10: Differential Equations & Differential Invariants
Logical Foundations of Cyber-Physical Systems

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1. Learning Objectives

2. A Gradual Introduction to Differential Invariants
   - Global Descriptive Power of Local Differential Equations
   - Intuition for Differential Invariants
   - Deriving Differential Equations

3. Differentials
   - Syntax
   - Semantics of Differential Symbols
   - Semantics of Differential Equations
   - Soundness
   - Example Proofs

4. Soundness Proof

5. Summary
Outline

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Learning Objectives
Differential Equations & Differential Invariants

discrete vs. continuous analogies
rigorous reasoning about ODEs
induction for differential equations
differential facet of logical trinity

understanding continuous dynamics
relate discrete+continuous

semantics of ODEs
operational CPS effects
Differential Facet of Logical Trinity

Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of $e = \bar{e}$ relate to the semantics of $e - \bar{e} = 0$, syntactically? What about derivatives?
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5. Summary
<table>
<thead>
<tr>
<th>ODE</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x' = 1, x(0) = x_0 )</td>
<td>( x(t) = x_0 + t )</td>
</tr>
<tr>
<td>( x' = 5, x(0) = x_0 )</td>
<td>( x(t) = x_0 + 5t )</td>
</tr>
<tr>
<td>( x' = x, x(0) = x_0 )</td>
<td>( x(t) = x_0 e^t )</td>
</tr>
<tr>
<td>( x' = x^2, x(0) = x_0 )</td>
<td>( x(t) = \frac{x_0}{1 - tx_0} )</td>
</tr>
<tr>
<td>( x' = \frac{1}{x}, x(0) = 1 )</td>
<td>( x(t) = \sqrt{1 + 2t} \ldots )</td>
</tr>
<tr>
<td>( y'(x) = -2xy, y(0) = 1 )</td>
<td>( y(x) = e^{-x^2} )</td>
</tr>
<tr>
<td>( x'(t) = tx, x(0) = x_0 )</td>
<td>( x(t) = x_0 e^{\frac{t^2}{2}} )</td>
</tr>
<tr>
<td>( x' = \sqrt{x}, x(0) = x_0 )</td>
<td>( x(t) = \frac{t^2}{4} \pm t \sqrt{x_0 + x_0} )</td>
</tr>
<tr>
<td>( x' = y, y' = -x, x(0) = 0, y(0) = 1 )</td>
<td>( x(t) = \sin t, y(t) = \cos t )</td>
</tr>
<tr>
<td>( x' = 1 + x^2, x(0) = 0 )</td>
<td>( x(t) = \tan t )</td>
</tr>
<tr>
<td>( x'(t) = \frac{2}{t^3} x(t) )</td>
<td>( x(t) = e^{-\frac{1}{t^2}} ) non-analytic</td>
</tr>
<tr>
<td>( x' = x^2 + x^4 )</td>
<td>non-elementary</td>
</tr>
<tr>
<td>( x'(t) = e^{t^2} )</td>
<td></td>
</tr>
</tbody>
</table>
Descriptive power of differential equations

1. Descriptive power: differential equations characterize continuous evolution only locally by the respective directions.
2. Simple differential equations describe complicated physical processes.
3. Complexity difference between local description and global behavior.
4. Analyzing ODEs via their solutions undoes their descriptive power.
5. Let’s exploit descriptive power of ODEs for proofs!

\[ x'' = -x \quad x(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \ldots \]

\[ x''(t) = e^{t^2} \quad \text{no elementary closed-form solution} \]
Global Descriptive Power of Local Differential Equations

You also prefer loop induction to unfolding all loop iterations, globally . . .

Descriptive power of differential equations

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Intuition for Differential Invariants

Differential Invariant

\[ \Gamma \vdash F, \Delta \quad F \vdash ???F \quad F \vdash P \]

\[ \Gamma \vdash [x' = f(x)]P, \Delta \]

\[ [\prime] [x' = f(x)]P \leftrightarrow \forall t \geq 0 [x := y(t)]P \quad (y' = f(y), y(0) = x) \]
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\((y' = f(y), y(0) = x)\)
Intuition for Differential Invariants

**Differential Invariant**

\[ \Gamma \vdash F, \Delta \quad F \vdash ??? \quad F \vdash P \]

\[ \Gamma \vdash [x' = f(x)]P, \Delta \]

Want: formula \( F \) remains true in the direction of the dynamics

\[ \neg F \quad \neg F \]

\[ ['] [x' = f(x)]P \leftrightarrow \forall t \geq 0 [x := y(t)]P \quad (y' = f(y), y(0) = x) \]

Next step is undefined for ODEs. But don’t need to know where exactly the system evolves to. Just that it remains somewhere in \( F \).

Show: only evolves into directions in which formula \( F \) stays true.
Guiding Example

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2 \]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] \]

\[ v^2 + w^2 = r^2 \]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2 \]

\[ v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v] v^2 + w^2 - r^2 = 0 \]
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5 Summary
Syntax With Primes

Syntax

\[ e ::= x \mid c \mid e + k \mid e - k \mid e \cdot k \mid e/k \]

Derivatives

\[
\begin{align*}
(e + k)' &= (e)' + (k)' \\
(e - k)' &= (e)' - (k)' \\
(e \cdot k)' &= (e)' \cdot k + e \cdot (k)' \\
(e/k)' &= \frac{(e)' \cdot k - e \cdot (k)'}{k^2}
\end{align*}
\]

...What do these primes mean? ...

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Syntax With Primes

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\begin{align*}
(e/k)' &= ((e)' \cdot k - e \cdot (k)')/k^2 \\
(c())' &= 0 \quad \text{for constants/numbers } c()
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\begin{align*}
(e/k)' &= ((e)' \cdot k - e \cdot (k)')/k^2 \quad \text{same singularities}
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Syntax With Primes

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\]

---

internalize primes into dL syntax

...What do these primes mean? ...
The Meaning of Primes

\[ \omega = \sum x \omega(x') \partial_{[e]} \partial_x (\omega) \]

Partial \[ \partial_{[e]} \partial_x (\omega) = \lim_{\kappa \to 0} \omega(x) \kappa^{-1} e - \omega(x) \kappa^{-1} e - \omega(x) \kappa^{-1} e + \omega(x) \kappa^{-1} e \]

what's the time derivative?

what's the time?

depends on the differential equation?

Not compositional!

well-defined in isolated state \( \omega \) at all?

No time-derivative without time!

meaning is a function of \( x \) and \( x' \).

Differential form!

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The Meaning of Primes

Semantics

\[ \omega[(e)'] = \]

what's the time derivative?
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The Meaning of Primes

Semantics

\[ \omega[(e)'] = \frac{d\omega[e]}{dt} \]
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what's the time derivative?
The Meaning of Primes

Semantics

\[ \omega[[e]'] = \frac{d\omega[e]}{dt} \]

what’s the time derivative? what’s the time?
The Meaning of Primes

Semantics

\[ \omega[(e)'] = \frac{d\omega[e]}{dt} \quad \text{nonsense!} \]

what’s the time derivative?  what’s the time?
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what’s the time derivative?  what’s the time?
depends on the differential equation?  Not compositional!
well-defined in isolated state \( \omega \) at all?  No time-derivative without time!
The Meaning of Differentials

\[ \omega[(e)'] = \sum_x \omega(x') \frac{\partial[e]}{\partial x} (\omega) \]

what’s the time derivative? depends on the differential equation? well-defined in isolated state \( \omega \) at all? meaning is a function of \( x \) and \( x' \).

what’s the time? Not compositional! No time-derivative without time! Differential form!
The Meaning of Differentials

**Semantics**

\[ \omega[(e)'] = \sum_x \omega(x') \frac{\partial[e]}{\partial x}(\omega) \]

**Partial**

\[ \frac{\partial[e]}{\partial x}(\omega) = \lim_{\kappa \to \omega(x)} \frac{\omega_\kappa[e] - \omega[e]}{\kappa - \omega(x)} \]

what’s the time derivative?
depends on the differential equation?
well-defined in isolated state \( \omega \) at all?
meaning is a function of \( x \) and \( x' \).

what’s the time?
Not compositional!
No time-derivative without time!
Differential form!
Definition (Hybrid program semantics) \( [[\cdot]] : \text{HP} \rightarrow \wp(S \times S) \)

\[
[[\dot{x} = f(x) \& Q]] = \{ (\varphi(0)|_{\{\dot{x}'\}^c}, \varphi(r)) : \varphi(z) \models \dot{x} = f(x) \land Q \text{ for all } 0 \leq z \leq r \}
\]

for a solution \( \varphi : [0, r] \rightarrow S \) of any duration \( r \in \mathbb{R} \)\)

where \( \varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z) \)

Initial value of \( x' \) in \( \omega \) is irrelevant since defined by ODE.

Final value of \( x' \) is carried over to the final state \( \nu \).
Definition (Hybrid program semantics) \( [[\cdot]] : \text{HP} \rightarrow \wp(S \times S) \)

\[
[[x' = f(x) \& Q]] = \{(\varphi(0)|_{x'} \subseteq \varphi(r)) : \varphi(z) \models x' = f(x) \& Q \text{ for all } 0 \leq z \leq r \}
\]

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Definition (Hybrid program semantics) \((\llbracket \cdot \rrbracket : \text{HP} \rightarrow \wp(S \times S))\)

\[
\llbracket x' = f(x) & Q \rrbracket = \{(\omega, \nu) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r \text{ for a solution } \varphi : [0, r] \rightarrow S \text{ of any duration } r \in \mathbb{R} \text{ with } \varphi(0) = \omega \text{ and } \varphi(r) = \nu\}
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where \(\varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)\)
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\text{with } \varphi(0) = \omega \text{ except on } x' \text{ and } \varphi(r) = \nu\}
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where \(\varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)\)

Initial value of \(x'\) in \(\omega\) is irrelevant since defined by ODE.
Final value of \(x'\) is carried over to the final state \(\nu\).
Definition (Hybrid program semantics) \( ([\cdot] : \text{HP} \rightarrow \wp(S \times S)) \)

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with \(\varphi(0) = \omega\) except on \(x'\) and \(\varphi(r) = \nu\)

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Initial value of \(x'\) in \(\omega\) is irrelevant since defined by ODE.
Final value of \(x'\) is carried over to the final state \(\nu\).
### Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

\[
\varphi(z)\langle (e)' \rangle = \frac{d\varphi(t)\langle e \rangle}{dt}(z)
\]

### Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \leftrightarrow [x' := f(x)]P$

### Lemma (Derivations) (Equations of Differentials)

\[
\begin{align*}
(e + k)' &= (e)' + (k)' \\
(e \cdot k)' &= (e)' \cdot k + e \cdot (k)' \\
(c())' &= 0 \\
(x)' &= x' 
\end{align*}
\]

for constants/numbers $c()$

for variables $x \in \mathcal{V}$
**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$
\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)
$$

**Lemma (Differential assignment) (Effect on Differentials)**

If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \iff [x' := f(x)]P$

**Axiomatics**

DE $[x' = f(x) \land Q]P \iff [x' = f(x) \land Q][x' := f(x)]P$

DI $([x' = f(x)]e = 0 \iff e = 0) \iff [x' = f(x)](e)' = 0$
### Lemma (Differential Lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$$

### Lemma (Differential Assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \iff [x' := f(x)] P$

### Axiomatics

**DE** $[x' = f(x) \land Q] P \iff [x' = f(x) \land Q][x' := f(x)] P$

**DI** $([x' = f(x)]e = 0 \iff e = 0) \iff [x' = f(x)](e)' = 0$

**Rate of change of $e$ along ODE is 0**
Differential Invariant

\[ dl \quad \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0} \]
Differential Invariant

\[
\frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}
\]

\[\text{DI } ([x' = f(x)]e = 0 \iff e = 0) \iff [x' = f(x)](e)' = 0\]

\[\text{DE } [x' = f(x)]P \iff [x' = f(x)][x' := f(x)]P\]
Differential Invariants for Differential Equations

Differential Invariant

\[ \frac{e = 0}{[x' := f(x)]e = 0} \]

\[ \frac{\vdash [x' := f(x)](e)' = 0}{\vdash e = 0 \vdash [x' = f(x)]e = 0} \]

\[ \text{DI } ([x' = f(x)]e = 0 \iff e = 0) \iff [x' = f(x)](e)' = 0 \]

\[ \text{DE } [x' = f(x)]P \iff [x' = f(x)][x' := f(x)]P \]

Proof (dl is a derived rule).

\[ \frac{\vdash e = 0}{\vdash [x' = f(x)]e = 0} \]
Differential Invariants for Differential Equations

Differential Invariant

\[
\frac{\vdash [x' := f(x)](e)' = 0}{\vdash e = 0 \vdash [x' = f(x)]e = 0}
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Proof (dl is a derived rule).

\[
\begin{align*}
\text{DE} & \quad \vdash [x' = f(x)](e)' = 0 \\
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\end{align*}
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Differential Invariant for Differential Equations

Differential Invariant

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Proof (dl is a derived rule).

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Differential Invariant

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\begin{align*}
\mathrm{dI} & \quad \vdash \ [x' := f(x)](e)' = 0 \\
\quad & \quad \quad \vdash \ [x' = f(x)]e = 0 \\
\end{align*}
\]

\[\mathrm{DI} \quad ([x' = f(x)]e = 0 \leftrightarrow e = 0) \leftrightarrow [x' = f(x)](e)' = 0\]

\[\mathrm{DE} \quad [x' = f(x)]P \leftrightarrow [x' = f(x)][x' := f(x)]P\]

\[\mathrm{Proof (dI is a derived rule).}\]

\[
\begin{align*}
\vdash & \ [x' := f(x)](e)' = 0 \\
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\end{align*}
\]

\[\mathrm{G} \quad P \quad \Rightarrow \quad [\alpha[P\]

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LFCPS/10: Differential Equations & Differential Invariants
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2 \]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2 \]

\[ \vdash v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v] v^2 + w^2 - r^2 = 0 \]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2 + w^2 = r^2 \]

\[ d(l v^2 + w^2 - r^2 = 0) \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0 \]

\[ \rightarrow R \]

\[ \vdash v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v]v^2 + w^2 - r^2 = 0 \]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2 + w^2 = r^2 \]

\[
\begin{align*}
\vdash [v' := w][w' := -v] & \quad 2vv' + 2ww' - 2rr' = 0 \\
\vdash [v' = w, w' = -v] & \quad v^2 + w^2 - r^2 = 0 \\
\vdash v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v]v^2 + w^2 - r^2 = 0
\end{align*}
\]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2 \]

\[ \begin{align*}
  &\mathbb{R} \\
  \Rightarrow & 2v(w) + 2w(-v) = 0 \\
  \iff & [v' := w][w' := -v] 2v v' + 2w w' - 2rr' = 0 \\
  \iff & [v' = w, w' = -v] v^2 + w^2 - r^2 = 0 \Rightarrow \frac{d}{dt} [v^2 + w^2 - r^2 = 0] \\
  \iff & [v' = w, w' = -v] v^2 + w^2 - r^2 = 0
\end{align*} \]
Guiding Example: Rotational Dynamics

\[ v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2 + w^2 = r^2 \]

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{R} \\
[=:]
\end{array} \\
\begin{array}{c}
dl \\
\rightarrow \mathbb{R}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\vdash 2v(w) + 2w(-v) = 0 \\
\vdash [v' := w][w' := -v]2vv' + 2ww' - 2rr' = 0 \\
\vdash v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v]v^2 + w^2 - r^2 = 0
\end{array}
\end{array}
\]

Simple proof without solving ODE, just by differentiating
Guiding Example: Rotational Dynamics

\[ v'^2 + w'^2 = r'^2 \rightarrow [v' = w, w' = -v] v'^2 + w'^2 = r'^2 \]

Simple proof without solving ODE, just by differentiating
Example Proof

\[ x^2 y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] \]

\[ x^2 y - 2 = 0 \]
Example Proof

\[
\begin{align*}
&\text{dI} \quad x^2 y - 2 = 0 \vdash [x' = -x^2, y' = 2xy] x^2 y - 2 = 0 \\
&\rightarrow R \quad y \quad x^2 y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2 y - 2 = 0
\end{align*}
\]
\[
\begin{align*}
\text{[:=]} & \quad \vdash [x' := -x^2, y' := 2xy] \quad 2x' y + x^2 y' - 0 = 0 \\
\text{dl} & \quad \vdash [x' = -x^2, y' = 2xy] \quad x^2 y - 2 = 0 \\
\rightarrow R & \quad \vdash x^2 y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] \quad x^2 y - 2 = 0
\end{align*}
\]
Example Proof

\[ \mathbb{R} \vdash 2x(-x^2)y + x^2(2xy) = 0 \]
\[ [:=} \quad \vdash [x':=-x^2][y':=2xy] 2xx'y + x^2y' - 0 = 0 \]
\[ dI \quad x^2y - 2 = 0 \vdash [x' = -x^2, y' = 2xy] x^2y - 2 = 0 \]
\[ \rightarrow \mathbb{R} \quad y \vdash x^2y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2y - 2 = 0 \]
Example Proof

$\mathbb{R}$

$\vdash 2x(-x^2)y + x^2(2xy) = 0$

$[:=$]

$\vdash [x' = -x^2, y' = 2xy] 2xx'y + x^2y' - 0 = 0$

$dl$

$x^2y - 2 = 0 \vdash [x' = -x^2, y' = 2xy] x^2y - 2 = 0$

$\rightarrow \mathbb{R}$

$\vdash x^2y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2y - 2 = 0$
Outline

1. Learning Objectives

2. A Gradual Introduction to Differential Invariants
   - Global Descriptive Power of Local Differential Equations
   - Intuition for Differential Invariants
   - Deriving Differential Equations

3. Differentials
   - Syntax
   - Semantics of Differential Symbols
   - Semantics of Differential Equations
   - Soundness
   - Example Proofs

4. Soundness Proof

5. Summary
### Differential Substitution Lemmas

**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[[e']] = \frac{d\varphi(t)[e]}{dt}(z)$$

**Syntactic**

**Analytic**

**Lemma (Differential assignment) (Effect on Differentials)**

If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \iff [x' := f(x)]P$

**Lemma (Derivations) (Equations of Differentials)**

- $(e + k)' = (e)' + (k)'$
- $(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
- $(c())' = 0$ \hspace{2cm} for constants/numbers $c()$
- $(x)' = x'$ \hspace{2cm} for variables $x \in \mathcal{V}$
Soundness Proof

**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$$

**Lemma (Differential assignment) (Effect on Differentials)**

If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \iff [x' := f(x)]P$

**Semantics**

$$\omega[(e)'] = \sum_x \omega(x') \frac{\partial[e]}{\partial x}(\omega)$$

**Definition (Hybrid program semantics)**

$$[(x' = f(x) \land Q)] = \{(\varphi(0)|_{x'} \subseteq, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r$$

for a $\varphi : [0, r] \rightarrow S$ where $\varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)$
**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

If \( \varphi \models x' = f(x) \land Q \) for duration \( r > 0 \), then for all \( 0 \leq z \leq r \), \( FV(e) \subseteq \{x\} \):

\[
\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)
\]

**Semantics**

\[
\omega[(e)'] = \sum_x \omega(x') \frac{\partial[e]}{\partial x}(\omega)
\]

**Definition (Hybrid program semantics)**

\[
[x' = f(x) \& Q] = \{(\varphi(0)|_{x'} \in \mathcal{C}, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r \}
\]

for a \( \varphi : [0, r] \to S \) where \( \varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z) \)
Lemma (Differential lemma)  (Differential value vs. Time-derivative)

If $\phi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$$

$$\frac{d\varphi(t)[e]}{dt}(z) \triangleq \sum_x \frac{\partial[e]}{\partial x}(\varphi(z)) \frac{d\varphi(t)(x)}{dt}(z)$$

Semantics

$$\omega[(e)'] = \sum_x \omega(x') \frac{\partial[e]}{\partial x}(\omega)$$

Definition (Hybrid program semantics)  ($[\cdot] : HP \to \wp(S \times S)$)

$[x' = f(x) \land Q] = \{(\varphi(0)\mid_{x'} \subseteq, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r$ for a $\varphi : [0, r] \rightarrow S$ where $\varphi(z)(x') \triangleq \frac{d\varphi(t)(x)}{dt}(z)\}$
Soundness Proof

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If \( \phi \models x' = f(x) \land Q \) for duration \( r > 0 \), then for all \( 0 \leq z \leq r \), \( FV(e) \subseteq \{x\} \):

\[
\phi(z) \[ (e)' \] = \frac{d\phi(t)[e]}{dt}(z)
\]

\[
\frac{d\phi(t)[e]}{dt}(z) \overset{\text{chain}}{=} \sum_x \frac{\partial[e]}{\partial x}(\phi(z)) \frac{d\phi(t)(x)}{dt}(z)
\]

Semantics

\[
\omega[(e)'] = \sum_x \omega(x') \frac{\partial[e]}{\partial x}(\omega)
\]

Definition (Hybrid program semantics) \( ([\cdot]) : \text{HP} \rightarrow \wp(S \times S) \)

\([x' = f(x) \land Q] = \{ (\phi(0)_{x'}, \phi(r)) : \phi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r \}

\]

for a \( \phi : [0, r] \rightarrow S \) where \( \phi(z)(x') \overset{\text{def}}{=} \frac{d\phi(t)(x)}{dt}(z) \)
Soundness Proof

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)\llbracket (e)' \rrbracket = \frac{d\varphi(t)\llbracket e \rrbracket}{dt}(z)$$

$$\frac{d\varphi(t)\llbracket e \rrbracket}{dt}(z) \triangleq \sum_x \frac{\partial \llbracket e \rrbracket}{\partial x}(\varphi(z)) \frac{d\varphi(t)(x)}{dt}(z) = \sum_x \frac{\partial \llbracket e \rrbracket}{\partial x}(\varphi(z))\varphi(z)(x')$$

Semantics

$$\omega\llbracket (e)' \rrbracket = \sum_x \omega(x') \frac{\partial \llbracket e \rrbracket}{\partial x}(\omega)$$

Definition (Hybrid program semantics) (\llbracket \cdot \rrbracket : HP \rightarrow \wp(S \times S))

$$\llbracket x' = f(x) \& Q \rrbracket = \{(\varphi(0)\llbracket x' \rrbracket, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r \}

\text{for a } \varphi : [0, r] \rightarrow S \text{ where } \varphi(z)(x') \triangleq \frac{d\varphi(t)(x)}{dt}(z)$$

André Platzer (CMU) LFCPS/10: Differential Equations & Differential Invariants LFCPS/10 17 / 19
Soundness Proof

Lemma (Differential lemma)  (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$$

$$\frac{d\varphi(t)[e]}{dt}(z) \overset{\text{chain}}{=} \sum_x \frac{\partial[e]}{\partial x}(\varphi(z)) \frac{d\varphi(t)(x)}{dt}(z) = \sum_x \frac{\partial[e]}{\partial x}(\varphi(z))\varphi(z)(x')$$

Semantics

$$\varphi(z)[(e)'] = \sum_x \varphi(z)(x') \frac{\partial[e]}{\partial x}(\varphi(z))$$

Definition (Hybrid program semantics)

$[x' = f(x) \& Q] = \{(\varphi(0)|_{x'} = 0, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r\}$

for a $\varphi : [0, r] \to S$ where $\varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)$
Soundness Proof

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$$\frac{d\varphi(t)[e]}{dt}(z) \triangleq \sum_x \frac{\partial[e]}{\partial x}(\varphi(z)) \frac{d\varphi(t)(x)}{dt}(z) = \sum_x \frac{\partial[e]}{\partial x}(\varphi(z))\varphi(z)(x')$$

**Semantics**

$$\varphi(z)[(e)'] = \sum_x \varphi(z)(x') \frac{\partial[e]}{\partial x}(\varphi(z))$$

**Definition (Hybrid program semantics)**

$$[x' = f(x) \& Q] = \{(\varphi(0)|_{x'}|^0, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \leq z \leq r \}$$

for a $\varphi : [0, r] \rightarrow S$ where $\varphi(z)(x') \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)$
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Differential Invariants for Differential Equations

Differential Invariant

\[ \dI \quad \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0} \]

\[ \DI \ ([x' = f(x)]e = 0 \leftrightarrow e = 0) \leftrightarrow [x' = f(x)](e)' = 0 \]

\[ \DE \ [x' = f(x)]P \leftrightarrow [x' = f(x)][x' := f(x)]P \]
Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$$

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If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \iff [x' := f(x)]P$

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$$(e + k)' = (e)' + (k)'$$
$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$
$$(c())' = 0 \quad \text{for constants/numbers } c()$$
$$(x)' = x' \quad \text{for variables } x \in \mathcal{V}$$
Appendix

- Differential Equations vs. Loops
- Differential Invariant Terms and Invariant Functions
Differential Equations vs. Loops

Lemma (Differential equations are their own loop)

\[ \lbrack (x' = f(x))^* \rbrack = \lbrack x' = f(x) \rbrack \]

<table>
<thead>
<tr>
<th>loop ( \alpha^* )</th>
<th>ODE ( x' = f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>repeat any number ( n \in \mathbb{N} ) of times</td>
<td>evolve for any duration ( r \in \mathbb{R} )</td>
</tr>
<tr>
<td>can repeat 0 times</td>
<td>can evolve for duration 0</td>
</tr>
<tr>
<td>effect depends on previous loop iteration</td>
<td>effect depends on the past solution</td>
</tr>
<tr>
<td>local generator is loop body ( \alpha )</td>
<td>local generator ( x' = f(x) )</td>
</tr>
<tr>
<td>full global execution trace</td>
<td>global solution ( \varphi : [0, r] \to S )</td>
</tr>
<tr>
<td>unwinding proof by iteration ( [\ast] )</td>
<td>proof by global solution with ( [\prime] )</td>
</tr>
<tr>
<td>inductive proof with loop invariant</td>
<td>proof with differential invariant</td>
</tr>
</tbody>
</table>
\[ R \vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0 \]

\[ \vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]
Theorem (Sophus Lie)

\( \text{DI} \subset \mathcal{Q} \vdash \) \[ \begin{aligned} [x' = 4y^3, y' = -4x^3] & \quad x^2 + y^2 = 0 \\
\therefore x^2 + y^2 = 0 & \Rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 
\end{aligned} \]
\[ x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \]

\[ x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]

\[ \vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]
\[
\begin{align*}
\vdash [x' &= 4y^3, y' = -4x^3] (4x^3x' + 4y^3y') = 0 \\
\vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \\
\vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \\
\vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0
\end{align*}
\]
| $\mathbb{R}$ | $\vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0$ |
| $[=:]$ | $\vdash [x' := 4y^3][y' := -4x^3](4x^3x' + 4y^3y') = 0$ |
| $dI$ | $x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0$ |
| cut, MR | $x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0$ |
| $\rightarrow R$ | $\vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0$ |
Generalizing Differential Invariants: Stronger Postcondition

\[ \mathbb{R} \vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0 \]

\[ [\vdash x':=4y^3][y':=−4x^3](4x^3x' + 4y^3y') = 0 \]

\[ dI \]

\[ x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \]

\[ \text{cut,MR} \]

\[ x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]

\[ \rightarrow \mathbb{R} \]

\[ \vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]
Generalizing Differential Invariants: Stronger Postcondition

\[
\begin{align*}
\mathbb{R} \vdash 4x^3(4y^3) + 4y^3(-4x^3) &= 0 \\
[=] \vdash [x’:=4y^3][y’:=-4x^3](4x^3x’ + 4y^3y’) &= 0 \\
dl \vdash x^4 + y^4 &= 0 \vdash [x’ = 4y^3, y’ = -4x^3] x^4 + y^4 = 0 \\
cut, MR \vdash x^2 + y^2 &= 0 \vdash [x’ = 4y^3, y’ = -4x^3] x^2 + y^2 = 0 \\
\rightarrow \mathbb{R} \vdash x^2 + y^2 = 0 \rightarrow [x’ = 4y^3, y’ = -4x^3] x^2 + y^2 &= 0
\end{align*}
\]
Generalizing Differential Invariants: Stronger Postcondition

\[ \mathbb{R} \vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0 \]

\[ [\vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0] \]

\[ \vdash [x' := 4y^3][y' := -4x^3](4x^3x' + 4y^3y') = 0 \]

\[ \vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \]

\[ \vdash x^2 + y^2 = 0 \]

\[ \vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]

\[ \vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \]

---

**Theorem (Sophus Lie)**

\[ Dl_c \]

\[ Q \vdash [x' := f(x)](e)' = 0 \]

\[ \vdash \forall c (e = c \rightarrow [x' = f(x) \& Q]e = c) \]

Premise and conclusion are equivalent if \( Q \) is a domain, i.e., characterizing a connected open set.
Generalizing Differential Invariants: Stronger Postcondition

\[
\begin{align*}
\mathbb{R} & \quad \Rightarrow 4x^3(4y^3) + 4y^3(-4x^3) = 0 \\
[=:] & \quad \Rightarrow [x' := 4y^3][y' := -4x^3](4x^3x' + 4y^3y') = 0 \\
dl & \quad x^4 + y^4 = 0 \Rightarrow [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \\
cut, MR & \quad x^2 + y^2 = 0 \Rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \\
\rightarrow \mathbb{R} & \quad \Rightarrow x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0
\end{align*}
\]

Theorem (Sophus Lie)

\[D I_c\]

\[
\begin{align*}
Q & \quad \vdash [x' := f(x)](e)' = 0 \\
\vdash \forall c \ (e = c \rightarrow [x' = f(x) \land Q]e = c)
\end{align*}
\]

Premise and conclusion are equivalent if \(Q\) is a domain, i.e., characterizing a connected open set.

Clu: \((e - c)' = (e)'\) independent of additive constants
Strengthening Induction Hypotheses

Stronger Induction Hypotheses

1. As usual in math and in proofs with loops:
2. Inductive proofs may need stronger induction hypotheses to succeed.
3. Differentially inductive proofs may need a stronger differential inductive structure to succeed.
4. Even if \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} = \{\{(x, y) \in \mathbb{R}^2 : x^4 + y^4 = 0\} \)
   have the same solutions, they have different differential structure.
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*Logical Foundations of Cyber-Physical Systems.*
URL: http://www.springer.com/978-3-319-63587-3,
doi:10.1007/978-3-319-63588-0.

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A complete uniform substitution calculus for differential dynamic logic.

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*Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.*
doi:10.1007/978-3-642-14509-4.

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Logics of dynamical systems.
André Platzer.

André Platzer.

André Platzer.