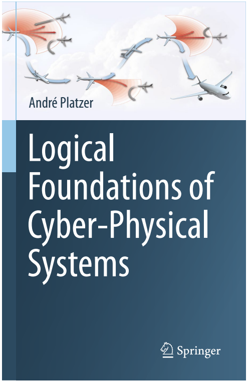
13: Differential Invariants & Proof Theory

Logical Foundations of Cyber-Physical Systems



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Outline

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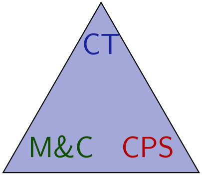
1. Learning Objectives
2. Recap: Proofs for Differential Equations
3. Differential Equation Proof Theory
   - Propositional Equivalences
   - Differential Invariants & Arithmetic
   - Differential Structure
   - Differential Invariant Equations
   - Equational Incompleteness
   - Strict Differential Invariant Inequalities
   - Differential Invariant Equations to Differential Invariant Inequalities
   - Differential Invariant Atoms
4. Differential Cut Power & Differential Ghost Power
5. Curves Playing with Norms and Degrees
6. Summary

limits of computation
proof theory for differential equations
provability of differential equations
nonprovability of differential equations
proofs about proofs
relativity theory of proofs
inform differential invariant search
intuition for differential equation proofs



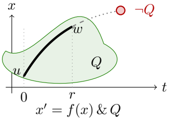
core argumentative principles
tame analytic complexity

improved analysis

Outline

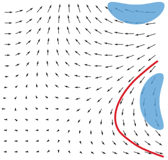
Differential Weakening

$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$



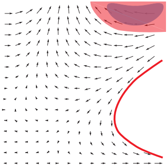
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \to F)$

DI $[x' = f(x) \,\&\, Q]F \leftarrow \big(Q \to F \land [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

Differential Invariants for Differential Equations

Differential Weakening

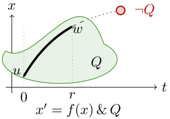
$$\frac{Q \vdash F}{P \vdash [x' = f(x)\,\&\,Q]F}$$

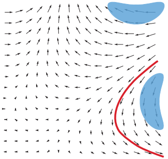
Differential Invariant

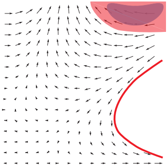
$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$







DW $[x' = f(x)\,\&\,Q]F \leftrightarrow [x' = f(x)\,\&\,Q](Q \rightarrow F)$

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DE $[x' = f(x)\,\&\,Q]F \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]F$

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

But generalizations are helpful to find the right F in the first place:

$$\text{cut,MR}\frac{A \vdash F \quad F \vdash [x' = f(x) \,\&\, Q]F \quad F \vdash B}{A \vdash [x' = f(x) \,\&\, Q]B}$$

Compare Provability with Classes Ω of Differential Invariants

$\mathcal{DI}_\Omega$: properties provable with differential invariants in $\Omega \subseteq \{\geq, >, =, \wedge, \vee\}$

$\mathcal{A} \leq \mathcal{B}$ iff all properties provable with $\mathcal{A}$ are also provable somehow with $\mathcal{B}$

$\mathcal{A} \not\leq \mathcal{B}$ otherwise, i.e., some property can be proved with $\mathcal{A}$ but not with $\mathcal{B}$

$\mathcal{A} \equiv \mathcal{B}$ iff $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$ so same deductive power

$\mathcal{A} < \mathcal{B}$ iff $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \not\leq \mathcal{A}$ so $\mathcal{A}$ has strictly less deductive power

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$\mathcal{DI}_{e=k} \equiv \mathcal{DI}_{e=0}$ by considering $(e-k)=0$

But generalizations are helpful to find the right F in the first place:

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Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is a propositional tautology then

$$\begin{array}{ll} & F \text{ differential invariant of } x' = f(x) \,\&\, Q \\ \textit{iff} & G \text{ differential invariant of } x' = f(x) \,\&\, Q \end{array}$$

Proof.

□

Can use any propositional normal form

Propositional Equivalences of Differential Invariants

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$$\dfrac{\text{dI}\ \dfrac{}{G \vdash [x' = f(x) \,\&\, Q]G}}{F \vdash [x' = f(x) \,\&\, Q]F}\ \text{MR,cut}$$

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Proof.

$$\dfrac{\dfrac{\dfrac{}{Q \vdash [x' := f(x)](G)'}\;[:=]}{G \vdash [x' = f(x)\,\&\,Q]G}\;\text{dI}}{F \vdash [x' = f(x)\,\&\,Q]F}\;\text{MR,cut}$$

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$F \leftrightarrow G$ propositionally equivalent, so $(F)' \leftrightarrow (G)'$ propositionally equivalent

□

Can use any propositional normal form

Propositional Equivalences of Differential Invariants

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$F \leftrightarrow G$ propositionally equivalent, so
$(F)' \leftrightarrow (G)'$ propositionally equivalent
since $(F_1 \land F_2)' \equiv (F_1)' \land (F_2)' \ \ldots$

□

Can use any propositional normal form

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is real-arithmetic equivalence then

$$\begin{aligned} & F \text{ differential invariant of } x' = f(x) \,\&\, Q \\ \text{iff} \quad & G \text{ differential invariant of } x' = f(x) \,\&\, Q \end{aligned}$$

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Proof.

$$\text{dI}\ \frac{}{-5\le x\land x\le 5 \vdash [x' = -x](-5\le x\land x\le 5)}$$

□

Arithmetic Equivalences of Differential Invariants

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Proof.

$$\dfrac{\dfrac{}{\vdash [x' := -x](0 \le x' \land x' \le 0)}\,[:=]}{-5 \le x \land x \le 5 \vdash [x' = -x](-5 \le x \land x \le 5)}\,\mathrm{dI}$$

□

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□

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Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is *real-arithmetic* *equivalence then*

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Proof.

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arithmetic equivalence $-5 \leq x \land x \leq 5 \leftrightarrow x^2 \leq 5^2$ □

Arithmetic Equivalences of Differential Invariants

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Proof.

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\dfrac{\text{not valid}}{\vdash 0 \leq -x \wedge -x \leq 0}}{\vdash [x' := -x](0 \leq x' \wedge x' \leq 0)}}{-5 \leq x \wedge x \leq 5 \vdash [x' = -x](-5 \leq x \wedge x \leq 5)} \qquad \text{dI}\ \dfrac{[:=]\ \dfrac{}{\vdash [x' := -x] 2x x' \leq 0}}{x^2 \leq 5^2 \vdash [x' = -x] x^2 \leq 5^2}$$

arithmetic equivalence $-5 \leq x \wedge x \leq 5 \leftrightarrow x^2 \leq 5^2$ □

Arithmetic Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is real-arithmetic equivalence then

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$$\dfrac{\dfrac{\dfrac{}{\vdash -x 2x \leq 0}\,\mathbb{R}}{\vdash [x' := -x]2xx' \leq 0}\,[:=]}{x^2 \leq 5^2 \vdash [x' = -x]x^2 \leq 5^2}\,\text{dI}$$

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arithmetic equivalence $-5 \le x \land x \le 5 \leftrightarrow x^2 \le 5^2$ □

Arithmetic Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is *real-arithmetic* *equivalence then*

$$F \text{ differential invariant of } x' = f(x)\,\&\,Q$$
$$\text{iff} \quad G \text{ differential invariant of } x' = f(x)\,\&\,Q$$

Proof.

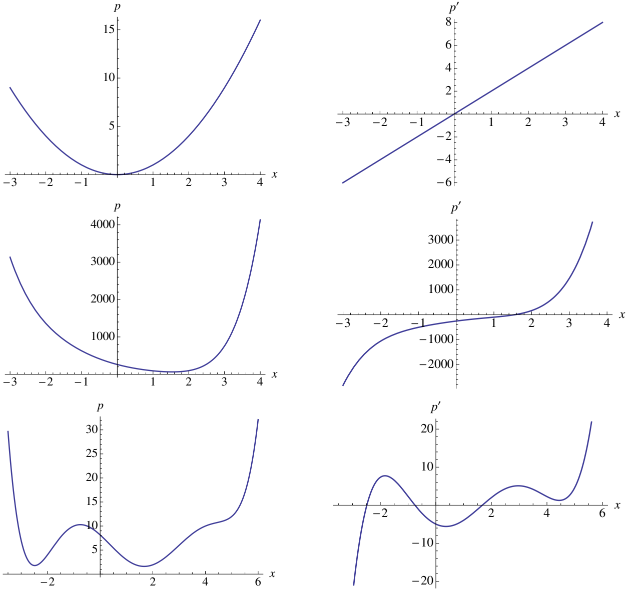
$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 0 \leq -x \wedge -x \leq 0}}{\vdash [x':=-x](0 \leq x' \wedge x' \leq 0)}[:=]}{-5 \leq x \wedge x \leq 5 \vdash [x' = -x](-5 \leq x \wedge x \leq 5)}\text{dI}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash -x2x \leq 0}\mathbb{R}}{\vdash [x':=-x]2xx' \leq 0}[:=]}{x^2 \leq 5^2 \vdash [x' = -x]x^2 \leq 5^2}\text{dI}$$

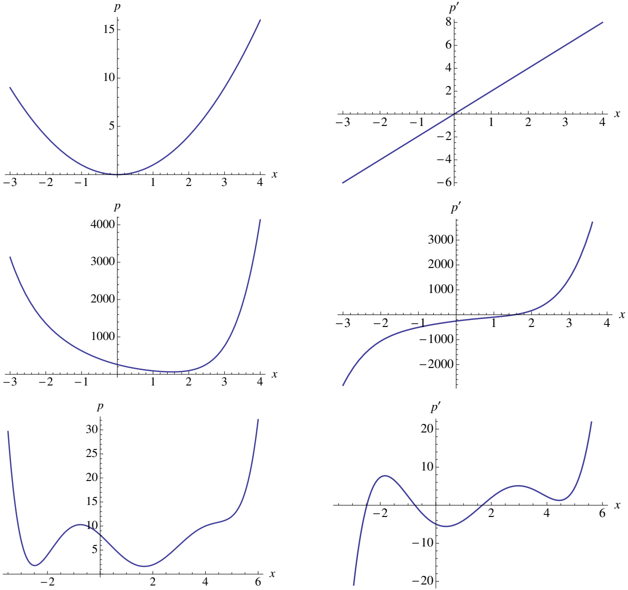
Despite arithmetic equivalence $-5 \leq x \wedge x \leq 5 \leftrightarrow x^2 \leq 5^2$ □

Differential structure matters! Higher degree helps here

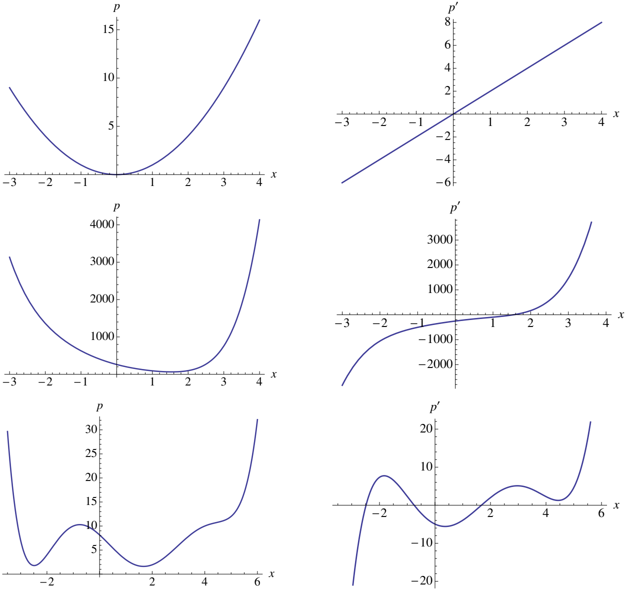
Different Differential Structure for Equivalent Solutions ≥0



Different Differential Structure for Equivalent Solutions ≥0



Same $p \geq 0$.
But different $p' \geq 0$.



Same $p \geq 0$.
But different $p' \geq 0$.

Can still normalize atomic formulas to $e = 0, e \geq 0, e > 0$

Differential Invariant Equations

Proposition (Equational deductive power [6, 2])

$$\mathcal{DI}_{=} \qquad \mathcal{DI}_{=,\wedge,\vee}$$

Proof core. Full: [6, 2].

-
-

Differential Invariant Equations

Proposition (Equational deductive power [6, 2])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

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atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2$
- $e_1 = e_2 \wedge k_1 = k_2$

Differential Invariant Equations

Proposition (Equational deductive power [6, 2])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$
- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

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atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

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Proposition (Equational deductive power [6, 2])

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Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x' := f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x' := f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$
- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Differential Invariant Equations

Proposition (Equational deductive power [6, 2])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

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- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Differential Invariant Equations

Proposition (Equational deductive power [6, 2])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\land,\lor}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \lor k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \land (k_1)' = (k_2)')$

  So $[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x':=f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$

- $e_1 = e_2 \land k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \land (k_1)' = (k_2)')$

□

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atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x':=f(x)]\big(((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0\big)$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x':=f(x)]\big((e_1)' = (e_2)' \wedge (k_1)' = (k_2)'\big)$

  So $[x':=f(x)]\big(((e_1 - e_2)^2 + (k_1 - k_2)^2)'=0\big)$

  $\equiv [x':=f(x)]\big(2(e_1-e_2)((e_1)'-(e_2)') + 2(k_1-k_2)((k_1)'-(k_2)')=0\big)$ □

Proposition (Equational deductive power [6, 2])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x':=f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x':=f(x)](((e_1 - e_2)^2 + (k_1 - k_2)^2)' = 0)$

  $\equiv [x':=f(x)](2(e_1 - e_2)((e_1)' - (e_2)') + 2(k_1 - k_2)((k_1)' - (k_2)') = 0)$ □

Proposition (Equational deductive power [6, 2])

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Proof core. Full: [6, 2].

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x' := f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x' := f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x' := f(x)](((e_1 - e_2)^2 + (k_1 - k_2)^2)' = 0)$

  $\equiv [x' := f(x)](2(e_1 - e_2)((e_1)' - (e_2)') + 2(k_1 - k_2)((k_1)' - (k_2)') = 0)$ □

Equational

Proposition (Equational [2])

$$\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} \quad \mathcal{DI} \quad \mathcal{DI}_{\geq} \quad \mathcal{DI}_{=}$$

Proof core.

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

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Proof core.

Provable with $\mathcal{DI}_{\geq}$ Unprovable with $\mathcal{DI}_{=}$

□

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Equations are not enough: $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_\geq \not\leq \mathcal{DI}_=$

Proof core.

Provable with $\mathcal{DI}_\geq$ Unprovable with $\mathcal{DI}_=$

$$\text{dI}\ \frac{}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_\geq \not\leq \mathcal{DI}_=$

Proof core.

Provable with $\mathcal{DI}_\geq$ — Unprovable with $\mathcal{DI}_=$

$$\dfrac{[:=]\ \dfrac{}{\vdash [x':=5]x' \geq 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$ — Unprovable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x' := 5]x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

□

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\land,\lor} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x':=5]x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

Unprovable with $\mathcal{DI}_{=}$

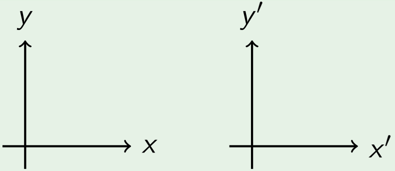
□

Example (Sets Bijective or Not)

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$$

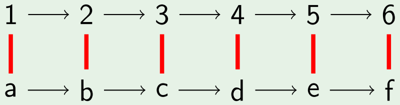
$$\mathsf{a} \longrightarrow \mathsf{b} \longrightarrow \mathsf{c} \longrightarrow \mathsf{d} \longrightarrow \mathsf{e} \longrightarrow \mathsf{f}$$

Example (Vector Spaces Isomorphic or Not)

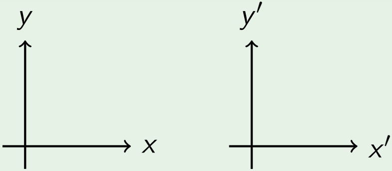


Proving Differences in Set Theory & Linear Algebra

Example (Sets Bijective or Not)



Example (Vector Spaces Isomorphic or Not)



Example (Sets Bijective or Not)

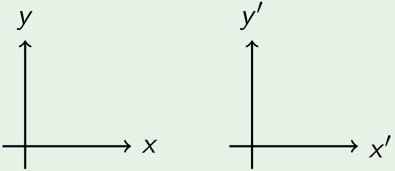
$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e \longrightarrow f$

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e$

Example (Vector Spaces Isomorphic or Not)



Example (Sets Bijective or Not)

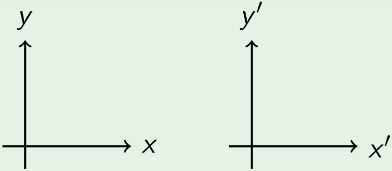
$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6 \qquad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$$

$$\mathrm{a} \longrightarrow \mathrm{b} \longrightarrow \mathrm{c} \longrightarrow \mathrm{d} \longrightarrow \mathrm{e} \longrightarrow \mathrm{f} \qquad \mathrm{a} \longrightarrow \mathrm{b} \longrightarrow \mathrm{c} \longrightarrow \mathrm{d} \longrightarrow \mathrm{e}$$

criterion: cardinality $|\{1, \dots, 6\}| = 6 \neq |\{a, b, c, d, e\}| = 5$

Need an indirect criterion especially if these sets are infinite

Example (Vector Spaces Isomorphic or Not)



Example (Sets Bijective or Not)

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$ $\quad$ $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

| | | | | |

$\mathrm{a} \longrightarrow \mathrm{b} \longrightarrow \mathrm{c} \longrightarrow \mathrm{d} \longrightarrow \mathrm{e} \longrightarrow \mathrm{f}$ $\quad$ $\mathrm{a} \longrightarrow \mathrm{b} \longrightarrow \mathrm{c} \longrightarrow \mathrm{d} \longrightarrow \mathrm{e}$

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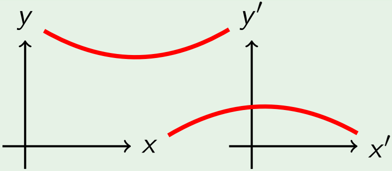
Example (Vector Spaces Isomorphic or Not)



Example (Sets Bijective or Not)

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e \longrightarrow f$

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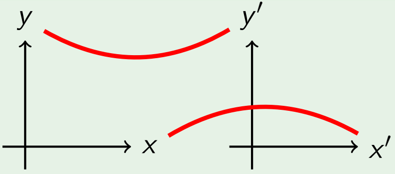
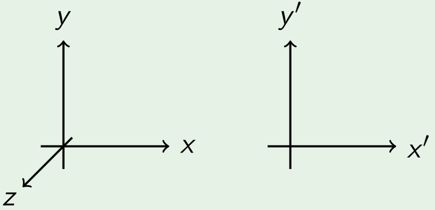
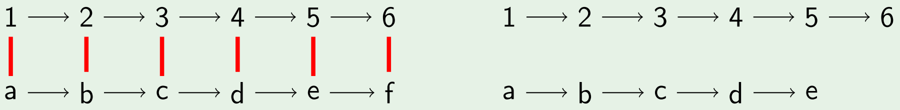
Example (Vector Spaces Isomorphic or Not)





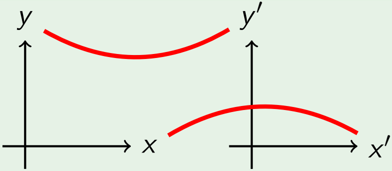
Example (Sets Bijective or Not)

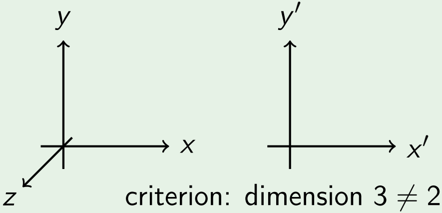


criterion: cardinality $|\{1, \dots, 6\}| = 6 \neq |\{a, b, c, d, e\}| = 5$

Need an indirect criterion especially if these sets are infinite

Example (Vector Spaces Isomorphic or Not)





criterion: dimension $3 \neq 2$

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$ | Unprovable with $\mathcal{DI}_{=}$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5 \geq 0}}{\vdash [x':=5]x' \geq 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_\geq \not\leq \mathcal{DI}_=$

Proof core.

Provable with $\mathcal{DI}_\geq$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x':=5]x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

Unprovable with $\mathcal{DI}_=$

$$\dfrac{}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{cut,MR}$$

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_\geq \not\leq \mathcal{DI}_=$

Proof core.

Provable with $\mathcal{DI}_\geq$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x' {:=} 5] x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5] x \geq 0}\ \text{dI}$$

Unprovable with $\mathcal{DI}_=$

$$\dfrac{\dfrac{}{p(x) = 0 \vdash [x' = 5] p(x) = 0}\ \text{dI}}{x \geq 0 \vdash [x' = 5] x \geq 0}\ \text{cut,MR}$$

□

Equational Incompleteness

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x':=5]x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

Unprovable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\overline{\vdash [x':=5](p(x))' = 0}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}\ \text{dI}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{cut,MR}$$

□

Proposition (Equational incompleteness [2])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$

$$\cfrac{\cfrac{\cfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x':=5]x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \mathrm{dI}$$

Unprovable with $\mathcal{DI}_{=}$

$$\cfrac{\cfrac{\cfrac{???}{\vdash [x':=5](p(x))' = 0}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}\ \mathrm{dI}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \mathrm{cut,MR}$$

□

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Proof core.

Provable with $\mathcal{DI}_\geq$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x':=5]x' \geq 0}\ [:=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

Unprovable with $\mathcal{DI}_=$

$$\dfrac{\dfrac{\dfrac{???}{\vdash [x':=5](p(x))' = 0}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}\ \text{dI}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{cut,MR}$$

Univariate polynomial $p(x)$ is 0 if 0 on all $x \geq 0$ □

Proposition (Strict barrier)

$\mathcal{DI}_>$ $\mathcal{DI}$ $\mathcal{DI}_=$ $\mathcal{DI}_>$

Proof core.

□

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

□

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$

Unprovable with $\mathcal{DI}_{>}$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$ Unprovable with $\mathcal{DI}_{>}$

$$\text{dI} \frac{}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}$$

□

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_> < \mathcal{DI}$ *because* $\mathcal{DI}_= \not\leq \mathcal{DI}_>$

Proof core.

Provable with $\mathcal{DI}_=$ — Unprovable with $\mathcal{DI}_>$

$$\dfrac{[:=]\dfrac{}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\,\text{dI}$$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$ Unprovable with $\mathcal{DI}_{>}$

$$\dfrac{\dfrac{\dfrac{}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v' := w][w' := -v] 2vv' + 2ww' = 0}\ [:=]}{v^2 + w^2 = c^2 \vdash [v' = w, w' = -v] v^2 + w^2 = c^2}\ \text{dI}$$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

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Proof core.

Provable with $\mathcal{DI}_{=}$

Unprovable with $\mathcal{DI}_{>}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}\ [:=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\ \text{dI}$$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

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Proof core.

Provable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}\ [:=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\ \text{dI}$$

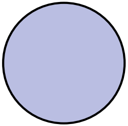
Unprovable with $\mathcal{DI}_{>}$

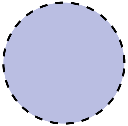
$e > 0$ is open set.

$v^2+w^2=c^2$ is a closed set

∎

closed $v^2+w^2 \leq 1$ with full boundary





open $v^2+w^2<1$ without boundary

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v' := w][w' := -v]2vv' + 2ww' = 0}\ [:=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\ \text{dI}$$

$v^2+w^2=c^2$ is a closed set

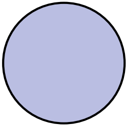
Unprovable with $\mathcal{DI}_{>}$

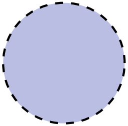
$e > 0$ is open set.

Only *true* and *false* are both

∎

closed $v^2+w^2 \leq 1$ with full boundary





open $v^2+w^2 < 1$ without boundary

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

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Proof core.

Provable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}\ [:=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\ \text{dI}$$

$v^2+w^2=c^2$ is a closed set

Unprovable with $\mathcal{DI}_{>}$

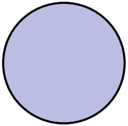
$e > 0$ is open set.

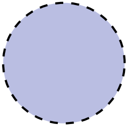
Only *true* and *false* are both

but don't help proof

□

closed $v^2+w^2 \leq 1$ with full boundary





open $v^2+w^2 < 1$ without boundary

Proposition (Equational)

$\mathcal{DI}_{=,\wedge,\vee} \quad \mathcal{DI}_{\geq}$

Proof core.

□

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

□

Differential Invariant Equations to Inequalities

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

Provable with $\mathcal{DI}_{\geq}$

□

Differential Invariant Equations to Inequalities

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

Provable with $\mathcal{DI}_{\geq}$

$$\text{dI}\,\frac{}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

□

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\text{dI}\;\dfrac{\dfrac{}{Q \vdash [x':=f(x)](e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

Provable with $\mathcal{DI}_{\geq}$

□

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\text{dI}\;\dfrac{\dfrac{*}{Q \vdash [x'{:=}f(x)](e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

Provable with $\mathcal{DI}_{\geq}$

□

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Provable with $\mathcal{DI}_{\geq}$

$$\text{dI}\,\dfrac{}{-e^2 \geq 0 \vdash [x' = f(x)\,\&\,Q](-e^2 \geq 0)}$$

□

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\text{dI}\dfrac{\dfrac{*}{Q \vdash [x':=f(x)](e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

Provable with $\mathcal{DI}_{\geq}$

$$\text{dI}\dfrac{\dfrac{}{Q \vdash [x':=f(x)] - 2e(e)' \geq 0}}{-e^2 \geq 0 \vdash [x' = f(x)\,\&\,Q](-e^2 \geq 0)}$$

□

Differential Invariant Equations to Inequalities

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\text{dI}\dfrac{\dfrac{*}{Q \vdash [x':=f(x)](e)' = 0}}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

Provable with $\mathcal{DI}_{\geq}$

$$\text{dI}\dfrac{\dfrac{*}{Q \vdash [x':=f(x)] - 2e(e)' \geq 0}}{-e^2 \geq 0 \vdash [x' = f(x)\,\&\,Q](-e^2 \geq 0)}$$

□

Local view of logic on differentials is crucial for this proof.

Degree increases

Differential Invariant Atoms

Theorem (Atomic)

$\mathcal{DI}_{\geq}$ $\mathcal{DI}_{\geq,\wedge,\vee}$ and $\mathcal{DI}_{>}$ $\mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

□

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\wedge,\vee}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

□

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\wedge,\vee}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

Provable with $\mathcal{DI}_{\geq,\wedge,\vee}$

Unprovable with $\mathcal{DI}_{\geq}$

□

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\wedge,\vee}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

Provable with $\mathcal{DI}_{\geq,\wedge,\vee}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0 \wedge y^2 \geq 0}\ \mathbb{R}}{\vdash [x':=5][y':=y^2](x'\geq 0 \wedge y'\geq 0)}\ [:=]}{x\geq 0\wedge y\geq 0 \vdash [x' = 5, y' = y^2](x\geq 0\wedge y\geq 0)}\ \text{dI}$$

Unprovable with $\mathcal{DI}_{\geq}$

□

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\wedge,\vee}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

Provable with $\mathcal{DI}_{\geq,\wedge,\vee}$

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Unprovable with $\mathcal{DI}_{\geq}$

$p(x, y)\geq 0 \leftrightarrow x\geq 0\wedge y\geq 0$

impossible since this implies

$p(x, 0)\geq 0 \leftrightarrow x\geq 0$

so $p(x, 0)$ is 0

□

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\land,\lor}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\land,\lor}$

Proof idea.

Provable with $\mathcal{DI}_{\geq,\land,\lor}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0 \land y^2 \geq 0}\ \mathbb{R}}{\vdash [x':=5][y':=y^2](x'\geq0 \land y'\geq0)}\ [:=]}{x\geq0\land y\geq0 \vdash [x' = 5, y' = y^2](x\geq0\land y\geq0)}\ \text{dI}$$

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Substantial remaining parts of the proof shown elsewhere [2]. □

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\wedge,\vee}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

Provable with $\mathcal{DI}_{\geq,\wedge,\vee}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0 \wedge y^2 \geq 0}\ \mathbb{R}}{\vdash [x':=5][y':=y^2](x'\geq 0 \wedge y'\geq 0)}\ [:=]}{x\geq 0\wedge y\geq 0 \vdash [x' = 5, y' = y^2](x\geq 0\wedge y\geq 0)}\ \text{dI}$$

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impossible since this implies

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so $p(x,0)$ is 0

Substantial remaining parts of the proof shown elsewhere [2]. □

dC still possible here but more involved argument separates.

Outline

Theorem (Gentzen's Cut Elimination) (1935)

$$\frac{A \vdash B \lor C \quad A \land C \vdash B}{A \vdash B}$$ *cut can be eliminated*

Theorem (No Differential Cut Elimination) (LMCS 2012)

Deductive power with differential cuts exceeds deductive power without.

$$\mathcal{DI} + DC > \mathcal{DI}$$

Theorem (Auxiliary Differential Variables) (LMCS 2012)

Deductive power with differential ghosts exceeds power without.

$$\mathcal{DI} + DC + DG > \mathcal{DI} + DC$$

$$\text{dI}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\cfrac{\cfrac{}{\vdash [x':=(x-2)^4+y^5][y':=y^2]3x^2x' \geq 0}\ [:=]}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1}\ \text{dI}$$

Ex: The Need for Differential Cuts

$$\dfrac{\dfrac{\dfrac{}{\vdash 3x^2((x-2)^4+y^5) \geq 0}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]3x^2x' \geq 0}\,[:=]}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x'=(x-2)^4+y^5, y'=y^2]x^3 \geq -1}\,\text{dI}$$

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 3x^2((x-2)^4+y^5) \geq 0}}{\vdash [x' := (x-2)^4+y^5][y' := y^2]3x^2x' \geq 0}\;[:=]}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1}\;\text{dI}$$

Ex: The Need for Differential Cuts

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 3x^2((x-2)^4+y^5)\geq 0}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]3x^2x'\geq 0}\;[:=]}{x^3\geq -1 \land y^5\geq 0 \vdash [x'=(x-2)^4+y^5, y'=y^2]x^3\geq -1}\;\text{dI}$$

Have to know something about y^5

$$\text{dC}\;\frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]{\color{red}y^5 \geq 0}}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \frac{[:=]\ \frac{}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x' := (x-2)^4 + y^5][y' := y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{\text{[:=]}\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5y^4y^2 \geq 0}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\dfrac{\text{dI}\quad \dfrac{}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1} \;\triangleright}{\text{dC}\quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5y^4y^2 \geq 0}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

$$\dfrac{\dfrac{\dfrac{}{y^5 \geq 0 \vdash [x':=(x-2)^4+y^5][y':=y^2]2x^2x' \geq 0}\,[:=]}{x^3 \geq -1 \vdash [x' = (x-2)^4+y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright}\,\text{dI}}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1}\,\text{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\,\mathbb{R}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0}\,[:=]}{y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]y^5 \geq 0}\,\text{dI}$$

$$\dfrac{\dfrac{\dfrac{\mathbb{R}\;\; y^5 \geq 0 \vdash 2x^2((x-2)^4 + y^5) \geq 0}{[:=]\;\; y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0}}{\mathrm{dI}\;\; x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\; \triangleright}}{\mathrm{dC}\;\; x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\dfrac{\dfrac{\dfrac{*}{\mathbb{R}\;\; \vdash 5y^4y^2 \geq 0}}{[:=]\;\; \vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0}}{\mathrm{dI}\;\; y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$*$$

$$\mathbb{R}\quad \overline{y^5 \geq 0 \vdash 2x^2((x-2)^4 + y^5) \geq 0}$$

$$[:=]\quad \overline{y^5 \geq 0 \vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0}$$

$$\text{dI}\quad \overline{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1} \quad \triangleright$$

$$\text{dC}\quad \overline{x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$*$$

$$\mathbb{R}\quad \overline{\vdash 5y^4y^2 \geq 0}$$

$$[:=]\quad \overline{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4y' \geq 0}$$

$$\text{dI}\quad \overline{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]\color{red}{y^5 \geq 0}}$$

Outline

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is *real-arithmetic* *equivalence then*

$$F \text{ differential invariant of } x' = f(x)\,\&\,Q$$
$$\text{iff} \quad G \text{ differential invariant of } x' = f(x)\,\&\,Q$$

Proof.

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 0 \leq -x \land -x \leq 0}}{\vdash [x':=-x](0\leq x' \land x' \leq 0)}\,[:=]}{-5\leq x \land x\leq 5 \vdash [x' = -x](-5\leq x \land x \leq 5)}\,\text{dI}
\qquad
\dfrac{\dfrac{\dfrac{*}{\vdash -x2x\leq 0}\,\mathbb{R}}{\vdash [x':=-x]2xx'\leq 0}\,[:=]}{x^2\leq 5^2 \vdash [x' = -x]x^2\leq 5^2}\,\text{dI}$$

Despite arithmetic equivalence $-5\leq x \land x\leq 5 \leftrightarrow x^2\leq 5^2$ □

Differential structure matters! Higher degree helps here

$$\text{dC}\ \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1]\|(x, y)\|_\infty \leq t}$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2\leq 1 \land x=y=t=0$$

$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{\equiv} -t \leq x \leq t \land -t \leq y \leq t$ Supremum norm

$\|(x, y)\|_2 \leq t \stackrel{\text{def}}{\equiv} x^2 + y^2 \leq t^2$ Euclidean norm

$$\dfrac{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \,\&\, v^2+w^2\leq 1]\|(x,y)\|_\infty \leq t}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1]\|(x,y)\|_\infty \leq t}\ \text{dC}\quad \text{dI} \triangleleft$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2\leq 1 \land x=y=t=0$$

$$\|(x,y)\|_\infty \leq t \stackrel{\text{def}}{\equiv} -t \leq x \leq t \land -t \leq y \leq t \qquad \text{Supremum norm}$$

$$\|(x,y)\|_2 \leq t \stackrel{\text{def}}{\equiv} x^2 + y^2 \leq t^2 \qquad \text{Euclidean norm}$$

$$\dfrac{\dfrac{\dfrac{}{v^2+w^2\leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\leq x'\leq t' \land -t'\leq y'\leq t')}[:=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\leq 1]\|(x,y)\|_\infty \leq t}\text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \leq t}\text{dC}$$

$$A \overset{\text{def}}{\equiv} v^2+w^2\leq 1 \land x=y=t=0$$

$$\|(x,y)\|_\infty \leq t \overset{\text{def}}{\equiv} -t\leq x\leq t \land -t\leq y\leq t$$ Supremum norm

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$$A \overset{\text{def}}{\equiv} v^2+w^2\le 1 \land x=y=t=0$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{*}{v^2+w^2\le 1 \vdash -1 \le v \le 1 \land -1 \le w \le 1}\ \mathbb{R}}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\le x'\le t' \land -t'\le y'\le t')}\ [:=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1\ \&\ v^2+w^2\le 1]\|(x,y)\|_\infty \le t}\ \text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \le t}\ \text{dC}
$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2\le 1 \land x=y=t=0$$

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$$

$$
\dfrac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1]\|(x,y)\|_2 \le t}\text{dC}
$$

$$A \overset{\text{def}}{\equiv} v^2{+}w^2{\le}1 \land x{=}y{=}t{=}0$$

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$$\dfrac{\dfrac{}{\triangleleft \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \,\&\, v^2{+}w^2{\le}1]\|(x,y)\|_2 \le t}\ \mathrm{dI}}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1]\|(x,y)\|_2 \le t}\ \mathrm{dC}$$

$$A \stackrel{\text{def}}{\equiv} v^2{+}w^2{\le}1 \land x{=}y{=}t{=}0$$

$$\|(x,y)\|_\infty \le t \stackrel{\text{def}}{\equiv} -t \le x \le t \land -t \le y \le t$$ Supremum norm

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$$\dfrac{\dfrac{\dfrac{}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx'+2yy'\le 2tt')}[:=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le 1]\|(x,y)\|_2 \le t}\text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \le t}\text{dC}$$

$$A \overset{\text{def}}{\equiv} v^2+w^2\le 1 \land x=y=t=0$$

$$\|(x,y)\|_\infty \le t \overset{\text{def}}{\equiv} -t\le x\le t \land -t\le y\le t \qquad \text{Supremum norm}$$

$$\|(x,y)\|_2 \le t \overset{\text{def}}{\equiv} x^2+y^2\le t^2 \qquad \text{Euclidean norm}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{v^2+w^2\le 1 \vdash -1 \le v \le 1 \land -1 \le w \le 1}\ \mathbb{R}}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\le x'\le t' \land -t'\le y'\le t')}\ [:=]}{\triangleleft \qquad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le 1]\|(x,y)\|_\infty \le t}\ \mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \le t}\ \mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{v^2+w^2\le 1 \vdash 2xv + 2yw \le 2t1}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx' + 2yy' \le 2tt')}\ [:=]}{\triangleleft \qquad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le 1]\|(x,y)\|_2 \le t}\ \mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \le t}\ \mathrm{dC}$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2\le 1 \land x=y=t=0$$

$$\|(x,y)\|_\infty \le t \stackrel{\text{def}}{\equiv} -t \le x \le t \land -t \le y \le t$$ Supremum norm

$$\|(x,y)\|_2 \le t \stackrel{\text{def}}{\equiv} x^2 + y^2 \le t^2$$ Euclidean norm

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{v^2+w^2\le 1 \vdash -1\le v\le 1 \land -1\le w\le 1}\ \mathbb{R}}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\le x'\le t' \land -t'\le y'\le t')}\ [:=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1\ \&\ v^2+w^2\le 1]\|(x,y)\|_\infty \le t}\ \mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \le t}\ \mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{not valid}}{v^2+w^2\le 1 \vdash 2xv+2yw \le 2t1}}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx'+2yy'\le 2tt')}\ [:=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1\ \&\ v^2+w^2\le 1]\|(x,y)\|_2 \le t}\ \mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \le t}\ \mathrm{dC}$$

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Curves Playing with Norms and Degrees

$$
\cfrac{\cfrac{\cfrac{\cfrac{*}{v^2{+}w^2{\le}1 \vdash -1 \le v \le 1 \land -1 \le w \le 1}\;\mathbb{R}}{v^2{+}w^2{\le}1 \vdash [x'{:=}v][y'{:=}w][v'{:=}\omega w][w'{:=}{-}\omega v][t'{:=}1](-t'{\le}x'{\le}t' \land -t'{\le}y'{\le}t')}\;[:=]}{\lhd \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \,\&\, v^2{+}w^2{\le}1] \|(x,y)\|_\infty \le t}\;\text{dI}}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x,y)\|_\infty \le t}\;\text{dC}
$$

Lower degree helps here

$$
\cfrac{\cfrac{\cfrac{\cfrac{\text{not valid}}{v^2{+}w^2{\le}1 \vdash 2xv + 2yw \le 2t1}}{v^2{+}w^2{\le}1 \vdash [x'{:=}v][y'{:=}w][v'{:=}\omega w][w'{:=}-\omega v][t'{:=}1](2xx' + 2yy' \le 2tt')}\;[:=]}{\lhd \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \,\&\, v^2{+}w^2{\le}1] \|(x,y)\|_2 \le t}\;\text{dI}}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x,y)\|_2 \le t}\;\text{dC}
$$

$$A \stackrel{\text{def}}{\equiv} v^2{+}w^2{\le}1 \land x{=}y{=}t{=}0$$

$\|(x,y)\|_\infty \le t \stackrel{\text{def}}{\equiv} -t \le x \le t \land -t \le y \le t$ Supremum norm

$\|(x,y)\|_2 \le t \stackrel{\text{def}}{\equiv} x^2 + y^2 \le t^2$ Euclidean norm

Interreducing Norms in Dimension n

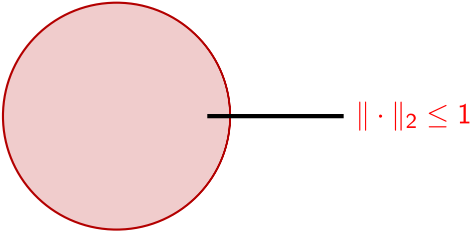
$$\forall x\,\forall y\,(\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

$$\forall x\,\forall y\,(\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$

Interreducing Norms in Dimension n

$$\forall x\, \forall y\, (\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

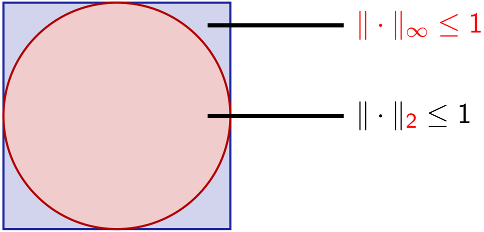
$$\forall x\, \forall y\, (\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$



Interreducing Norms in Dimension n

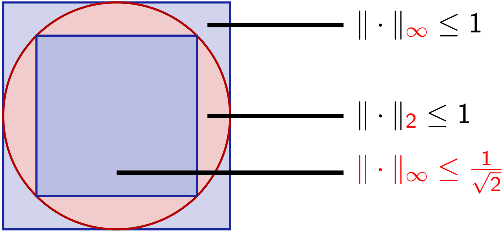
$$\forall x\, \forall y\, (\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

$$\forall x\, \forall y\, (\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$



$$\forall x\,\forall y\,(\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

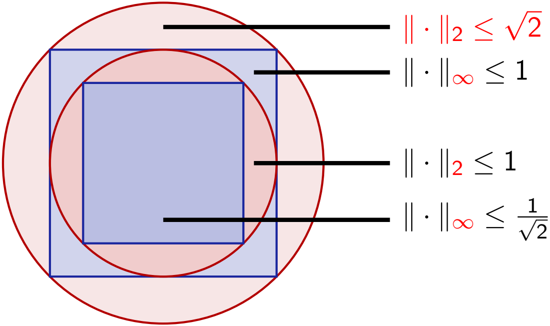
$$\forall x\,\forall y\,(\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$



Interreducing Norms in Dimension n

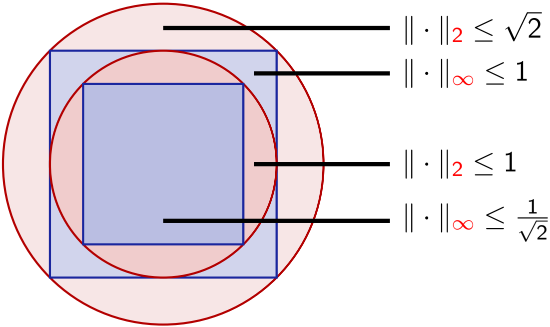
$$\forall x\,\forall y\,(\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

$$\forall x\,\forall y\,(\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$



$$\forall x\,\forall y\,(\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

$$\forall x\,\forall y\,(\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$

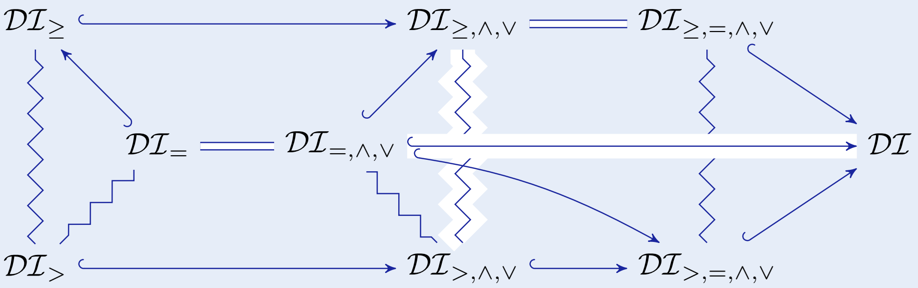


Benefit from norm relations but be mindful of approximation error factors

Outline

Summary: Differential Invariance Chart

Theorem (Differential Invariance Chart)



- Rich theory and structure behind differential invariants
- Scrutinize what property can be proved with what invariant
- Use provability sanity checks like open/closed/univariate
- Real differential semialgebraic geometry
- Exploit differential cuts to obtain more knowledge

André Platzer.
Logical Foundations of Cyber-Physical Systems.
Springer, Switzerland, 2018.
URL: http://www.springer.com/978-3-319-63587-3,
doi:10.1007/978-3-319-63588-0.

André Platzer.
The structure of differential invariants and differential cut elimination.
Log. Meth. Comput. Sci., 8(4:16):1–38, 2012.
doi:10.2168/LMCS-8(4:16)2012.

André Platzer.
Foundations of cyber-physical systems.
Lecture Notes 15-424/624/824, Carnegie Mellon University, 2017.
URL: http://lfcps.org/course/fcps17.html.

André Platzer.
A complete uniform substitution calculus for differential dynamic logic.
J. Autom. Reas., 59(2):219–265, 2017.
doi:10.1007/s10817-016-9385-1.

André Platzer.
A differential operator approach to equational differential invariants.
In Lennart Beringer and Amy Felty, editors, *ITP*, volume 7406 of *LNCS*, pages 28–48, Berlin, 2012. Springer.
doi:10.1007/978-3-642-32347-8_3.

André Platzer.
Differential-algebraic dynamic logic for differential-algebraic programs.
J. Log. Comput., 20(1):309–352, 2010.
doi:10.1093/logcom/exn070.