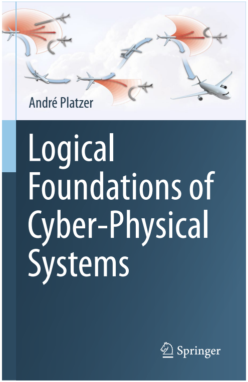
10: Differential Equations & Differential Invariants

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

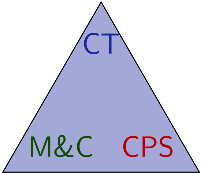
Outline

Outline

Learning Objectives

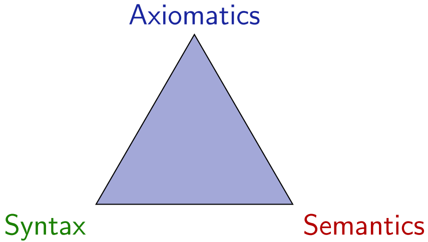
Differential Equations & Differential Invariants

discrete vs. continuous analogies
rigorous reasoning about ODEs
induction for differential equations
differential facet of logical trinity



understanding continuous dynamics
relate discrete+continuous

semantics of ODEs
operational CPS effects



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of $e = \tilde{e}$ relate to the semantics of $e - \tilde{e} = 0$, syntactically? What about derivatives?

Outline

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - t x_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1+2t}\dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

Descriptive power of differential equations

1. Descriptive power: differential equations characterize continuous evolution only locally by the respective directions.
2. Simple differential equations describe complicated physical processes.
3. Complexity difference between local description and global behavior
4. Analyzing ODEs via their solutions undoes their descriptive power.
5. Let's exploit descriptive power of ODEs for proofs!

$x'' = -x$	$x(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots$
$x''(t) = e^{t^2}$	no elementary closed-form solution

Global Descriptive Power of Local Differential Equations

You also prefer loop induction to unfolding all loop iterations, globally . . .

Descriptive power of differential equations

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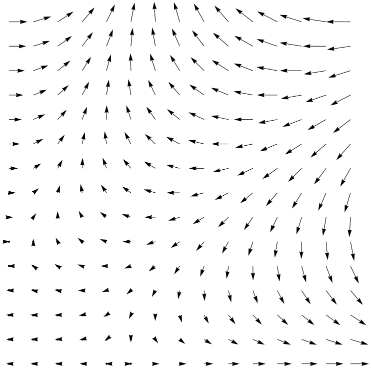
$x''(t) = e^{t^2}$ no elementary closed-form solution

Differential Invariant

$$\frac{\Gamma \vdash F, \Delta \quad F \vdash \textcolor{red}{???}F \quad F \vdash P}{\Gamma \vdash [x' = f(x)]P, \Delta}$$



$$['] \ [x' = f(x)]P \leftrightarrow \forall t{\geq}0\, [x := y(t)]P \qquad (y' = f(y),\ y(0) = x)$$

Intuition for Differential Invariants

Differential Invariant

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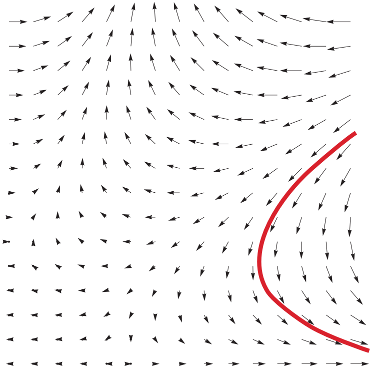

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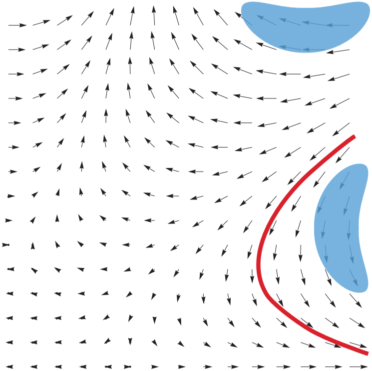

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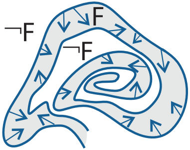
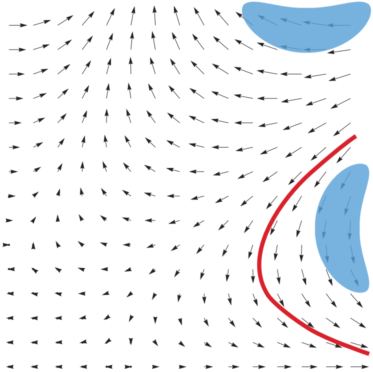
Want: formula F remains true in the direction of the dynamics





$$['] \; [x' = f(x)]P \leftrightarrow \forall t{\geq}0\, [x := y(t)]P \qquad (y' = f(y),\ y(0) = x)$$

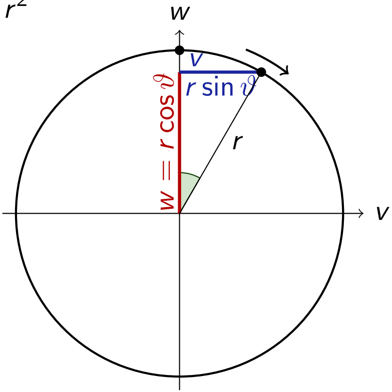
Next step is undefined for ODEs. But don't need to know where exactly the system evolves to. Just that it remains somewhere in F.
Show: only evolves into directions in which formula F stays true.

Guiding Example

$v^2+w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2+w^2 = r^2$

Guiding Example: Rotational Dynamics

$$v^2+w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2+w^2 = r^2$$



$v^2+w^2 = r^2 \to [v' = w, w' = -v]v^2+w^2 = r^2$

$$\to\text{R}\ \frac{}{\vdash v^2+w^2-r^2=0 \to [v' = w, w' = -v]v^2+w^2-r^2=0}$$

Outline

Syntax With Primes

Syntax $e ::= x \mid c \mid e + k \mid e - k \mid e \cdot k \mid e/k$

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Derivatives

$$(e+k)' = (e)' + (k)'$$
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$$(e\cdot k)' = (e)'\cdot k + e\cdot (k)'$$
$$(e/k)' = \big((e)'\cdot k - e\cdot (k)'\big)/k^2$$
$$(c())' = 0 \qquad \text{for constants/numbers } c()$$

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. . . What do these primes mean? . . .

Syntax With Primes

Syntax $e ::= x \mid c \mid e + k \mid e - k \mid e \cdot k \mid e/k \mid (e)'$

internalize primes into dL syntax

Derivatives

$$
\begin{aligned}
(e+k)' &= (e)' + (k)' \\
(e-k)' &= (e)' - (k)' \\
(e \cdot k)' &= (e)' \cdot k + e \cdot (k)' \\
(e/k)' &= ((e)' \cdot k - e \cdot (k)')/k^2 && \text{same singularities} \\
(c())' &= 0 && \text{for constants/numbers } c()
\end{aligned}
$$

. . . What do these primes mean? . . .

The Meaning of Primes

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Semantics

$\omega[\![(e)']\!] =$

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$$\omega[\![(e)']\!] = \frac{d\omega[\![e]\!]}{dt}$$

The Meaning of Primes

Semantics $\omega[\![(e)']\!] = \frac{d\omega[\![e]\!]}{dt}$

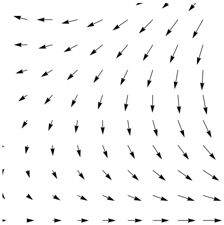
what's the time derivative?

Semantics $\omega[\![(e)']\!] = \frac{\mathrm{d}\omega[\![e]\!]}{\mathrm{d}t}$

what's the time derivative?

what's the time?

Semantics $\omega[\![(e)']\!] = \frac{\mathrm{d}\omega[\![e]\!]}{\mathrm{d}t}$ nonsense!



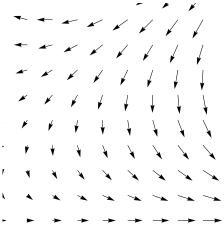
what's the time derivative?

depends on the differential equation?

what's the time?

Semantics

$\omega[\![(e)']\!] =$



what's the time derivative? what's the time?

depends on the differential equation? Not compositional!

Semantics

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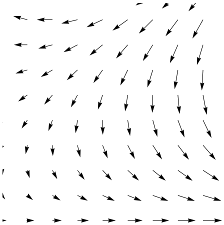

what's the time derivative?
depends on the differential equation?
well-defined in isolated state ω at all?

what's the time?
Not compositional!

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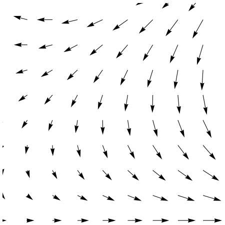
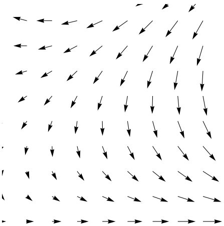


what's the time derivative? — what's the time?

depends on the differential equation? — Not compositional!

well-defined in isolated state ω at all? — No time-derivative without time!

Semantics $\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$



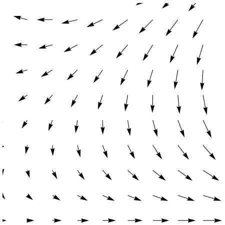
what's the time derivative?	what's the time?
depends on the differential equation?	Not compositional!
well-defined in isolated state ω at all?	No time-derivative without time!
meaning is a function of x and x'.	Differential form!

Semantics

$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Partial

$$\frac{\partial [\![e]\!]}{\partial x}(\omega) = \lim_{\kappa \to \omega(x)} \frac{\omega_x^\kappa[\![e]\!] - \omega[\![e]\!]}{\kappa - \omega(x)}$$



$\to \mathbb{R}$

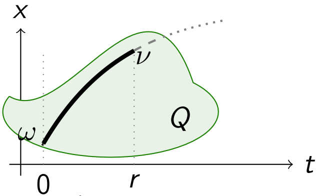
what's the time derivative?
depends on the differential equation?
well-defined in isolated state ω at all?
meaning is a function of x and x'.

what's the time?
Not compositional!
No time-derivative without time!
Differential form!

Definition (Hybrid program semantics) $(\llbracket\cdot\rrbracket : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

$$\llbracket x' = f(x) \,\&\, Q \rrbracket = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \le z \le r \text{ for a solution } \varphi : [0,r] \to \mathcal{S} \text{ of any duration } r \in \mathbb{R}\}$$

where $\varphi(z)(x') \stackrel{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)$



$x' = f(x) \,\&\, Q$

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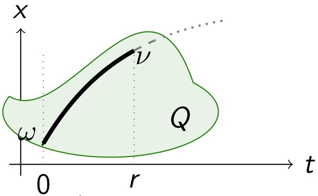

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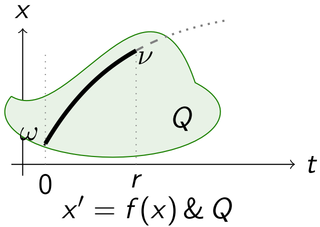

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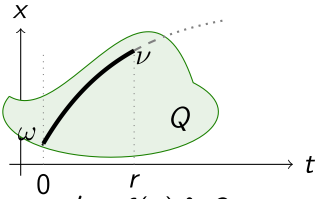

Initial value of x' in ω is irrelevant since defined by ODE.
Final value of x' is carried over to the final state ν.

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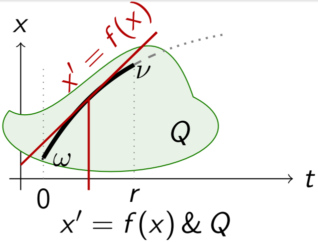

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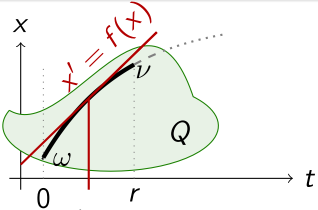

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Differential Substitution Lemmas

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$, *then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

Syntactic ′ → $$\varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$ ← Analytic ′

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e+k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \quad \textit{for constants/numbers } c()$$

$$(x)' = x' \quad \textit{for variables } x \in \mathcal{V}$$

Differential Substitution Lemmas

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Axiomatics

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

DI $\big([x' = f(x)]e = 0 \leftrightarrow e = 0\big) \leftarrow [x' = f(x)](e)' = 0$

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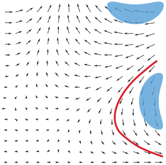
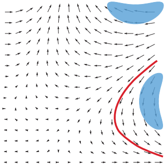
rate of change of e along ODE is 0



Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$



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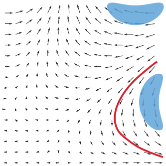
$$\text{DE}\quad [x' = f(x)]P \leftrightarrow [x' = f(x)][x' := f(x)]P$$



Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

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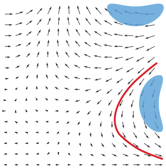
$$\text{DE}\ [x' = f(x)]P \leftrightarrow [x' = f(x)][x' := f(x)]P$$

Proof (dI is a derived rule).

$$\text{DI}\ \frac{}{e = 0 \vdash [x' = f(x)]e = 0}$$

□



Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

$$\text{DI}\ \big([x' = f(x)]e = 0 \leftrightarrow e = 0\big) \leftarrow [x' = f(x)](e)' = 0$$

$$\text{DE}\ [x' = f(x)]P \leftrightarrow [x' = f(x)][x' := f(x)]P$$

Proof (dI is a derived rule).

$$\text{DI}\ \cfrac{\text{DE}\ \cfrac{}{\vdash [x' = f(x)](e)' = 0}}{e = 0 \vdash [x' = f(x)]e = 0}$$

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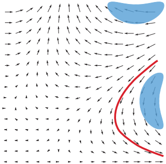
Differential Invariants for Differential Equations



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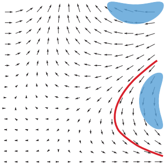
□

Differential Invariants for Differential Equations



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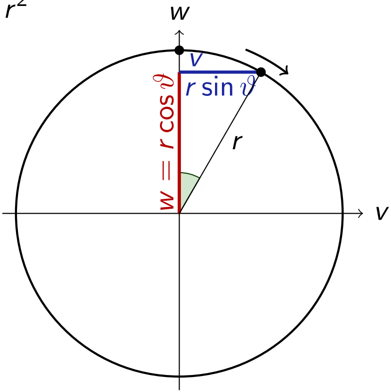
$$\text{G}\quad \frac{P}{[\alpha]P}$$

□

$$v^2+w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2+w^2 = r^2$$



$v^2+w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2+w^2 = r^2$

$$\rightarrow\text{R}\;\frac{}{\vdash v^2+w^2-r^2=0 \rightarrow [v' = w, w' = -v]v^2+w^2-r^2=0}$$

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$$\text{dI}\ \dfrac{}{v^2+w^2-r^2=0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}$$
$$\to\text{R}\ \dfrac{v^2+w^2-r^2=0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}{\vdash v^2+w^2-r^2=0 \to [v' = w, w' = -v]v^2+w^2-r^2=0}$$

$v^2+w^2 = r^2 \to [v' = w, w' = -v]v^2+w^2 = r^2$

$$\dfrac{\dfrac{\dfrac{}{\vdash [v':=w][w':=-v]2vv' + 2ww' - 2rr' = 0}\,[:=]}{v^2+w^2-r^2=0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}\,\text{dI}}{\vdash v^2+w^2-r^2=0 \to [v' = w, w' = -v]v^2+w^2-r^2=0}\,\to\text{R}$$

Guiding Example: Rotational Dynamics

$$v^2+w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2+w^2 = r^2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 2v(w) + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' - 2rr' = 0}\ [:=]}{v^2+w^2-r^2=0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}\ \mathrm{dI}}{\vdash v^2+w^2-r^2=0 \rightarrow [v' = w, w' = -v]v^2+w^2-r^2=0}\ \rightarrow\!\mathrm{R}$$

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$$
\begin{array}{ll}
 & \qquad * \\
\hline
\mathbb{R} & \vdash 2v(w) + 2w(-v) = 0 \\
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\hline
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$$

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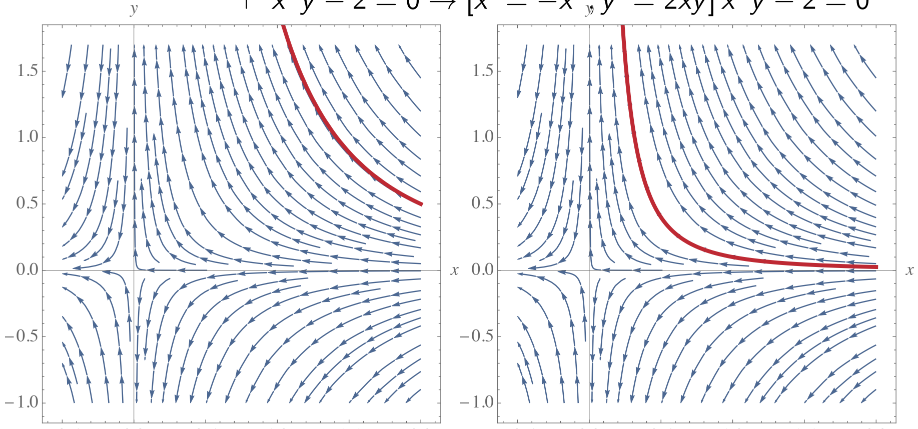
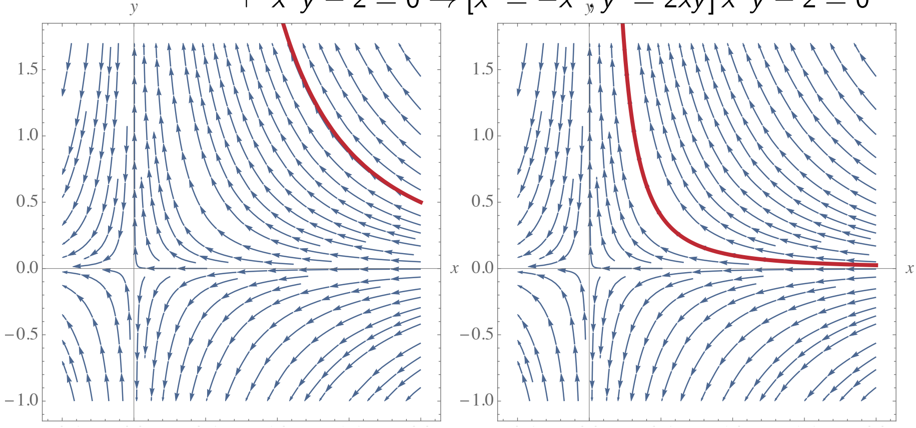
Simple proof without solving ODE, just by differentiating

$$\rightarrow\text{R}\quad \dfrac{}{\vdash x^2y-2=0 \rightarrow [x'=-x^2, y'=2xy]\, x^2y-2=0}$$



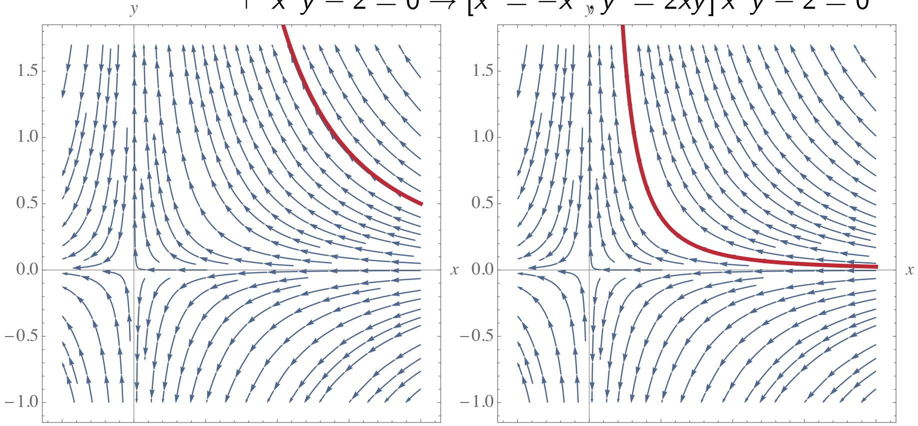
Example Proof

$$
\dfrac{\dfrac{}{x^2y-2=0 \vdash [x'=-x^2, y'=2xy]\, x^2y-2=0}\;\text{dI}}{\vdash x^2y-2=0 \rightarrow [x'=-x^2, y'=2xy]\, x^2y-2=0}\;{\rightarrow}\text{R}
$$



Example Proof

$$\text{[:=]}\ \dfrac{\dfrac{\vdash [x':=-x^2][y':=2xy]\,2xx'y + x^2y' - 0 = 0}{x^2y-2=0 \vdash [x'=-x^2, y'=2xy]\,x^2y-2=0}\ \text{dI}}{\vdash x^2y-2=0 \rightarrow [x'=-x^2, y'=2xy]\,x^2y-2=0}\ \rightarrow\text{R}$$

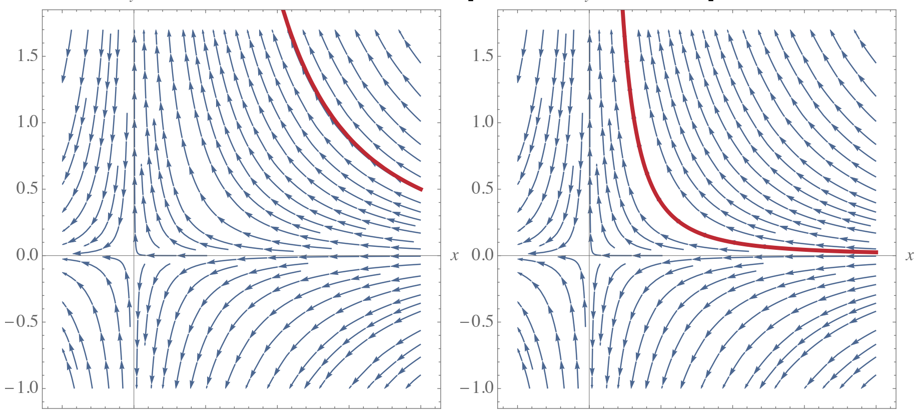


$$\mathbb{R}\quad \vdash 2x(-x^2)y + x^2(2xy) = 0$$

$$[:=]\quad \vdash [x':=-x^2][y':=2xy]\, 2xx'y + x^2y' - 0 = 0$$

$$\mathrm{dI}\quad x^2y - 2 = 0 \vdash [x' = -x^2, y' = 2xy]\, x^2y - 2 = 0$$

$$\rightarrow\mathrm{R}\quad \vdash x^2y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy]\, x^2y - 2 = 0$$



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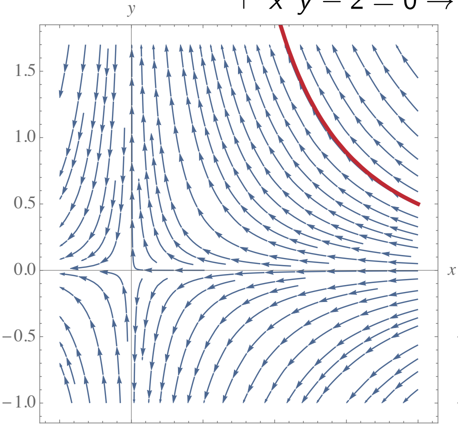

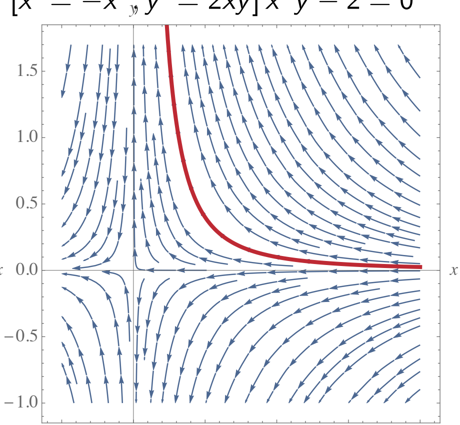

Outline

Differential Substitution Lemmas

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r>0$, *then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

Syntactic ′ → $$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$ ← Analytic ′

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \wedge Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e+k)' = (e)' + (k)'$$

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$$(c())' = 0 \qquad \textit{for constants/numbers } c()$$

$$(x)' = x' \qquad \textit{for variables } x \in \mathcal{V}$$

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Semantics

$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Definition (Hybrid program semantics) ($[\![\cdot]\!] : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S})$)

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0{\leq}z{\leq}r$$
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Soundness Proof

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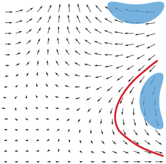
Outline



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6 Appendix
- Differential Equations vs. Loops
- Differential Invariant Terms and Invariant Functions

Lemma (Differential equations are their own loop)

$$[\![(x' = f(x))^*]\!] = [\![x' = f(x)]\!]$$

loop α^*	ODE $x' = f(x)$
repeat any number $n \in \mathbb{N}$ of times	evolve for any duration $r \in \mathbb{R}$
can repeat 0 times	can evolve for duration 0
effect depends on previous loop iteratior	effect depends on the past solution
local generator is loop body α	local generator $x' = f(x)$
full global execution trace	global solution $\varphi : [0, r] \to \mathcal{S}$
unwinding proof by iteration $[*]$	proof by global solution with $[']$
inductive proof with loop invariant	proof with differential invariant

$$\to\text{R}\ \frac{}{\vdash x^2 + y^2 = 0 \to [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}$$

$$\dfrac{\overline{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\;\text{cut,MR}}{\vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\;\rightarrow\text{R}$$

$$\dfrac{\dfrac{\dfrac{}{x^4+y^4=0 \vdash [x'=4y^3, y'=-4x^3]\, x^4+y^4=0}\ \text{dI}}{x^2+y^2=0 \vdash [x'=4y^3, y'=-4x^3]\, x^2+y^2=0}\ \text{cut,MR}}{\vdash x^2+y^2=0 \rightarrow [x'=4y^3, y'=-4x^3]\, x^2+y^2=0}\ {\rightarrow}\text{R}$$

Generalizing Differential Invariants: Stronger Postcondition

$$
\dfrac{\dfrac{\dfrac{\overline{\vdash [x':=4y^3][y':=-4x^3](4x^3x' + 4y^3y') = 0}\;[:=]}{x^4+y^4=0 \vdash [x' = 4y^3, y' = -4x^3]\, x^4+y^4=0}\;\mathrm{dI}}{x^2+y^2=0 \vdash [x' = 4y^3, y' = -4x^3]\, x^2+y^2=0}\;\mathrm{cut,MR}}{\vdash x^2+y^2=0 \rightarrow [x' = 4y^3, y' = -4x^3]\, x^2+y^2=0}\;{\rightarrow}\mathrm{R}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0}\;\mathbb{R}}{\vdash [x':=4y^3][y':=-4x^3](4x^3x' + 4y^3y') = 0}\;[:=]}{x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^4 + y^4 = 0}\;\text{dI}}{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\;\text{cut,MR}}{\vdash x^2 + y^2 = 0 \to [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\;\to\text{R}$$

Generalizing Differential Invariants: Stronger Postcondition

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0}\ \mathbb{R}}{\vdash [x':=4y^3][y':=-4x^3](4x^3x' + 4y^3y') = 0}\ [:=]}{x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^4 + y^4 = 0}\ \text{dI}}{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\ \text{cut,MR}}{\vdash x^2 + y^2 = 0 \to [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\ {\to}\text{R}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash 4x^3(4y^3)+4y^3(-4x^3)=0}\,\mathbb{R}}{\vdash [x':=4y^3][y':=-4x^3](4x^3x'+4y^3y')=0}\,[:=]}{x^4+y^4=0 \vdash [x'=4y^3,y'=-4x^3]\,x^4+y^4=0}\,\mathrm{dI}}{x^2+y^2=0 \vdash [x'=4y^3,y'=-4x^3]\,x^2+y^2=0}\,\mathrm{cut,MR}}{\vdash x^2+y^2=0 \rightarrow [x'=4y^3,y'=-4x^3]\,x^2+y^2=0}\,\rightarrow\mathrm{R}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0}\ \mathbb{R}}{\vdash [x' := 4y^3][y' := -4x^3](4x^3x' + 4y^3y') = 0}\ [:=]}{x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^4 + y^4 = 0}\ \text{dI}}{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\ \text{cut,MR}}{\vdash x^2 + y^2 = 0 \to [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\ \to\text{R}
$$

Theorem (Sophus Lie)

$$
DI_c \quad \frac{Q \vdash [x' := f(x)](e)' = 0}{\vdash \forall c\, \big(e = c \to [x' = f(x)\,\&\,Q]e = c\big)}
$$

premise and conclusion are equivalent if Q is a domain, i.e., characterizing a connected open set.

Generalizing Differential Invariants: Stronger Postcondition

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0}\;\mathbb{R}}{\vdash [x':=4y^3][y':=-4x^3](4x^3x' + 4y^3y') = 0}\;[:=]}{x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^4 + y^4 = 0}\;\text{dI}}{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\;\text{cut,MR}}{\vdash x^2 + y^2 = 0 \to [x' = 4y^3, y' = -4x^3]\, x^2 + y^2 = 0}\;\to\text{R}$$

Theorem (Sophus Lie)

$$DI_c \quad \dfrac{Q \vdash [x':=f(x)](e)' = 0}{\vdash \forall c\, \big(e = c \to [x' = f(x)\,\&\, Q]e = c\big)}$$

premise and conclusion are equivalent if Q is a domain, i.e., characterizing a connected open set.

Clou: $(e - c)' = (e)'$ independent of additive constants

Strengthening Induction Hypotheses

Stronger Induction Hypotheses

1. As usual in math and in proofs with loops:
2. Inductive proofs may need stronger induction hypotheses to succeed.
3. Differentially inductive proofs may need a stronger differential inductive structure to succeed.
4. Even if $\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 0\} = \{\{(x,y) \in \mathbb{R}^2 \ : \ x^4 + y^4 = 0\}$ have the same solutions, they have different differential structure.

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