega \in [\![[\alpha \cup \beta]P]\!]$ iff both $\omega \in [\![[\alpha]P]\!]$ and $\omega \in [\![[\beta]P]\!]$. □

Lemma

$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$ *is sound.*

Soundness of Dynamic Axioms

Lemma

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$ *is sound.*

Proof using $[\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$.

$(\omega, \nu) \in [\![\alpha \cup \beta]\!]$ iff $(\omega, \nu) \in [\![\alpha]\!]$ or $(\omega, \nu) \in [\![\beta]\!]$.
Thus, $\omega \in [\![[\alpha \cup \beta]P]\!]$ iff both $\omega \in [\![[\alpha]P]\!]$ and $\omega \in [\![[\beta]P]\!]$. □

Lemma

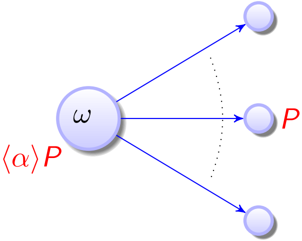
$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$ *is sound.*

Proof using $[\![\alpha; \beta]\!] = [\![\alpha]\!] \circ [\![\beta]\!]$.

$(\omega, \nu) \in [\![\alpha; \beta]\!]$ iff $(\omega, \mu) \in [\![\alpha]\!]$ and $(\mu, \nu) \in [\![\beta]\!]$ for some state μ.
Thus, $\omega \in [\![[\alpha; \beta]P]\!]$ iff $\mu \in [\![[\beta]P]\!]$ for all μ with $(\omega, \mu) \in [\![\alpha]\!]$.
That is, $\omega \in [\![[\alpha; \beta]P]\!]$ iff $\omega \in [\![[\alpha][\beta]P]\!]$. □

Axioms for Diamonds

$\langle\cdot\rangle \quad \langle\alpha\rangle P \leftrightarrow$

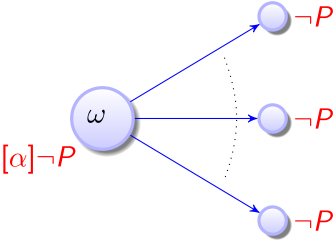


Semantics

$$[\![\langle\alpha\rangle P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\}$$
$$[\![[\alpha]P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\}$$

Axioms for Diamonds

$\langle\cdot\rangle$ $\langle\alpha\rangle P \leftrightarrow$

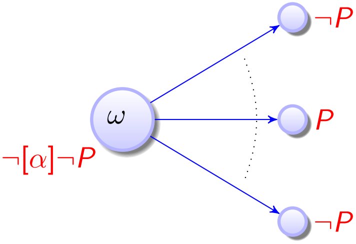


Semantics

$$\begin{aligned}
[\![\langle\alpha\rangle P]\!] &= \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\}\\
[\![[\alpha]P]\!] &= \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\}
\end{aligned}$$

Axioms for Diamonds

$\langle\cdot\rangle \quad \langle\alpha\rangle P \leftrightarrow$

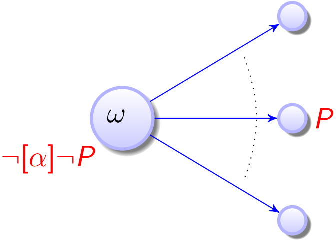


Semantics

$$\begin{aligned}
[\![\langle\alpha\rangle P]\!] &= \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] &= \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\}
\end{aligned}$$

Axioms for Diamonds: Duality

$\langle\cdot\rangle$ $\langle\alpha\rangle P \leftrightarrow \neg[\alpha]\neg P$

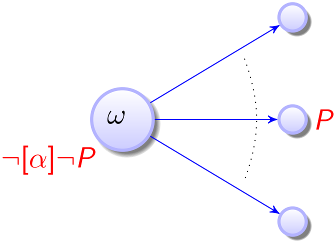


Semantics

$$\begin{aligned} [\![\langle\alpha\rangle P]\!] &= \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \\ [\![[\alpha]P]\!] &= \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\} \end{aligned}$$

Axioms for Diamonds: Duality

$\langle\cdot\rangle \quad \langle\alpha\rangle P \leftrightarrow \neg[\alpha]\neg P$



Duality axiom $\langle\cdot\rangle$ relates $\langle\alpha\rangle$ to $[\alpha]$ for arbitrary HP α

Semantics

$$\begin{aligned} [\![\langle\alpha\rangle P]\!] &= \{\omega \ : \ \nu \in [\![P]\!] \text{ for some } \nu : \ (\omega,\nu) \in [\![\alpha]\!]\} \\ [\![[\alpha]P]\!] &= \{\omega \ : \ \nu \in [\![P]\!] \text{ for all } \nu : \ (\omega,\nu) \in [\![\alpha]\!]\} \end{aligned}$$

Outline

1. Learning Objectives
2. Approach & Reminder
3. Intermediate Conditions for CPS
4. Dynamic Axioms for Dynamical Systems
    - Nondeterministic Choices
    - Assignments
    - Differential Equations
    - Tests
    - Sequential Compositions
    - Loops
    - Soundness
    - Diamonds
5. First Bouncing Ball Proof
6. Summary

A Proof of a Single-hop Bouncing Ball

$$\text{[;]}\ \frac{}{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B_{(x,v)}}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$
$$B_{(x,v)} \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$
$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A Proof of a Single-hop Bouncing Ball

$$[;]\quad [\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$$

$$[;]\ \dfrac{[\cup]\ \dfrac{}{A \rightarrow [x''=-g][?x=0;\, v:=-cv \cup ?x \geq 0]B(x,v)}}{A \rightarrow [x''=-g;\,(?x=0;\, v:=-cv \cup ?x \geq 0)]B(x,v)}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$\{x''=-g\} \overset{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A Proof of a Single-hop Bouncing Ball

$$[\cup]\quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$$

$$\begin{array}{rl}
[;] & \overline{A \to [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)} \\
[\cup] & \overline{A \to [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)} \\
[;] & \overline{A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}
\end{array}$$

$$\begin{aligned}
A &\overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \\
B(x,v) &\overset{\text{def}}{\equiv} 0 \leq x \land x \leq H \\
\{x'' = -g\} &\overset{\text{def}}{\equiv} \{x' = v, v' = -g\}
\end{aligned}$$

A Proof of a Single-hop Bouncing Ball

$[;]\quad [\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

$$\begin{array}{rl}
\hline
[?],[?] & A \to [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[;] & A \to [x'' = -g]\big({\color{red}[?x = 0; v := -cv]}B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[\cup] & A \to [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\
\hline
[;] & A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A Proof of a Single-hop Bouncing Ball

$$[?]\quad [?Q]P \leftrightarrow (Q \rightarrow P)$$

$$\begin{array}{ll}
\text{[:=]} & \overline{A \rightarrow [x''=-g]\big((x=0 \rightarrow [v:=-cv]B(x,v)) \land (x \geq 0 \rightarrow B(x,v))\big)} \\
\text{[?],[?]} & \overline{A \rightarrow [x''=-g]\big([?x=0][v:=-cv]B(x,v) \land [?x \geq 0]B(x,v)\big)} \\
\text{[;]} & \overline{A \rightarrow [x''=-g]\big([?x=0; v:=-cv]B(x,v) \land [?x \geq 0]B(x,v)\big)} \\
\text{[}\cup\text{]} & \overline{A \rightarrow [x''=-g][?x=0; v:=-cv \cup ?x \geq 0]B(x,v)} \\
\text{[;]} & \overline{A \rightarrow [x''=-g; (?x=0; v:=-cv \cup ?x \geq 0)]B(x,v)}
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x''=-g\} \overset{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A Proof of a Single-hop Bouncing Ball

$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$\begin{array}{rl}
 & \\ \hline
[\,'\,] & A \to [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\ \hline
[:=] & A \to [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\ \hline
[?],[?] & A \to [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \hline
[;] & A \to [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \hline
[\cup] & A \to [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\ \hline
[;] & A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$\begin{aligned}
A &\overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \\
B(x,v) &\overset{\text{def}}{\equiv} 0 \leq x \land x \leq H \\
\{x'' = -g\} &\overset{\text{def}}{\equiv} \{x' = v, v' = -g\}
\end{aligned}$$

A Proof of a Single-hop Bouncing Ball

$$[']\quad [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$$

$$\begin{array}{rl}
\hline
[;] & A \to \forall t{\geq}0\, [x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\
\hline
['] & A \to [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[:=] & A \to [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[?],[?] & A \to [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[;] & A \to [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[\cup] & A \to [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\
\hline
[;] & A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A Proof of a Single-hop Bouncing Ball

$$[;]\quad [\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$$

$$\begin{array}{r l}
 & \\ \cline{2-2}
[:=] & A \to \forall t{\geq}0\, [x := H-\frac{g}{2}t^2][v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\ \cline{2-2}
[;] & A \to \forall t{\geq}0\, [x := H-\frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\ \cline{2-2}
['] & A \to [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\ \cline{2-2}
[:=] & A \to [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\ \cline{2-2}
[?],[?] & A \to [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \cline{2-2}
[;] & A \to [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \cline{2-2}
[\cup] & A \to [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\ \cline{2-2}
[;] & A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A Proof of a Single-hop Bouncing Ball

$$\begin{array}{rl}
 & \hline
\text{[:=]} & A \to \forall t{\geq}0\, [x := H - \frac{g}{2}t^2]\big((x{=}0 \to B(x,-c(-gt))) \land (x{\geq}0 \to B(x,-gt))\big) \\ \hline
\text{[:=]} & A \to \forall t{\geq}0\, [x := H - \frac{g}{2}t^2][\textcolor{red}{v := -gt}]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\ \hline
\text{[;]} & A \to \forall t{\geq}0\, [x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x,-cv)) \land (x{\geq}0 \to B(x,v))\big) \\ \hline
\text{[$'$]} & A \to [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\ \hline
\text{[:=]} & A \to [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\ \hline
\text{[?],[?]} & A \to [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \hline
\text{[;]} & A \to [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\ \hline
\text{[$\cup$]} & A \to [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\ \hline
\text{[;]} & A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A Proof of a Single-hop Bouncing Ball

$$A \to \forall t{\geq}0\,\Big(\big(H-\tfrac{g}{2}t^2{=}0 \to B(H-\tfrac{g}{2}t^2,-c(-gt))\big) \land \big(H-\tfrac{g}{2}t^2{\geq}0 \to B(H-\tfrac{g}{2}t^2,-gt$$

$$\text{[:=]}\quad A \to \forall t{\geq}0\,[x := H-\tfrac{g}{2}t^2]\Big(\big(x{=}0 \to B(x,-c(-gt))\big) \land \big(x{\geq}0 \to B(x,-gt)\big)\Big)$$

$$\text{[:=]}\quad A \to \forall t{\geq}0\,[x := H-\tfrac{g}{2}t^2][v := -gt]\Big(\big(x{=}0 \to B(x,-cv)\big) \land \big(x{\geq}0 \to B(x,v)\big)\Big)$$

$$\text{[;]}\quad A \to \forall t{\geq}0\,[x := H-\tfrac{g}{2}t^2;\, v := -gt]\Big(\big(x{=}0 \to B(x,-cv)\big) \land \big(x{\geq}0 \to B(x,v)\big)\Big)$$

$$\text{[′]}\quad A \to [x'' = -g]\Big(\big(x = 0 \to B(x,-cv)\big) \land \big(x \geq 0 \to B(x,v)\big)\Big)$$

$$\text{[:=]}\quad A \to [x'' = -g]\Big(\big(x = 0 \to [v := -cv]B(x,v)\big) \land \big(x \geq 0 \to B(x,v)\big)\Big)$$

$$\text{[?],[?]}\quad A \to [x'' = -g]\Big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\Big)$$

$$\text{[;]}\quad A \to [x'' = -g]\Big([?x = 0;\, v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\Big)$$

$$\text{[}\cup\text{]}\quad A \to [x'' = -g][?x = 0;\, v := -cv \cup ?x \geq 0]B(x,v)$$

$$\text{[;]}\quad A \to [x'' = -g;\, (?x = 0;\, v := -cv \cup ?x \geq 0)]B(x,v)$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A Proof of a Single-hop Bouncing Ball

$$
\begin{array}{rl}
 & A \rightarrow \forall t{\geq}0\, \big((H-\frac{g}{2}t^2{=}0 \rightarrow B(H-\frac{g}{2}t^2,-c(-gt))\big) \land \big(H-\frac{g}{2}t^2{\geq}0 \rightarrow B(H-\frac{g}{2}t^2,-gt \\
\hline
[:=] & A \rightarrow \forall t{\geq}0\, [x := H-\frac{g}{2}t^2]\big((x{=}0 \rightarrow B(x,-c(-gt))) \land (x{\geq}0 \rightarrow B(x,-gt))\big) \\
\hline
[:=] & A \rightarrow \forall t{\geq}0\, [x := H-\frac{g}{2}t^2][v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \land (x{\geq}0 \rightarrow B(x,v))\big) \\
\hline
[;] & A \rightarrow \forall t{\geq}0\, [x := H-\frac{g}{2}t^2;\, v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \land (x{\geq}0 \rightarrow B(x,v))\big) \\
\hline
['] & A \rightarrow [x'' = -g]\big((x = 0 \rightarrow B(x,-cv)) \land (x \geq 0 \rightarrow B(x,v))\big) \\
\hline
[:=] & A \rightarrow [x'' = -g]\big((x = 0 \rightarrow [v := -cv]B(x,v)) \land (x \geq 0 \rightarrow B(x,v))\big) \\
\hline
[?],[?] & A \rightarrow [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[;] & A \rightarrow [x'' = -g]\big([?x = 0;\, v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[\cup] & A \rightarrow [x'' = -g][?x = 0;\, v := -cv \cup ?x \geq 0]B(x,v) \\
\hline
[;] & A \rightarrow [x'' = -g;\, (?x = 0;\, v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

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A Proof of a Single-hop Bouncing Ball

Resolving abbreviations at the top premise yields provable arithmetic:

$$\begin{aligned}
&0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow \\
&\qquad \forall t{\ge}0\, \Big(\big(H - \frac{g}{2}t^2 = 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H \big) \\
&\qquad\qquad \land \big(H - \frac{g}{2}t^2 \ge 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H \big) \Big)
\end{aligned}$$

A Proof of a Single-hop Bouncing Ball

Resolving abbreviations at the top premise yields provable arithmetic:

$$
\begin{aligned}
&0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow \\
&\qquad \forall t{\ge}0 \left(\left(H - \frac{g}{2}t^2 = 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H\right) \right. \\
&\qquad\qquad \left. \land \left(H - \frac{g}{2}t^2 \ge 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H\right)\right)
\end{aligned}
$$

A Proof of a Single-hop Bouncing Ball

Resolving abbreviations at the top premise yields provable arithmetic:

$$\begin{aligned}
&0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow \\
&\qquad \forall t{\ge}0 \, \Big(\big(H - \frac{g}{2}t^2 = 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H \big) \\
&\qquad\qquad \land \big(H - \frac{g}{2}t^2 \ge 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H \big) \Big)
\end{aligned}$$

A Proof of a Single-hop Bouncing Ball

Resolving abbreviations at the top premise yields provable arithmetic:

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow$$
$$\forall t{\geq}0 \left((H - \frac{g}{2}t^2 = 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \leq H) \right.$$
$$\left. \land (H - \frac{g}{2}t^2 \geq 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \leq H) \right)$$

Exciting!

We have just formally verified our very first CPS!

A Proof of a Single-hop Bouncing Ball

Resolving abbreviations at the top premise yields provable arithmetic:

$$\begin{aligned}&0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow\\ &\qquad \forall t{\ge}0 \, \Big(\big(H - \frac{g}{2}t^2 = 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H\big)\\ &\qquad\qquad \land \big(H - \frac{g}{2}t^2 \ge 0 \rightarrow 0 \le H - \frac{g}{2}t^2 \land H - \frac{g}{2}t^2 \le H\big)\Big)\end{aligned}$$

Exciting!

We have just formally verified our very first CPS!

Okay, it was a grotesquely simplified single-hop bouncing ball.
But the axioms of our proof technique were completely general, so they carry us forward to true CPSs.

Outline

1. Learning Objectives
2. Approach & Reminder
3. Intermediate Conditions for CPS
4. Dynamic Axioms for Dynamical Systems
   - Nondeterministic Choices
   - Assignments
   - Differential Equations
   - Tests
   - Sequential Compositions
   - Loops
   - Soundness
   - Diamonds
5. First Bouncing Ball Proof
6. **Summary**

Summary: Important Differential Dynamic Logic Axioms

equations of truth

$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$

$[?]$ $[?Q]P \leftrightarrow (Q \rightarrow P)$

$[']$ $[x' = f(x)]p(x) \leftrightarrow \forall t \geq 0\, [x := y(t)]p(x)$ $\quad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]$ $[\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

$\langle\cdot\rangle$ $\langle\alpha\rangle P \leftrightarrow \neg[\alpha]\neg P$

One axiom for each HP operator
Using an axiom from left to right simplifies the HP structure

Summary: Important Differential Dynamic Logic Axioms

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$[\cup]\ [\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

$[;]\ [\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

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Summary: Important Differential Dynamic Logic Axioms

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One axiom for each HP operator
Using an axiom from left to right simplifies the HP structure except . . .

Summary: Important Differential Dynamic Logic Axioms

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$$[:=]\ [x := e]p(x) \leftrightarrow p(e)$$

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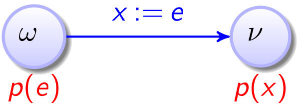
One axiom for each HP operator
Using an axiom from left to right simplifies the HP structure except . . .

Admissibility Caveats

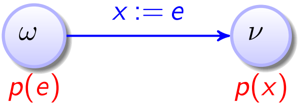
$[:=]\quad [x := e]p(x) \leftrightarrow p(e)$



- Elegant understanding is via uniform substitutions in Part IV. Till then:

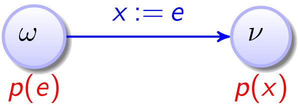
Admissibility Caveats

$[:=]\quad [x := e]p(x) \leftrightarrow p(e)$



- Elegant understanding is via uniform substitutions in Part IV. Till then:
  1. $p(e)$ stands for the same formula as $p(x)$

Admissibility Caveats



[:=] $[x := e]p(x) \leftrightarrow p(e)$

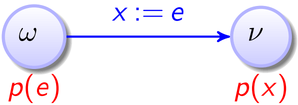
Elegant understanding is via uniform substitutions in Part IV. Till then:

1. $p(e)$ stands for the same formula as $p(x)$
2. except all free occurrences of x are replaced by e

- $[x := x + y]\, x \leq y^2 \leftrightarrow x + y \leq y^2$
- $[x := x^2]\forall x\,(x \geq 0) \leftrightarrow \forall x\,(x^2 \geq 0)$
- $[x := x + y]\forall y\,(x \leq y^2) \leftrightarrow \forall y\,(x + y \leq y^2)$
- $[x := x + y][y := 5]x{\geq}0 \leftrightarrow [y := 5]x + y{\geq}0$
- $[y := 2b][(x := x{+}y; x' = y)^*]x{\geq}y \leftrightarrow [(x := x{+}2b; x' = 2b)^*]x{\geq}2b$

Admissibility Caveats

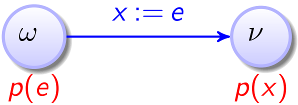
[:=] $[x := e]p(x) \leftrightarrow p(e)$



- Elegant understanding is via uniform substitutions in Part IV. Till then:
  1. $p(e)$ stands for the same formula as $p(x)$
  2. except all free occurrences of x are replaced by e
  3. but x cannot occur in $p(x)$ bound in a quantifier or a modality with an assignment or ODE for x
  - $[x := x + y]\, x \leq y^2 \leftrightarrow x + y \leq y^2$
  - $[x := x^2]\forall x\, (x \geq 0) \leftrightarrow \forall x\, (x^2 \geq 0)$
  - $[x := x + y]\forall y\, (x \leq y^2) \leftrightarrow \forall y\, (x + y \leq y^2)$
  - $[x := x + y][y := 5]x{\geq}0 \leftrightarrow [y := 5]x + y{\geq}0$
  - $[y := 2b][(x := x{+}y; x' = y)^*]x{\geq}y \leftrightarrow [(x := x{+}2b; x' = 2b)^*]x{\geq}2b$

Admissibility Caveats

$[:=]\quad [x := e]p(x) \leftrightarrow p(e)$



Elegant understanding is via uniform substitutions in Part IV. Till then:

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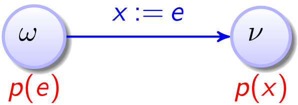
Admissibility Caveats



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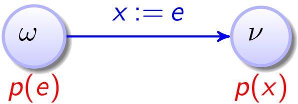

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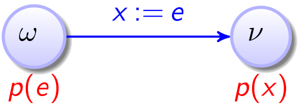

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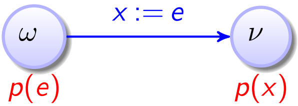

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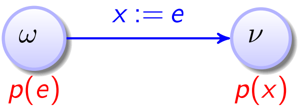

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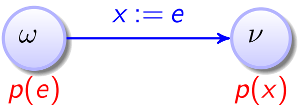

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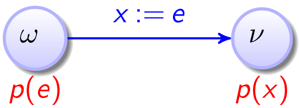

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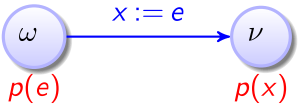

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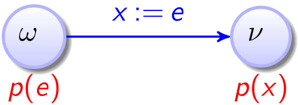

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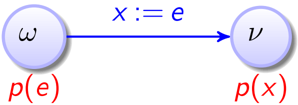

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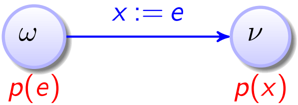

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If you bind a free variable, you go to logic jail!

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