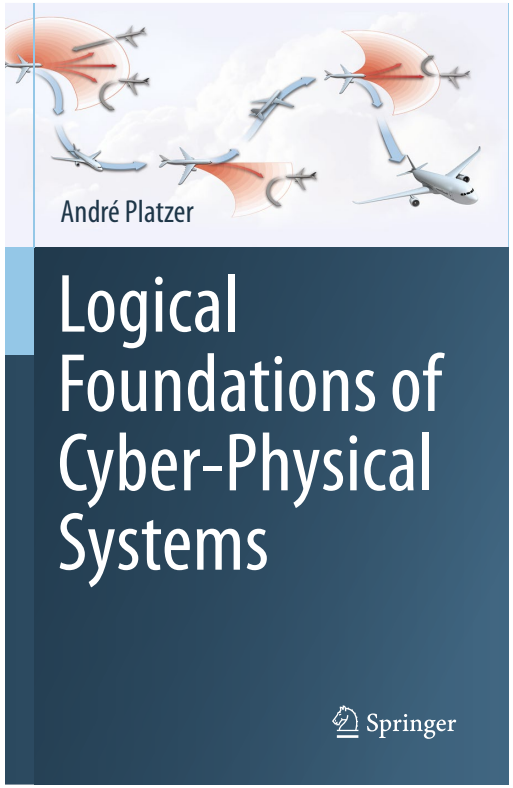
# 14: Hybrid Systems & Games

## Logical Foundations of Cyber-Physical Systems




André Platzer

Carnegie Mellon University
Computer Science Department

# Outline

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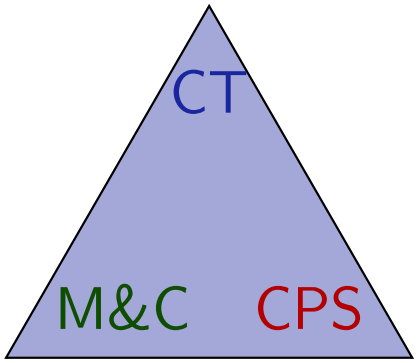
# Learning Objectives

## Hybrid Systems & Games

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
best/worst-case analysis
models of alternating computation




adversarial dynamics
conflicting actions
multi-agent systems
angelic/demonic choice

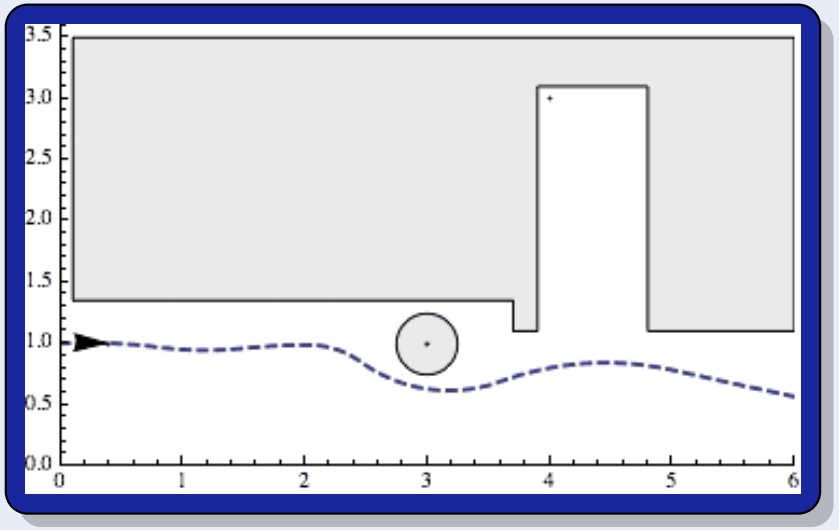
multi-agent state change
CPS semantics
reflections on choices

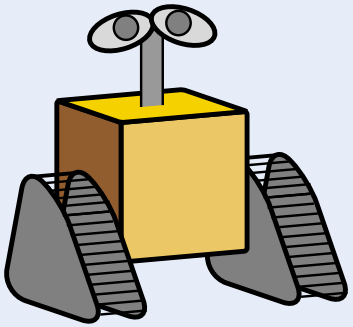
# Outline

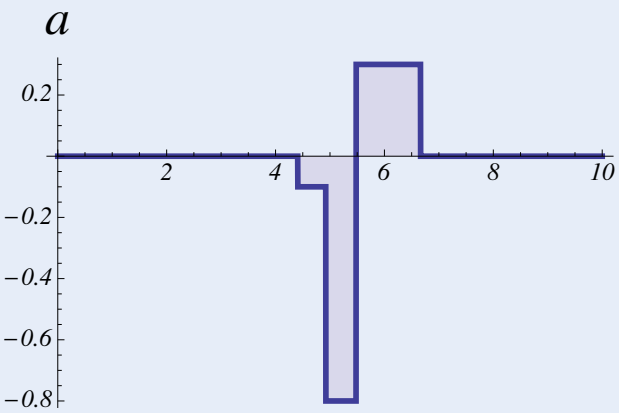
## Challenge (Hybrid Systems)

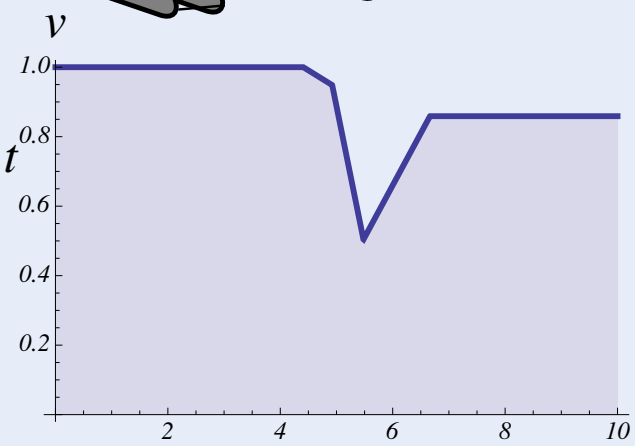
Fixed rule describing state evolution with both

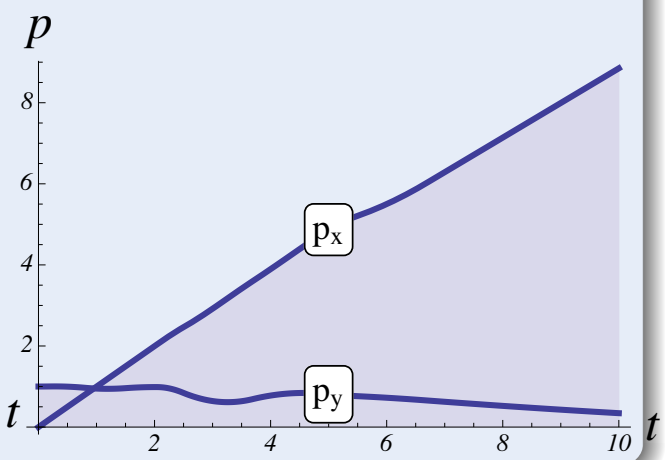
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)





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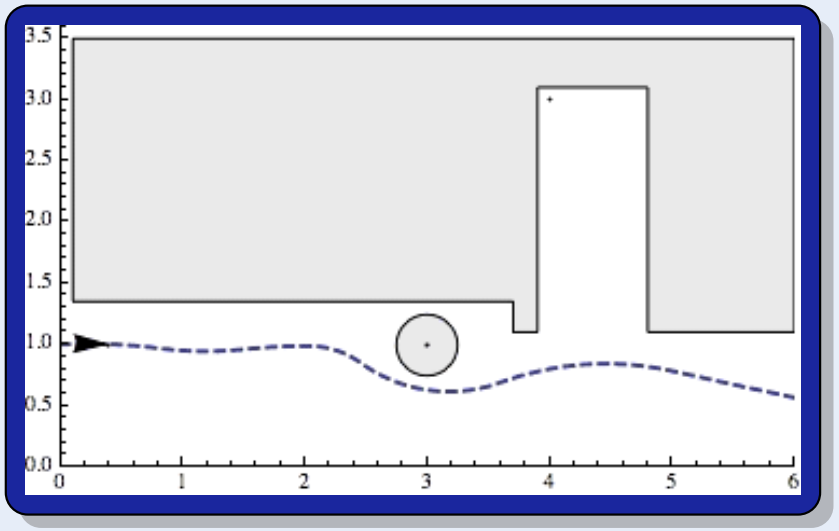
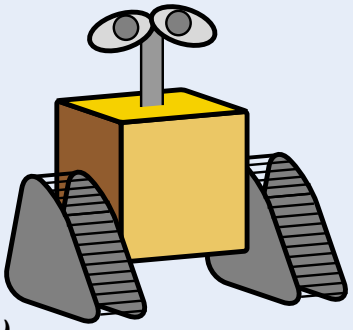
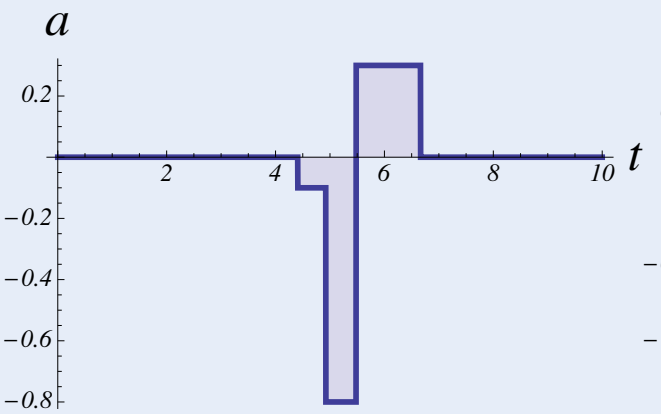
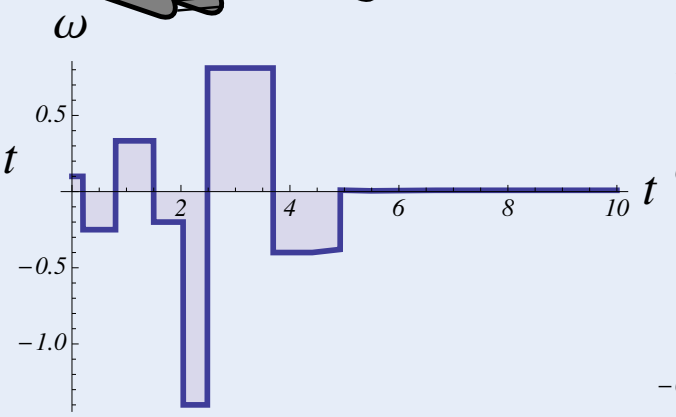
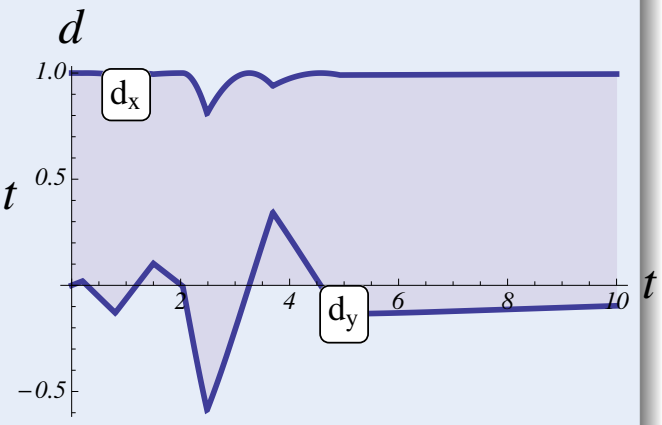




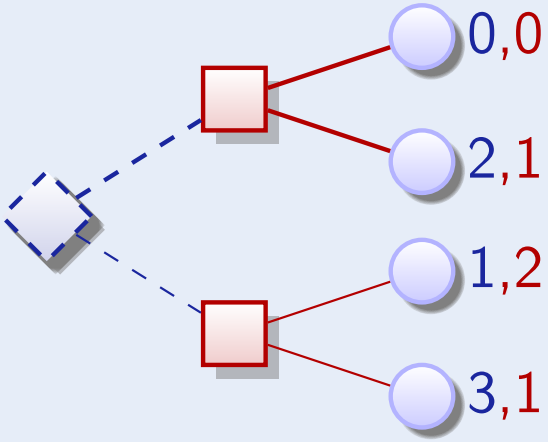



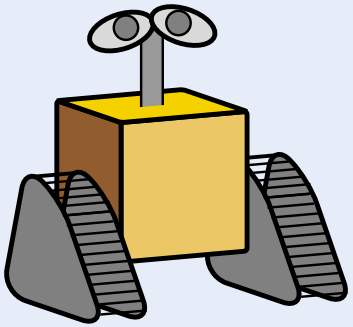



## Challenge (Games)

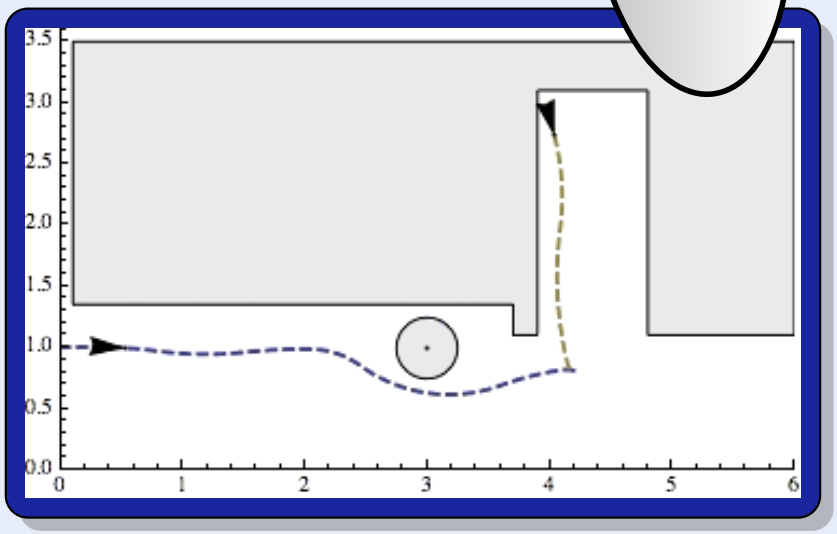
Game rules describing play evolution with both

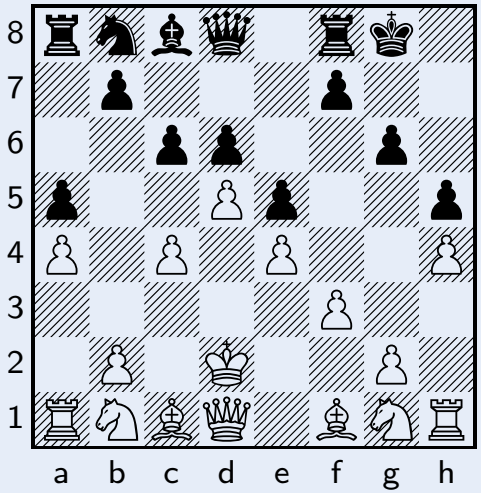
- Angelic choices (player ◇ Angel)
- Demonic choices (player □ Demon)



| ◇\□ | Tr | Pl |
|---|---|---|
| Trash | 1,2 | 0,0 |
| Plant | 0,0 | 2,1 |

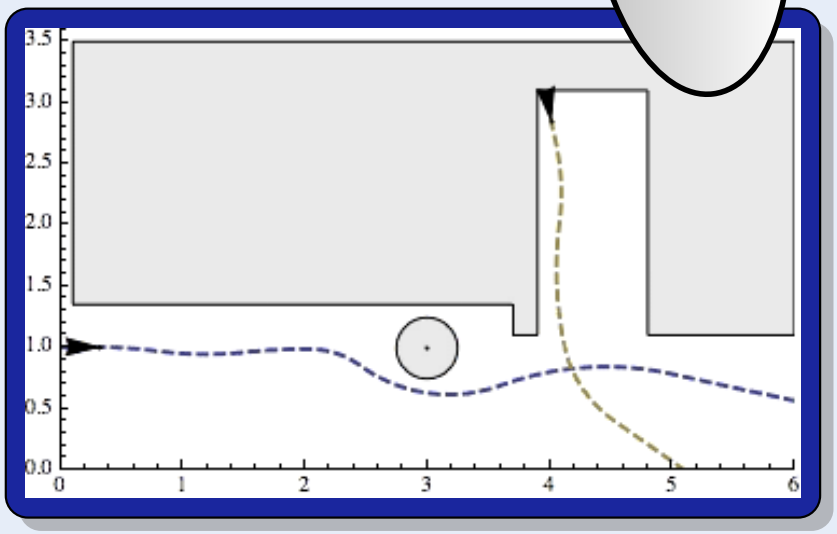


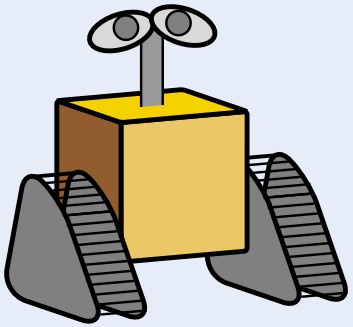



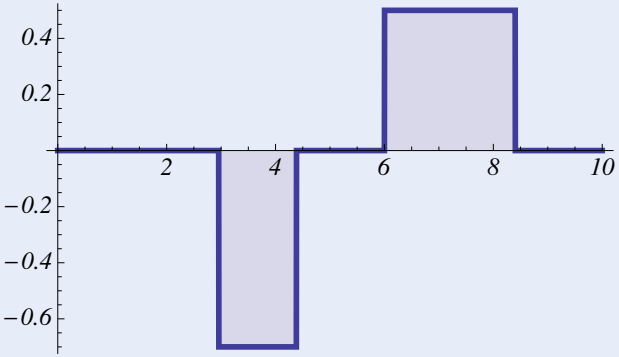
## Challenge (Hybrid Games)

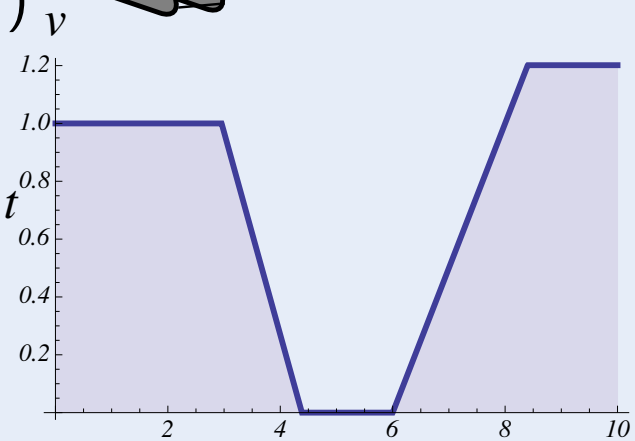
Game rules describing play evolution with

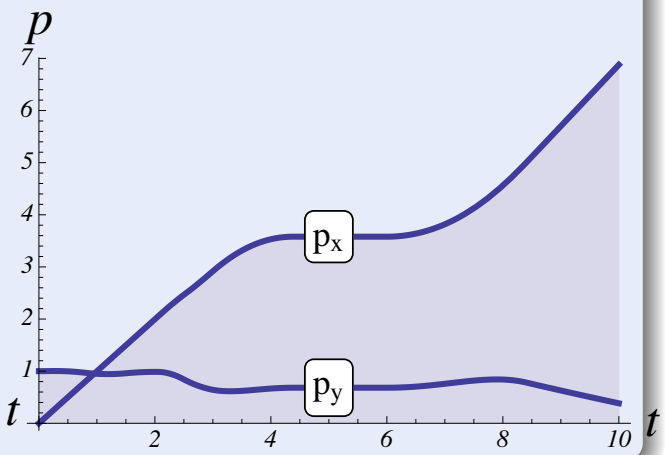
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)





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Game rules describing play evolution with

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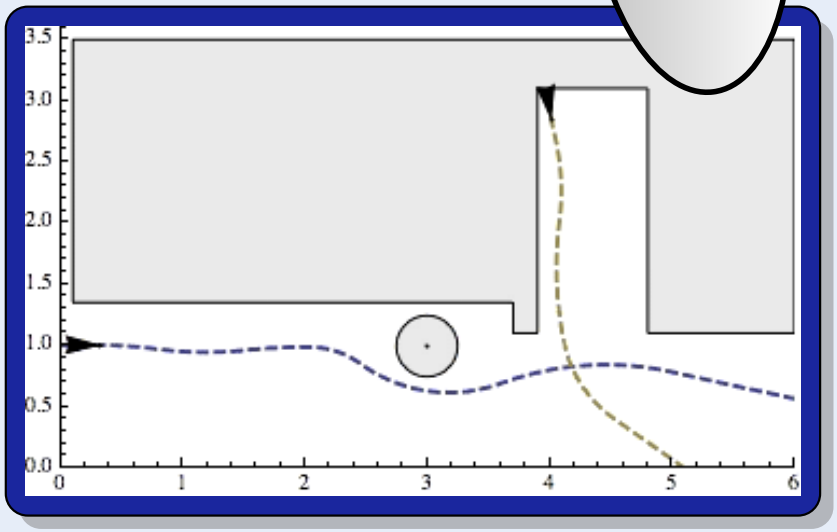


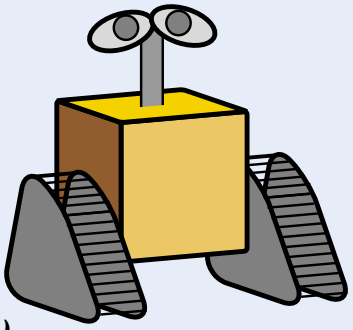

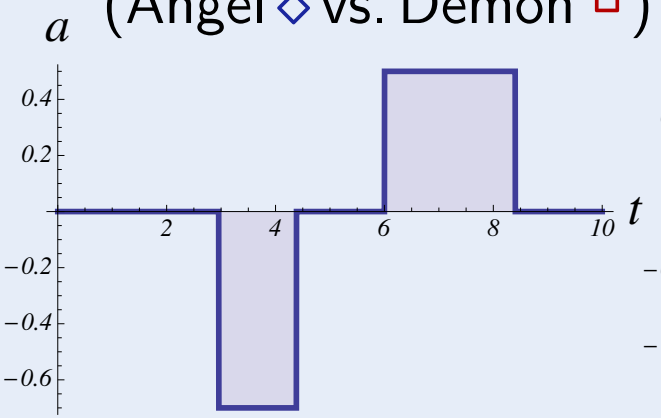


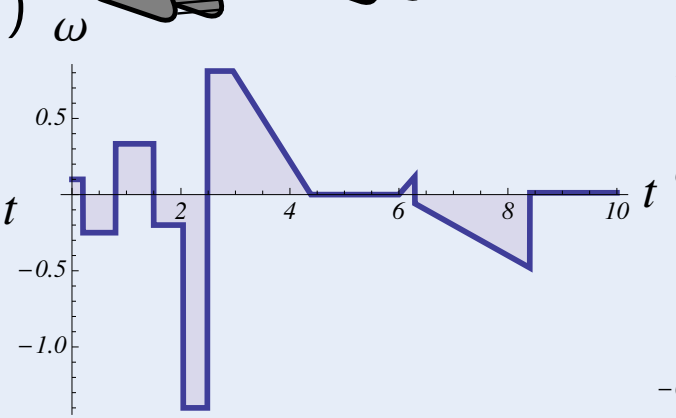


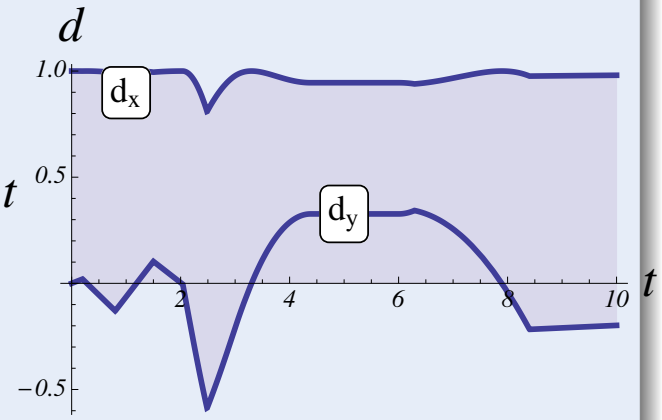

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Game rules describing play evolution with

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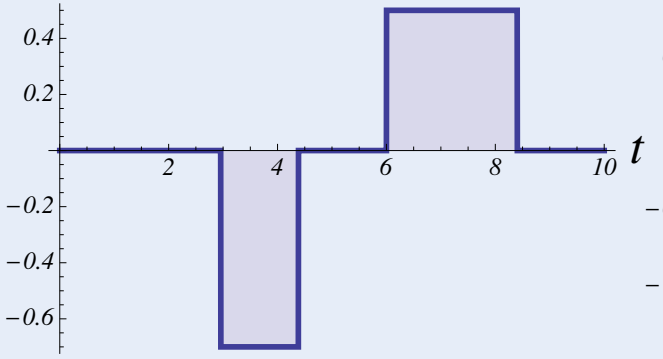
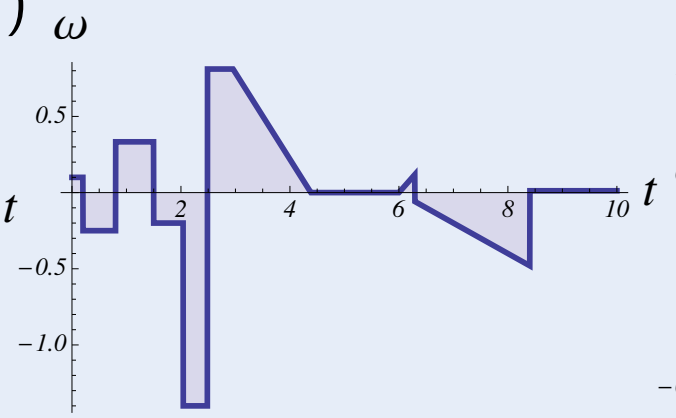
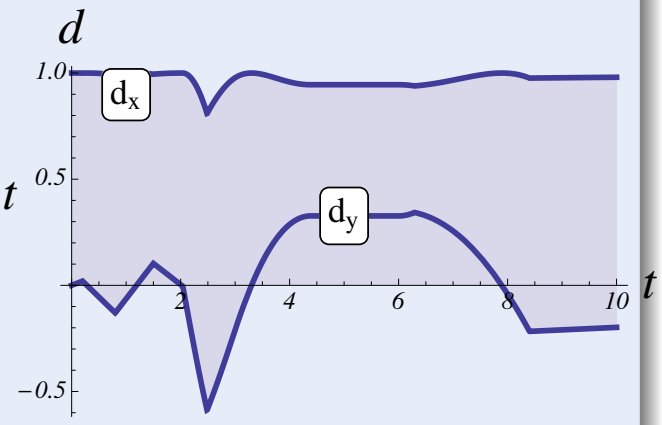


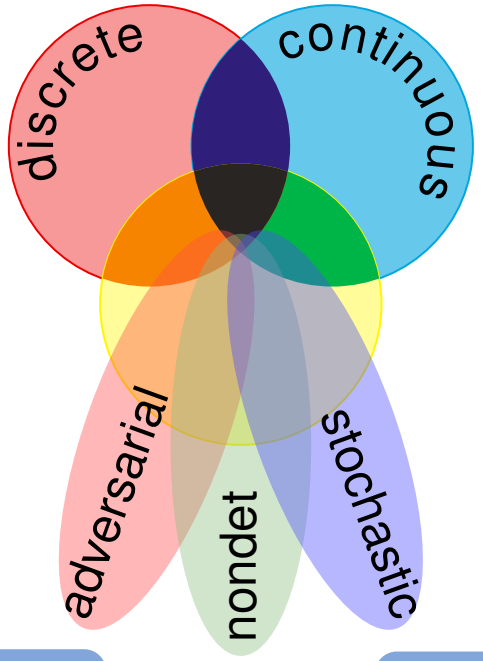



# CPSs are Multi-Dynamical Systems

## CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.




## CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

## Tame Parts

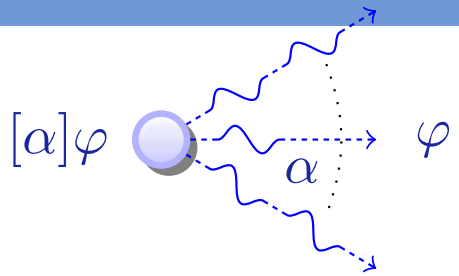
Exploiting compositionality tames CPS complexity.

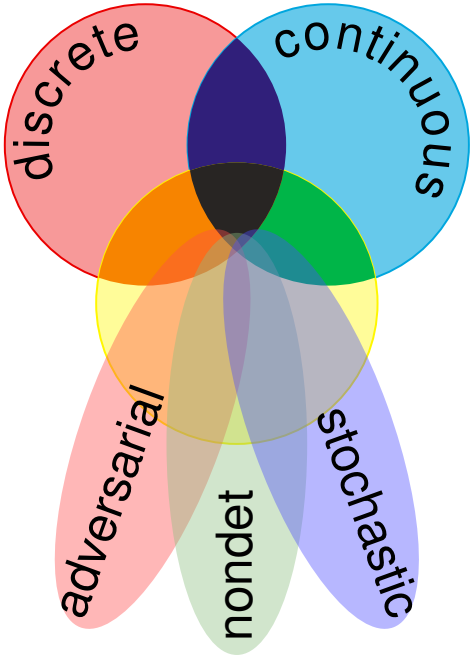
Analytic simplification

# Dynamic Logics for Dynamical Systems

**differential dynamic logic**

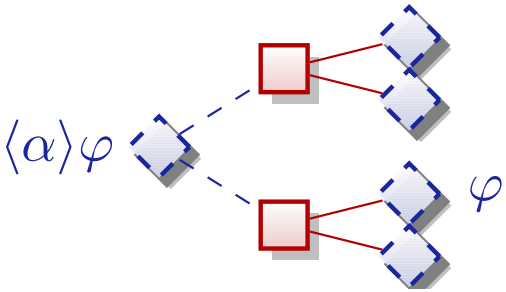
$\mathsf{dL} = \mathsf{DL} + \mathsf{HP}$



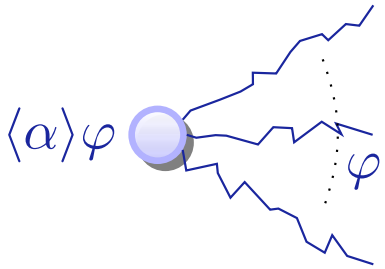

**differential game logic**

$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG}$




**stochastic differential DL**

$\mathsf{Sd}\mathcal{L} = \mathsf{DL} + \mathsf{SHP}$




**quantified differential DL**

$\mathsf{Qd}\mathcal{L} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$

# Outline

# Differential Dynamic Logic dL: Syntax

**Definition (Hybrid program $\alpha$)**

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

**Definition (dL Formula $P$)**

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

# Differential Dynamic Logic dL: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

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All Reals | Some Reals | All Runs | Some Runs

# Differential Dynamic Logic dL: Nondeterminism

Nondet. Choice

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Nondeterminism during HP runs

# Differential Dynamic Logic dL: Nondeterminism

Differential Equation

Nondet. Choice

Nondet. Repeat

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All Choices

Some Choice

# Differential Dynamic Logic dL: Nondeterminism

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

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Modality decides the mode: help/hurt

All Choices

Some Choice

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Modality decides the mode: help/hurt

All Choices

Some Choice

$[\alpha_1]\langle\alpha_2\rangle[\alpha_3]\langle\alpha_4\rangle P$ only fixed interaction depth

# Control & Dual Control Operators

**◇ Angel Ops**

| | |
|---|---|
| $\cup$ | choice |
| $*$ | repeat |
| $x' = f(x)$ | evolve |
| $?Q$ | challenge |

Let Angel be one player

# Control & Dual Control Operators

**◇ Angel Ops**

| | |
|---|---|
| $\cup$ | choice |
| $*$ | repeat |
| $x' = f(x)$ | evolve |
| $?Q$ | challenge |

**□ Demon Ops**

| | |
|---|---|
| $\cap$ | choice |
| $\times$ | repeat |
| $x' = f(x)^{\mathrm{d}}$ | evolve |
| $?Q^{\mathrm{d}}$ | challenge |

Let Angel be one player

Let Demon be another player

# Control & Dual Control Operators

| ◇ Angel Ops | |
|---|---|
| $\cup$ | choice |
| $*$ | repeat |
| $x' = f(x)$ | evolve |
| $?Q$ | challenge |

$\xrightarrow{\mathsf{d}}$ $\xleftarrow{\mathsf{d}}$

| □ Demon Ops | |
|---|---|
| $\cap$ | choice |
| $\times$ | repeat |
| $x' = f(x)^{\mathsf{d}}$ | evolve |
| $?Q^{\mathsf{d}}$ | challenge |

Duality operator $^{\mathsf{d}}$ passes control between players

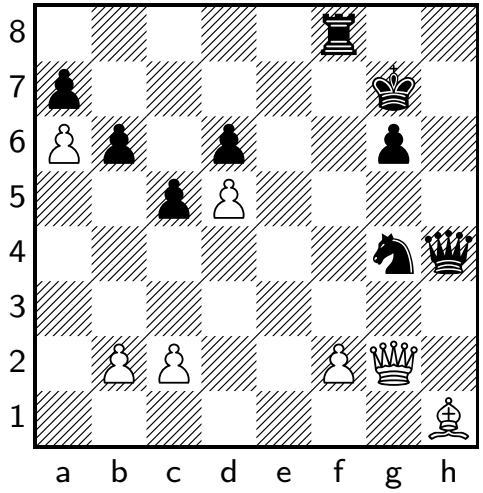
# Game Operators

| ◇ Angel Ops | | ◻ Demon Ops | |
|---|---|---|---|
| $\cup$ | choice | $\cap$ | choice |
| $*$ | repeat | $\times$ | repeat |
| $x' = f(x)$ | evolve | $x' = f(x)^{\mathrm{d}}$ | evolve |
| $?Q$ | challenge | $?Q^{\mathrm{d}}$ | challenge |

d

d

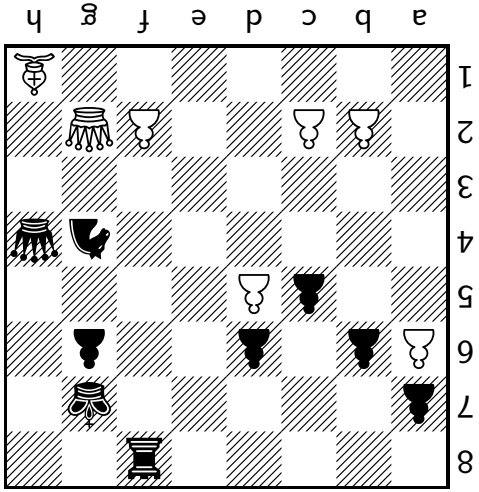
Duality operator $^{\mathrm{d}}$ passes control between players



# Game Operators

| ◇ Angel Ops | | d | □ Demon Ops | |
|---|---|---|---|---|
| $\cup$ | choice | → | $\cap$ | choice |
| $*$ | repeat | | $\times$ | repeat |
| $x' = f(x)$ | evolve | ← | $x' = f(x)^{\mathsf{d}}$ | evolve |
| $?Q$ | challenge | d | $?Q^{\mathsf{d}}$ | challenge |

Duality operator $^{\mathsf{d}}$ passes control between players



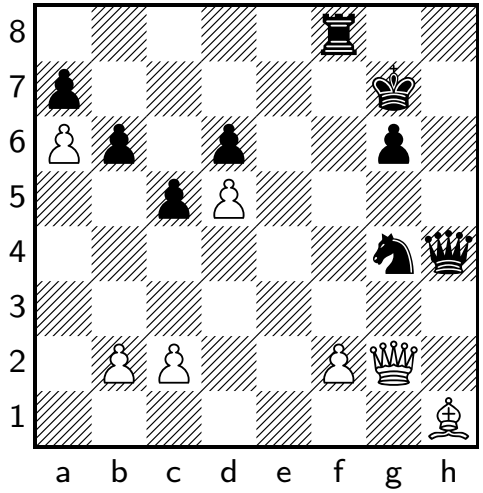
# Game Operators

| ◇ Angel Ops | | ◻ Demon Ops | |
|---|---|---|---|
| $\cup$ | choice | $\cap$ | choice |
| $*$ | repeat | $\times$ | repeat |
| $x' = f(x)$ | evolve | $x' = f(x)^{\mathrm{d}}$ | evolve |
| $?Q$ | challenge | $?Q^{\mathrm{d}}$ | challenge |

d

d

Duality operator $^{\mathrm{d}}$ passes control between players



# Definable Game Operators

| ◇ Angel Ops | | ◻ Demon Ops | |
|---|---|---|---|
| $\cup$ | choice | $\cap$ | choice |
| $*$ | repeat | $\times$ | repeat |
| $x' = f(x)$ | evolve | $x' = f(x)^{\mathsf{d}}$ | evolve |
| $?Q$ | challenge | $?Q^{\mathsf{d}}$ | challenge |

$\mathsf{d}$ ⟶ ⟵ $\mathsf{d}$

$$\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv \\
\text{while}(Q)\,\alpha &\equiv \\
\alpha \cap \beta &\equiv \\
\alpha^{\times} &\equiv \\
(x' = f(x) \,\&\, Q)^{\mathsf{d}} &\quad x' = f(x) \,\&\, Q \\
(x := e)^{\mathsf{d}} &\quad x := e \\
?Q^{\mathsf{d}} &\quad ?Q
\end{aligned}$$

# Definable Game Operators

| ◇ Angel Ops | | □ Demon Ops | |
|---|---|---|---|
| $\cup$ | choice | $\cap$ | choice |
| $*$ | repeat | $\times$ | repeat |
| $x' = f(x)$ | evolve | $x' = f(x)^{\mathsf{d}}$ | evolve |
| $?Q$ | challenge | $?Q^{\mathsf{d}}$ | challenge |

$\mathsf{d}$ →

← $\mathsf{d}$

$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv$$

$$\alpha \cap \beta \equiv$$

$$\alpha^{\times} \equiv$$

$$(x' = f(x)\,\&\,Q)^{\mathsf{d}} \quad x' = f(x)\,\&\,Q$$

$$(x := e)^{\mathsf{d}} \quad x := e$$

$$?Q^{\mathsf{d}} \quad ?Q$$

# Definable Game Operators

| ◇ Angel Ops | | ◻ Demon Ops | |
|---|---|---|---|
| $\cup$ | choice | $\cap$ | choice |
| $*$ | repeat | $\times$ | repeat |
| $x' = f(x)$ | evolve | $x' = f(x)^{\mathsf{d}}$ | evolve |
| $?Q$ | challenge | $?Q^{\mathsf{d}}$ | challenge |

$\mathsf{d}$ ⇄ $\mathsf{d}$

$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv$$

$$\alpha^\times \equiv$$

$$(x' = f(x) \,\&\, Q)^{\mathsf{d}} \quad x' = f(x) \,\&\, Q$$

$$(x := e)^{\mathsf{d}} \quad x := e$$

$$?Q^{\mathsf{d}} \quad ?Q$$

# Definable Game Operators

| ◇ Angel Ops | |
|---|---|
| $\cup$ | choice |
| $*$ | repeat |
| $x' = f(x)$ | evolve |
| $?Q$ | challenge |

d

d

| □ Demon Ops | |
|---|---|
| $\cap$ | choice |
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| $x' = f(x)^{\mathsf{d}}$ | evolve |
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$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

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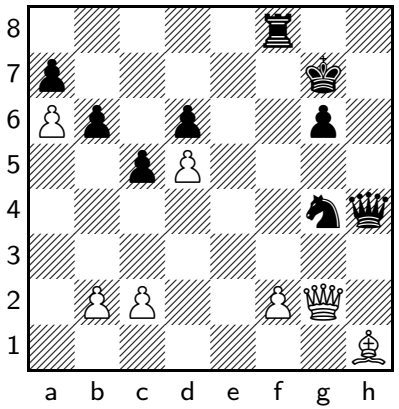

# Definable Game Operators

| ◇ Angel Ops | | ◻ Demon Ops | |
|---|---|---|---|
| $\cup$ | choice | $\cap$ | choice |
| $*$ | repeat | $\times$ | repeat |
| $x' = f(x)$ | evolve | $x' = f(x)^{\mathsf{d}}$ | evolve |
| $?Q$ | challenge | $?Q^{\mathsf{d}}$ | challenge |

$\mathsf{d}$ $\rightarrow$

$\leftarrow$ $\mathsf{d}$

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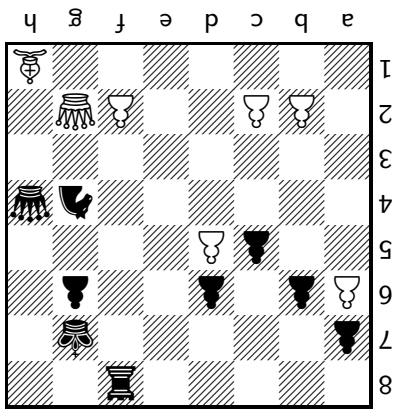

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$\xrightarrow{\mathsf{d}}$ $\xleftarrow{\mathsf{d}}$

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← $\mathsf{d}$

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# Definable Game Operators

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$\mathsf{d}$ ⟶ ⟵ $\mathsf{d}$

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# Outline

# Hybrid Games: Syntax

**Definition (Hybrid game $\alpha$)**

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^{\mathsf{d}}$$

# Hybrid Games: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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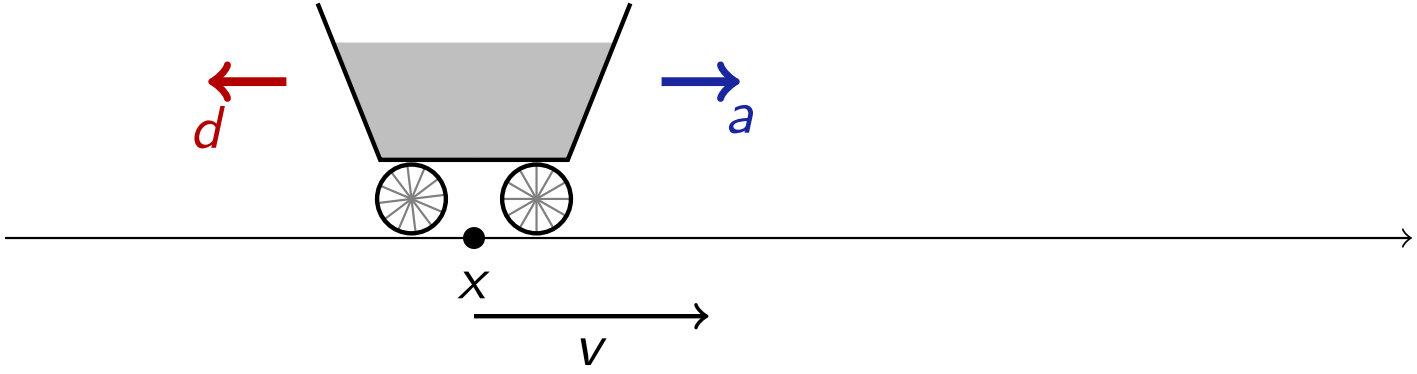

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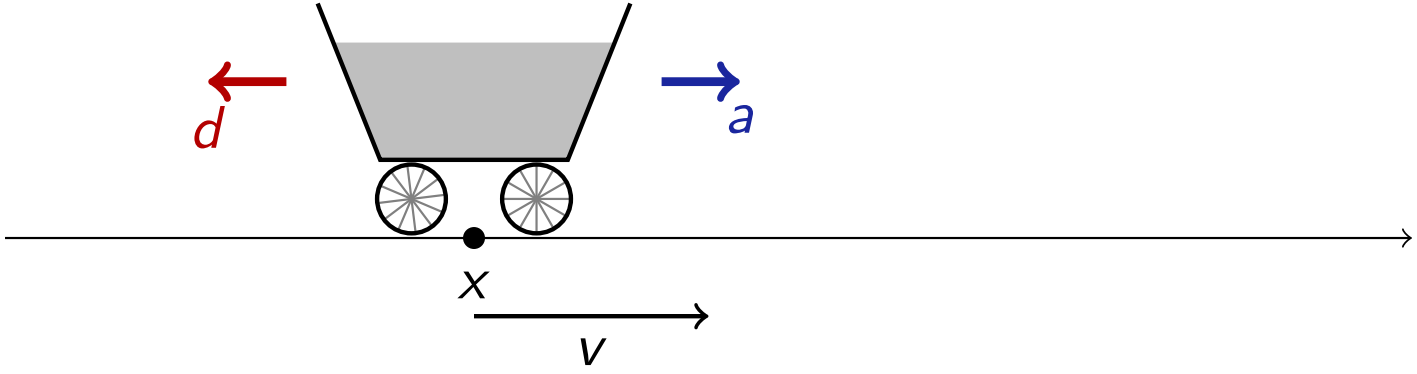


$$\big((a := 1 \cup a := -1); (d := 1 \cup d := -1)^{\mathrm{d}}; \{x' = v, v' = a + d\}\big)^*$$

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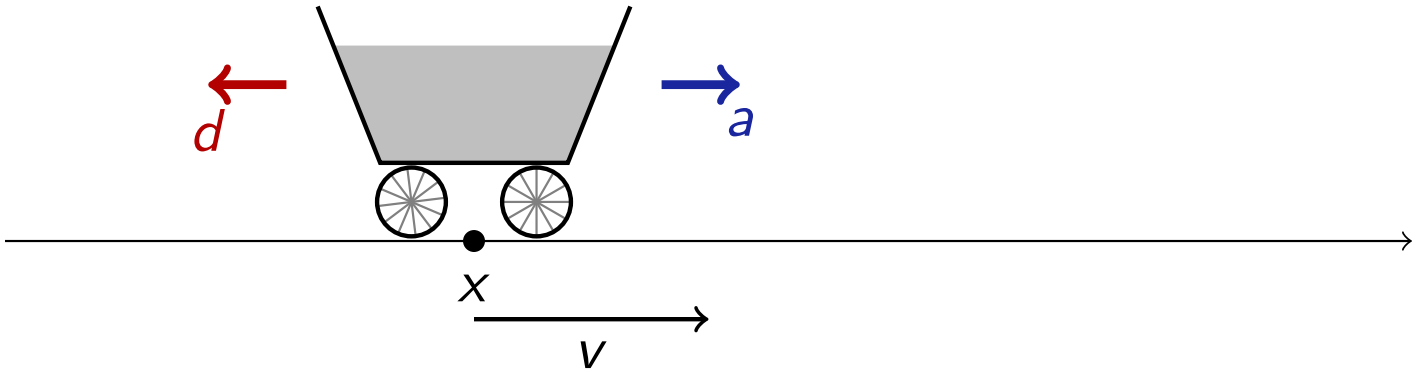


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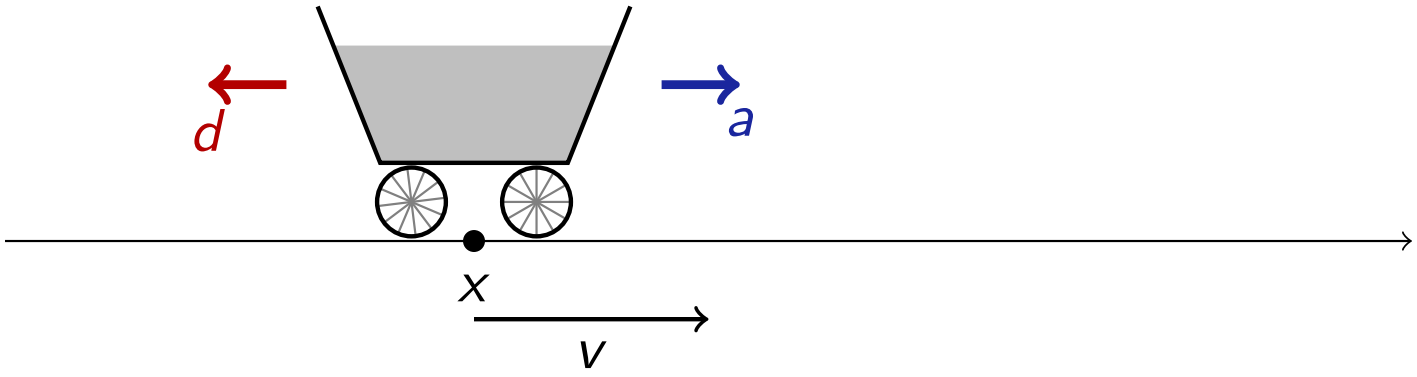


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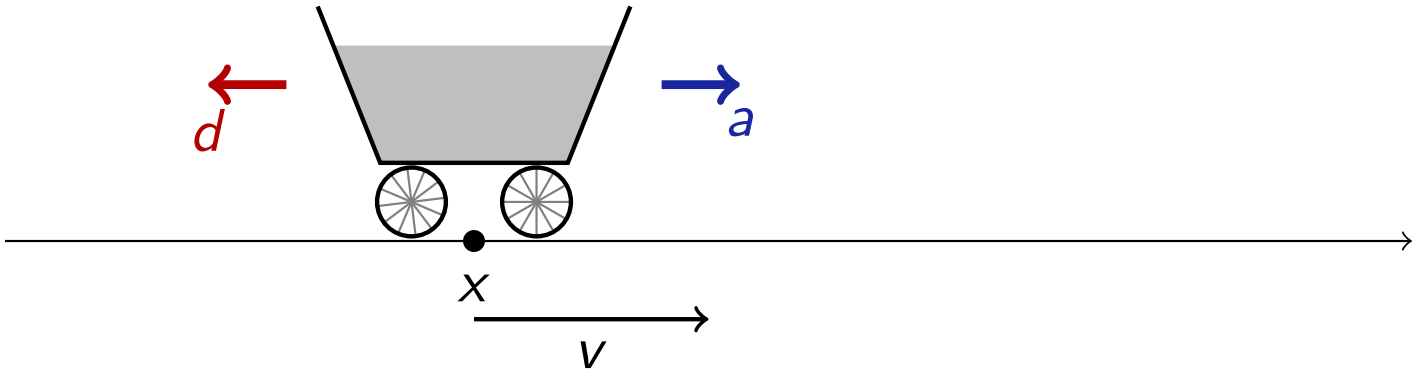


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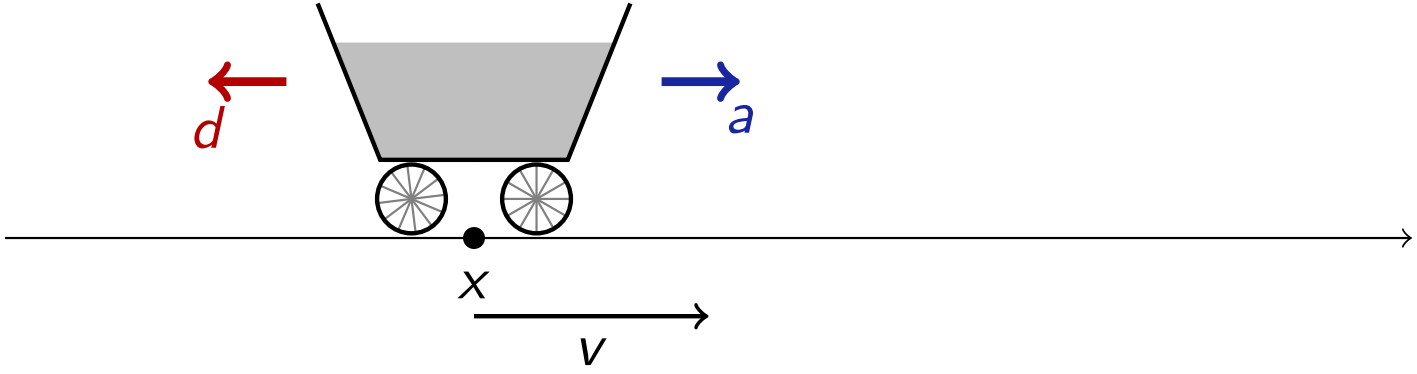


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Hybrid systems can't say that $a$ is Angel's choice and $d$ is Demon's

# Differential Game Logic: Syntax

**Definition (Hybrid game $\alpha$)**

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^{\mathsf{d}}$$

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**Definition (dGL Formula $P$)**

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

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All Reals | Some Reals

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All Reals | Some Reals | Angel Wins | Demon Wins

# Simple Examples

$$\langle (x := x + 1; (x' = 1)^{\mathsf{d}} \cup x := x - 1)^* \rangle \, (0 \leq x < 1)$$

$$\langle (x := x + 1; (x' = 1)^{\mathsf{d}} \cup (x := x - 1 \cap x := x - 2))^* \rangle (0 \leq x < 1)$$

# Simple Examples

$\models \langle (x := x + 1; (x' = 1)^{\mathsf{d}} \cup x := x - 1)^* \rangle (0 \leq x < 1)$

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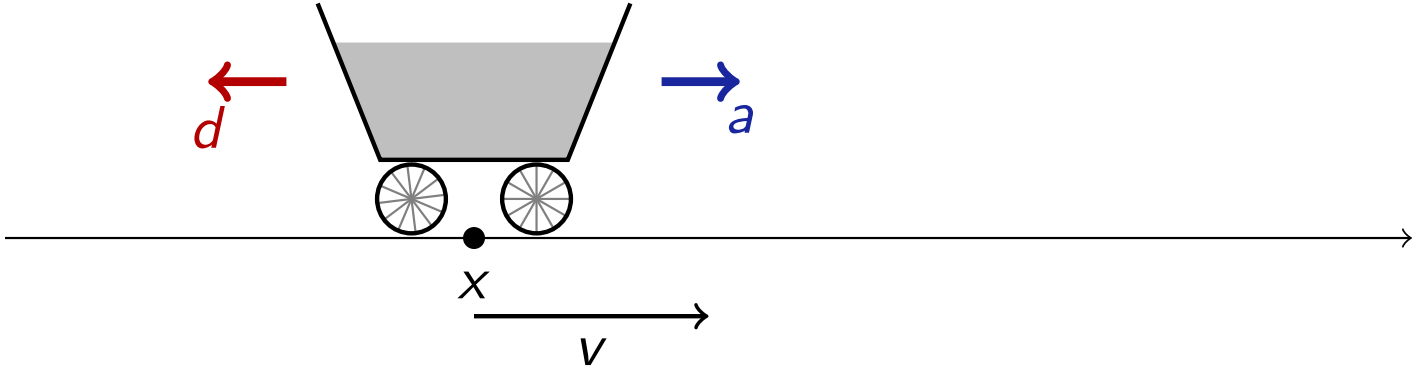
# Simple Examples

$$\models \langle (x := x + 1; (x' = 1)^{\mathsf{d}} \cup x := x - 1)^* \rangle (0 \leq x < 1)$$

$$\not\models \langle (x := x + 1; (x' = 1)^{\mathsf{d}} \cup (x := x - 1 \cap x := x - 2))^* \rangle (0 \leq x < 1)$$

# Example: Push-around Cart




$$v \geq 1 \rightarrow$$

$$\left[\left(\left(d := 1 \cup d := -1\right)^{\mathsf{d}}; \left(a := 1 \cup a := -1\right); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

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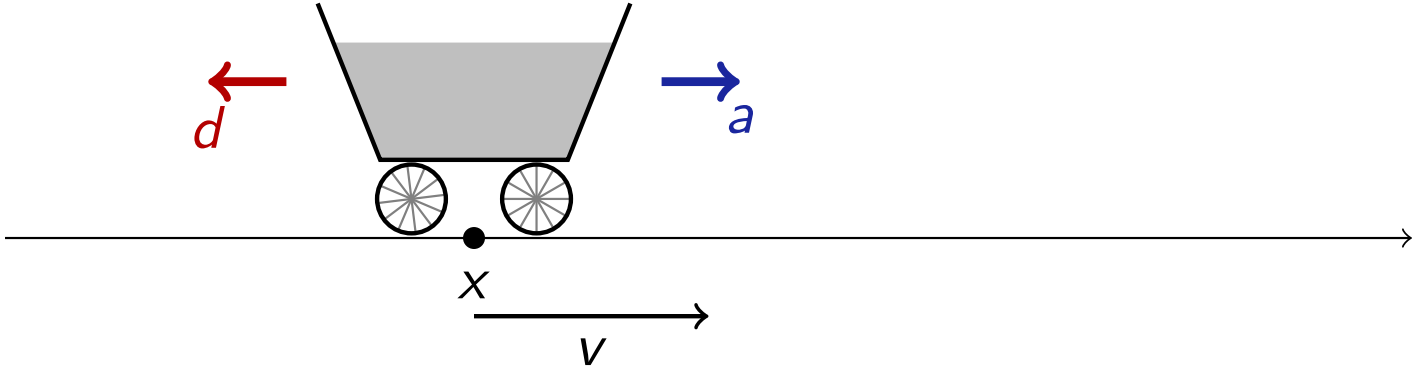


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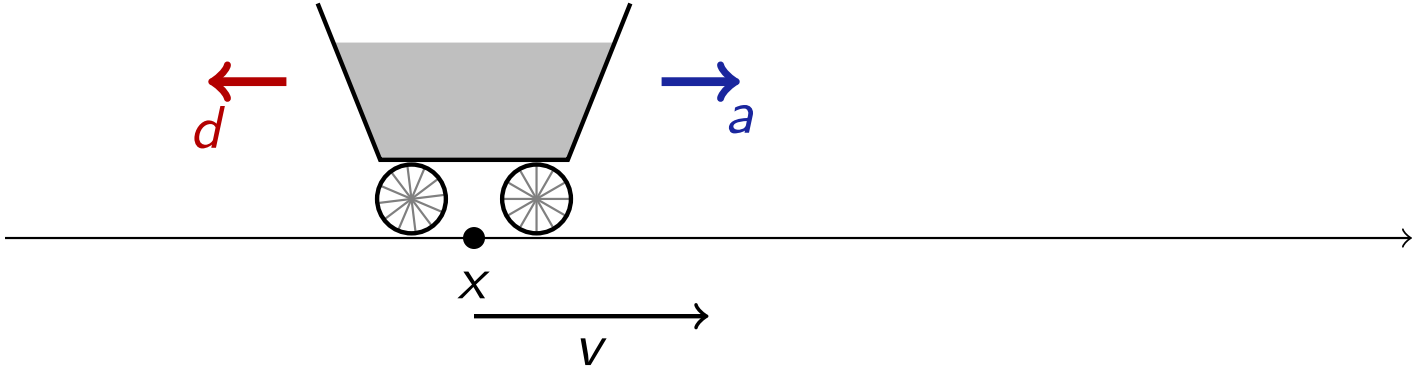


$\models v \geq 1 \rightarrow$ $\quad$ $d$ before $a$ can compensate

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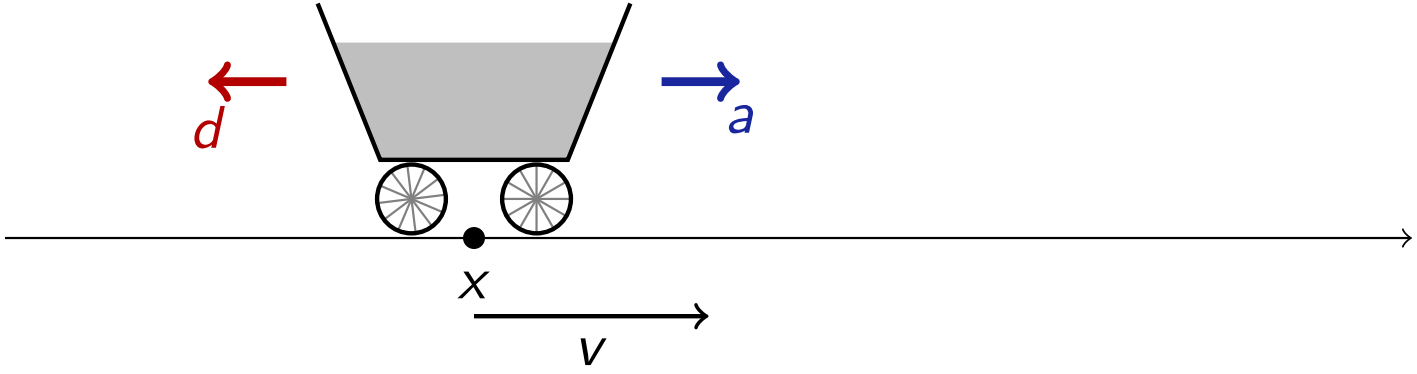


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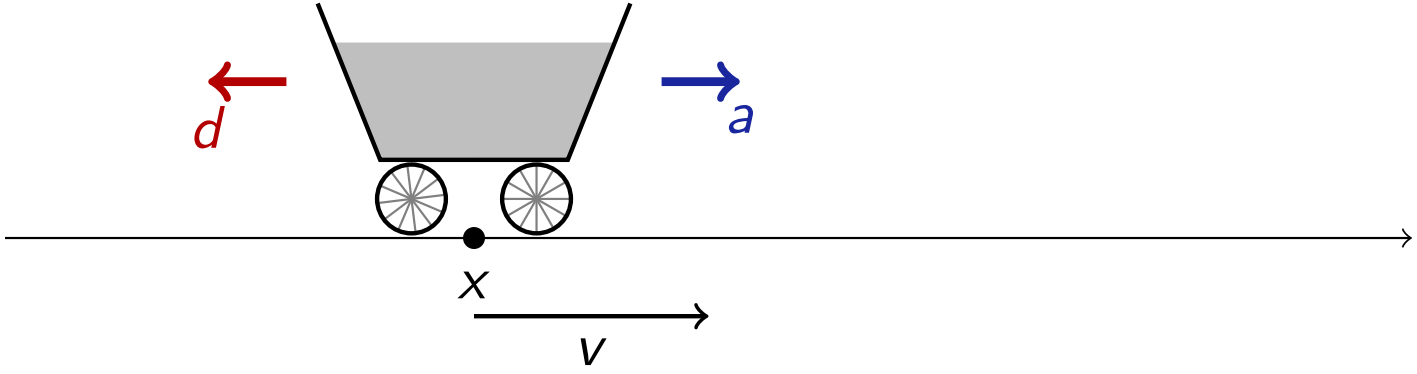


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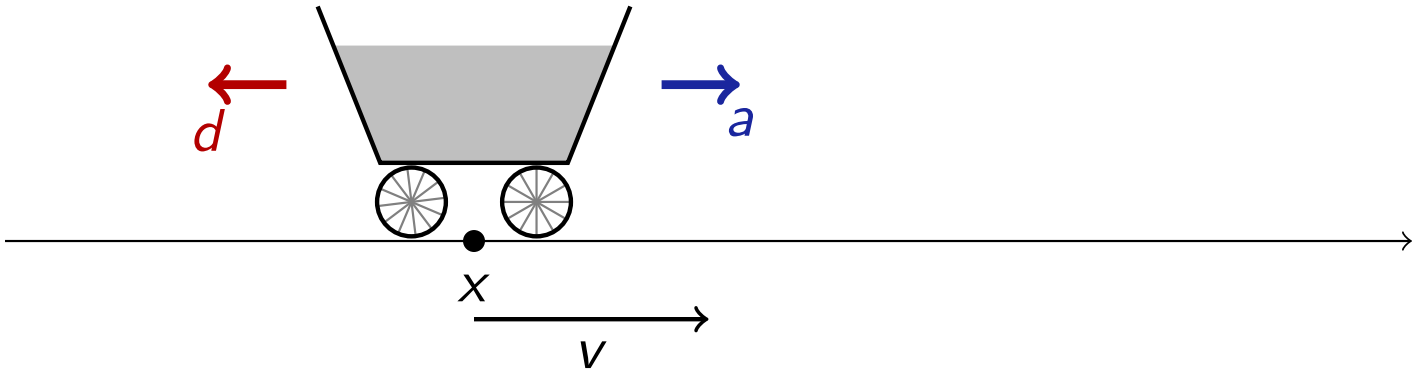


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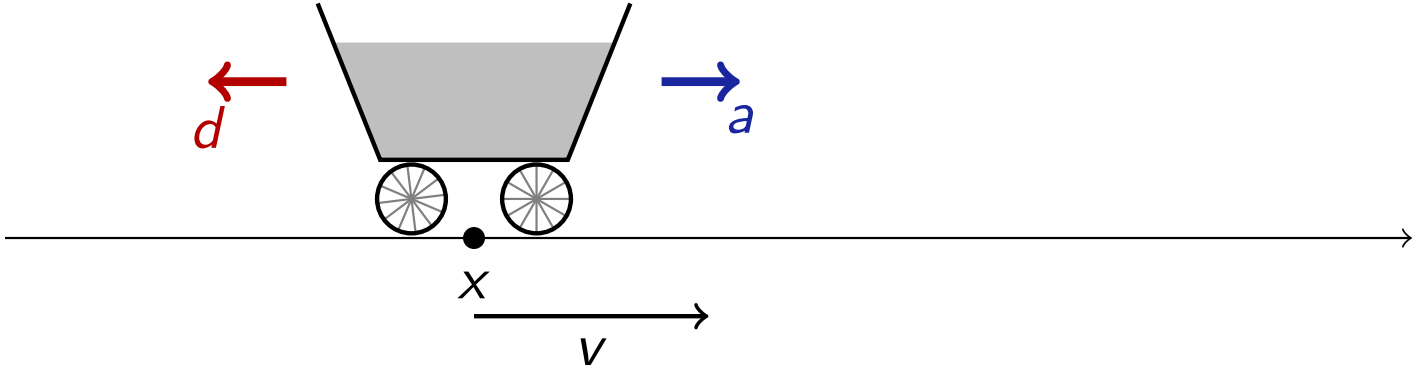


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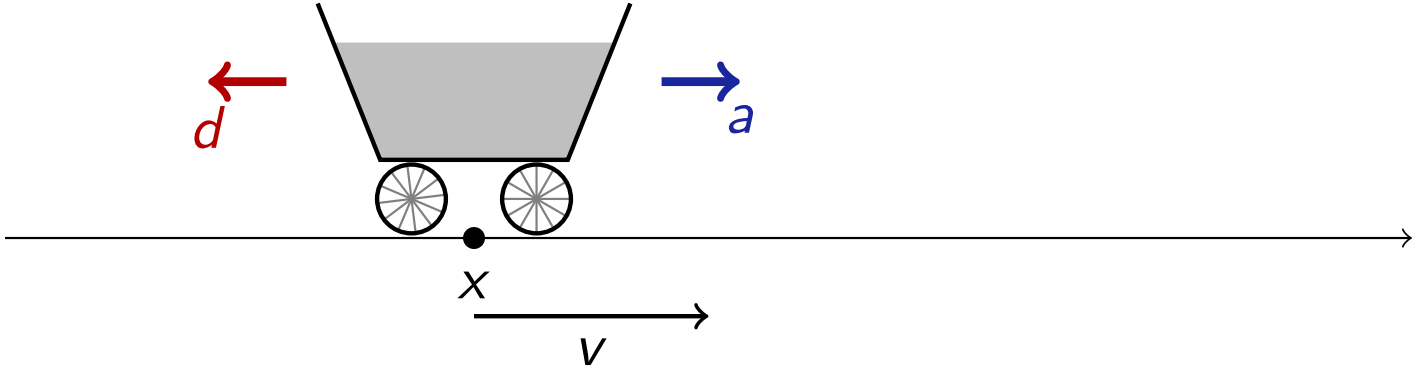


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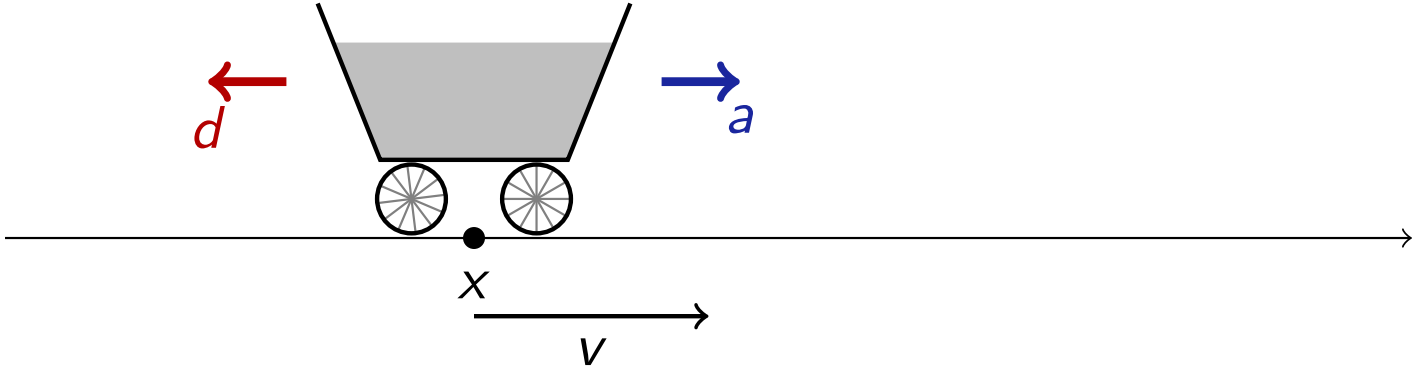


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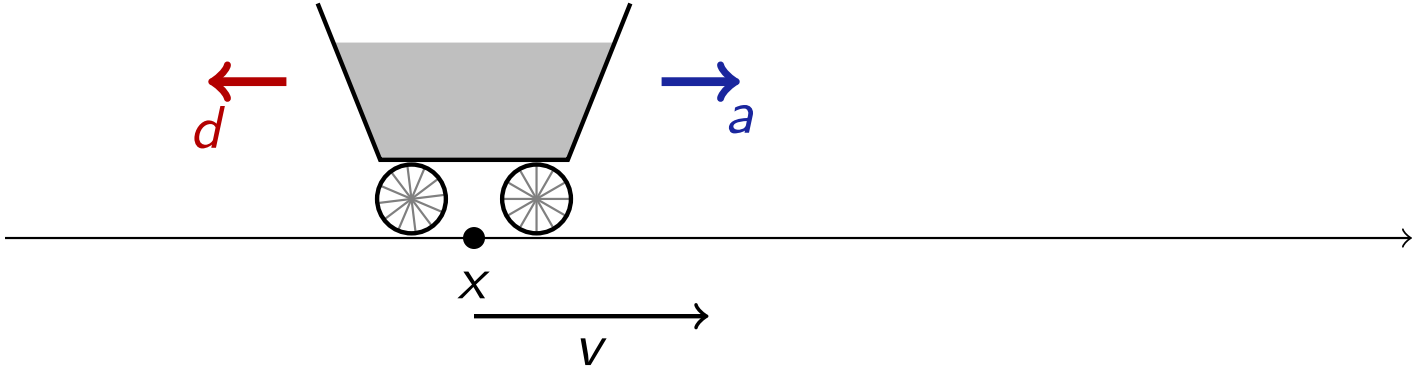


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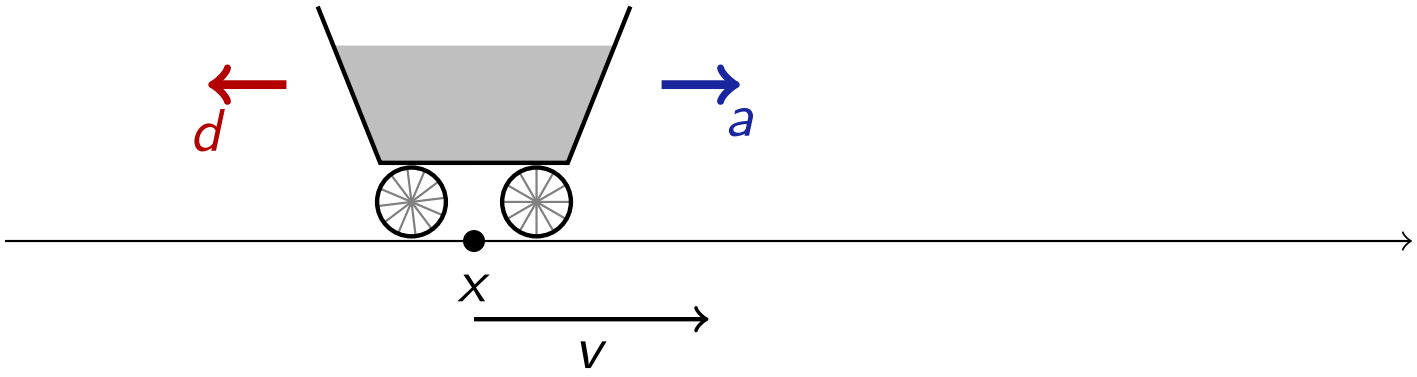


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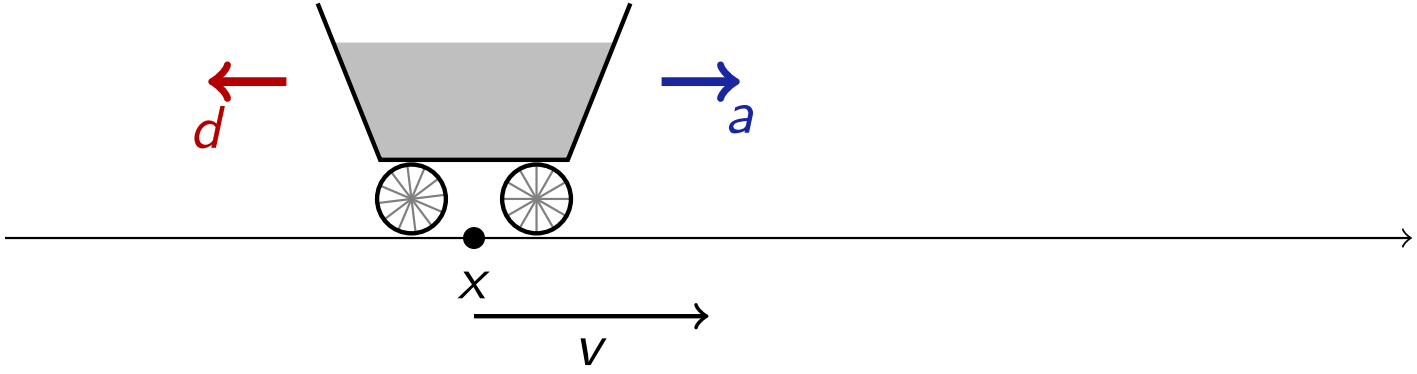
$$\left\langle\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

$$\models \left\langle\left((d := 1 \cap d := -1);\ (a := 2 \cup a := -2);\ \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

$$\left\langle\left((d := 2 \cap d := -2);\ (a := 2 \cup a := -2);\right.\right.$$
$$\left.\left. t := 0;\ \{x' = v, v' = a + d, t' = 1 \,\&\, t \leq 1\}\right)^*\right\rangle x^2 \geq 100$$

# Example: Push-around Cart




$\models v \geq 1 \rightarrow$ $\quad$ $d$ before $a$ can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

$\not\models$ $\quad$ counterstrategy $d := -1$

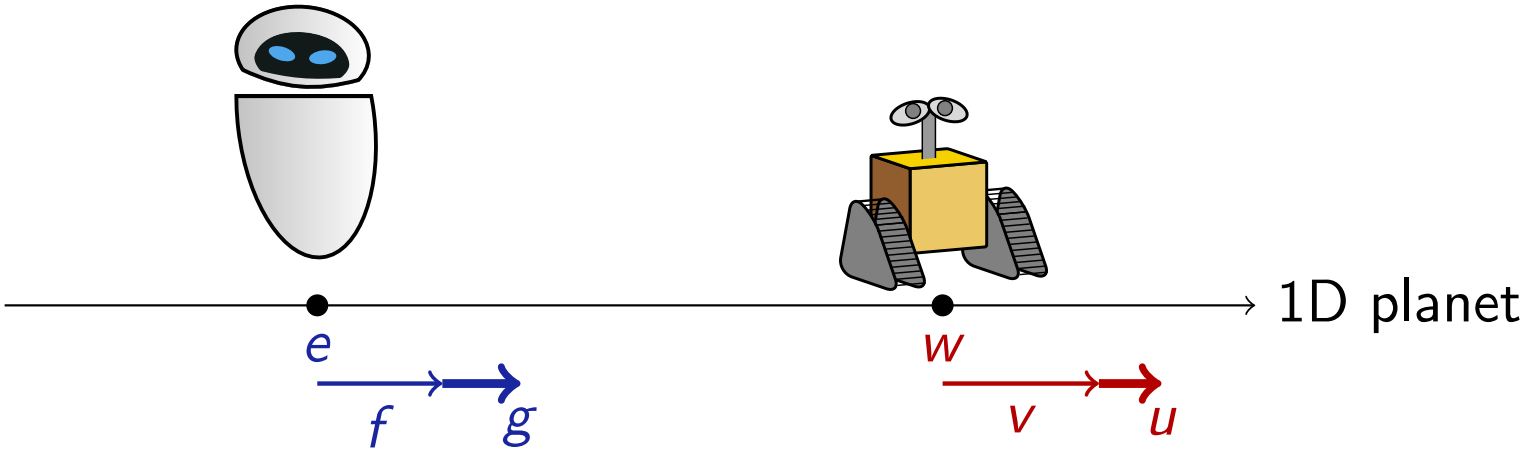
$$\left\langle\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

$\models \left\langle\left((d := 1 \cap d := -1);\ (a := 2 \cup a := -2);\ \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$

$\models \left\langle\left((d := 2 \cap d := -2);\ (a := 2 \cup a := -2);\right.\right.$ $\quad$ $a := d$ then $a := 2 \operatorname{sign} v$

$$\left.\left. t := 0;\ \{x' = v, v' = a + d, t' = 1 \,\&\, t \leq 1\}\right)^*\right\rangle x^2 \geq 100$$

# Example: WALL•E and EVE Robot Dance

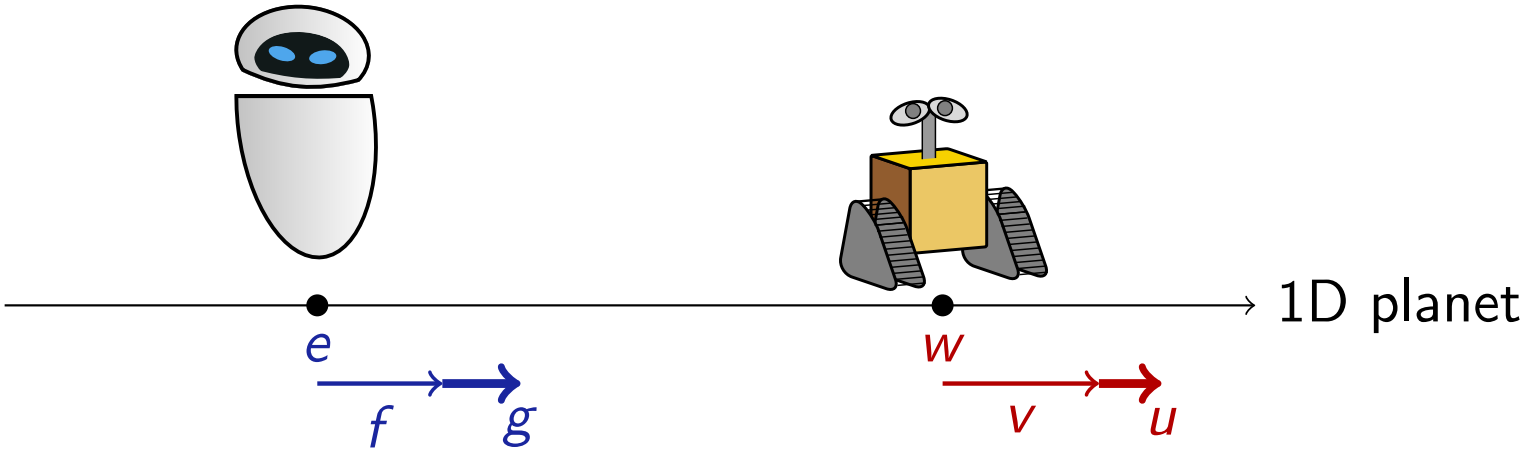



$$(w-e)^2 \leq 1 \wedge v = f \rightarrow$$
$$\langle ((u := 1 \cap u := -1);$$
$$(g := 1 \cup g := -1);$$
$$t := 0;$$
$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \leq 1\}^{\mathrm{d}}$$
$$)^{\times} \rangle\, (w-e)^2 \leq 1$$

EVE at $e$ plays Angel's part controlling $g$

WALL•E at $w$ plays Demon's part controlling $u$

# Example: WALL•E and EVE Robot Dance and the World

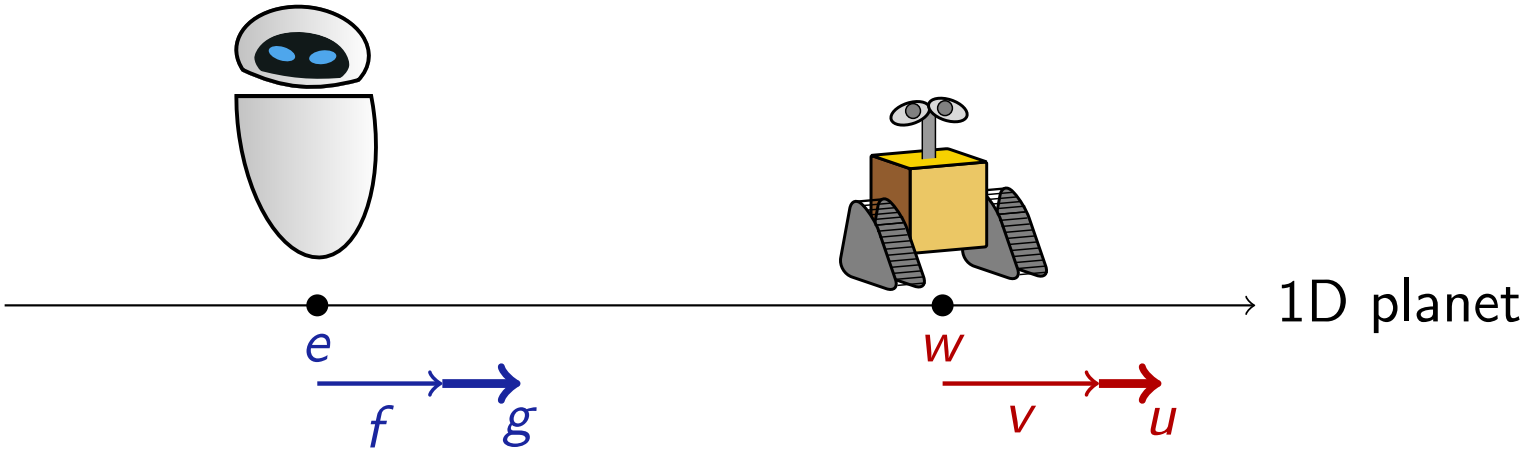



$$(w-e)^2 \leq 1 \wedge v = f \rightarrow$$
$$\langle ((u := 1 \cap u := -1);$$
$$(g := 1 \cup g := -1);$$
$$t := 0;$$
$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \leq 1\}^{\mathrm{d}}$$
$$)^{\times}\rangle\ (w-e)^2 \leq 1$$

EVE at $e$ plays Angel's part controlling $g$

WALL•E at $w$ plays Demon's part controlling $u$ and world time

# Example: WALL•E and EVE

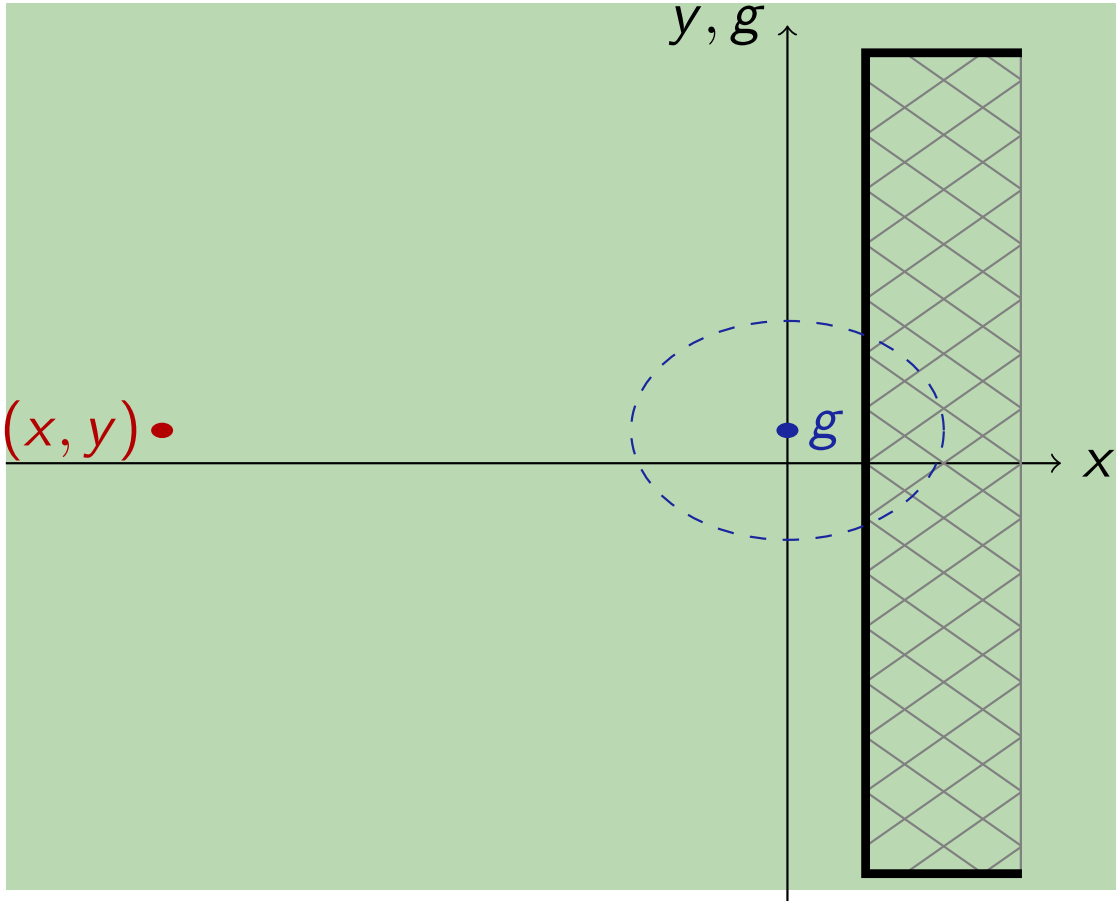



$$(w-e)^2 \leq 1 \wedge v = f \rightarrow$$
$$[((u := 1 \cap u := -1);$$
$$(g := 1 \cup g := -1);$$
$$t := 0;$$
$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \leq 1\}$$
$$)^{\times}]\ (w-e)^2 > 1$$

WALL•E at $w$ plays Demon's part controlling $u$ and world time

EVE at $e$ plays Angel's part controlling $g$

# Example: Goalie in Robot Soccer




$$x < 0 \wedge v > 0 \wedge y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y - g)^2 \leq 1$$

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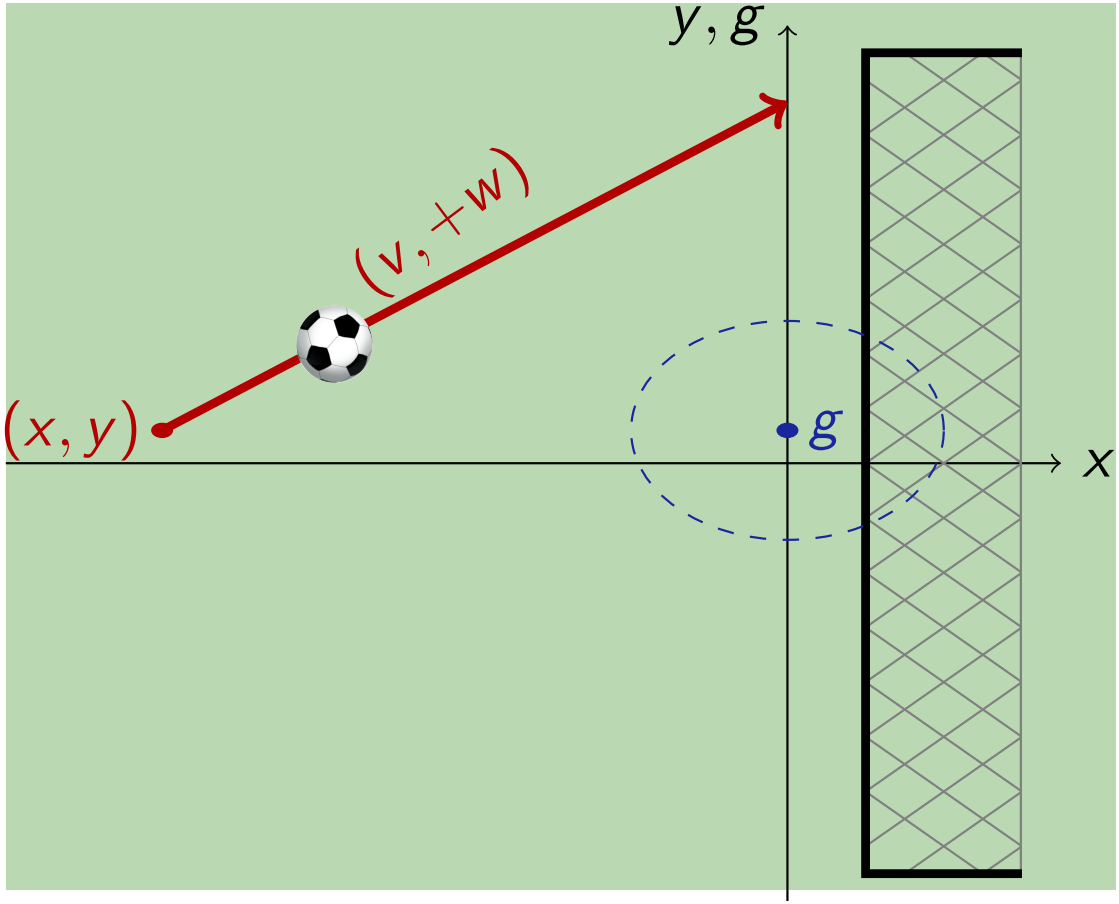


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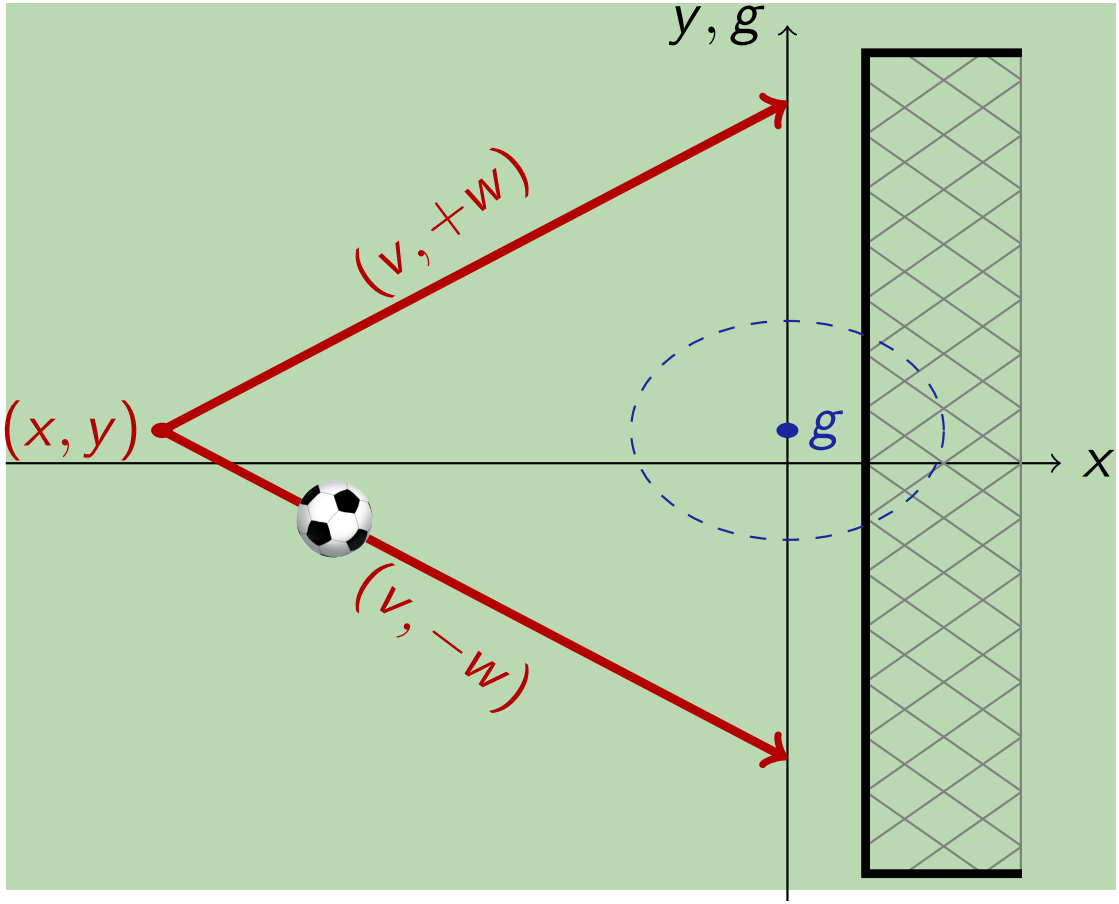


$$x < 0 \wedge v > 0 \wedge y = g \rightarrow$$
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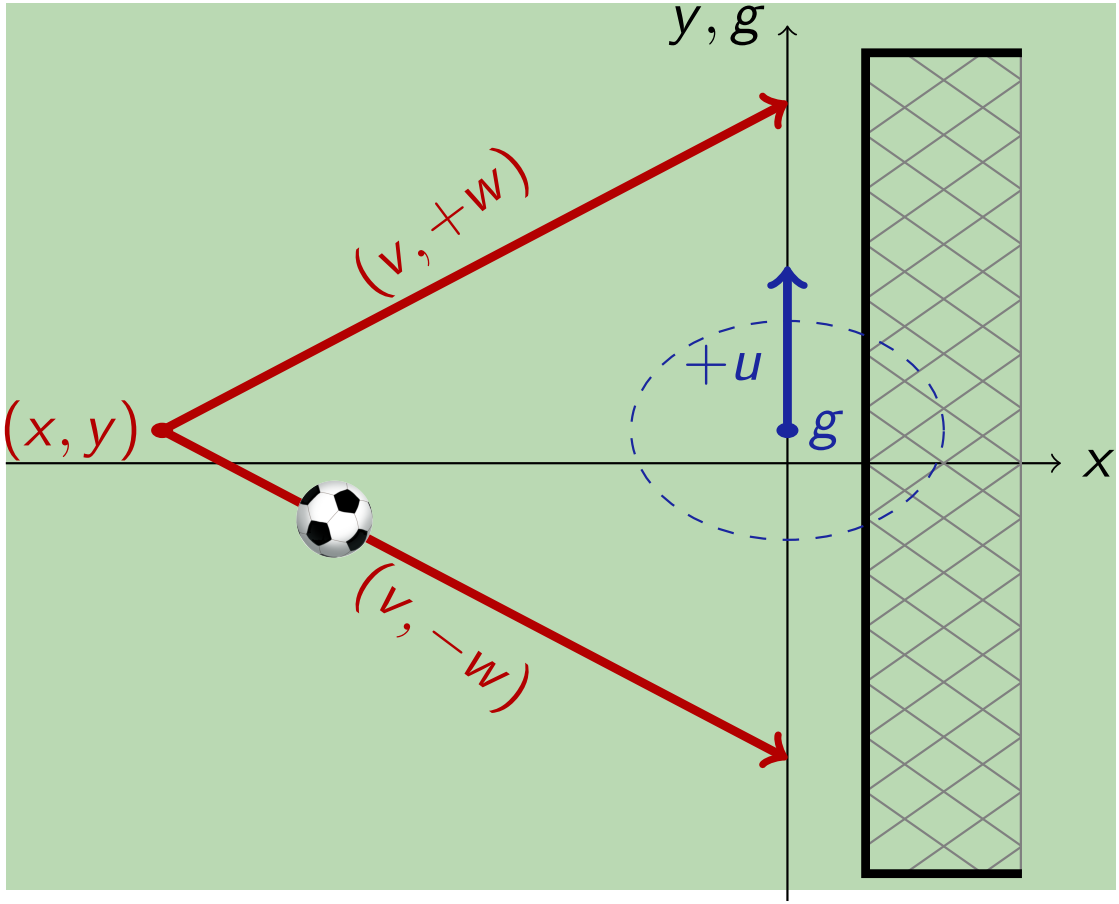


$$x < 0 \wedge v > 0 \wedge y = g \rightarrow$$
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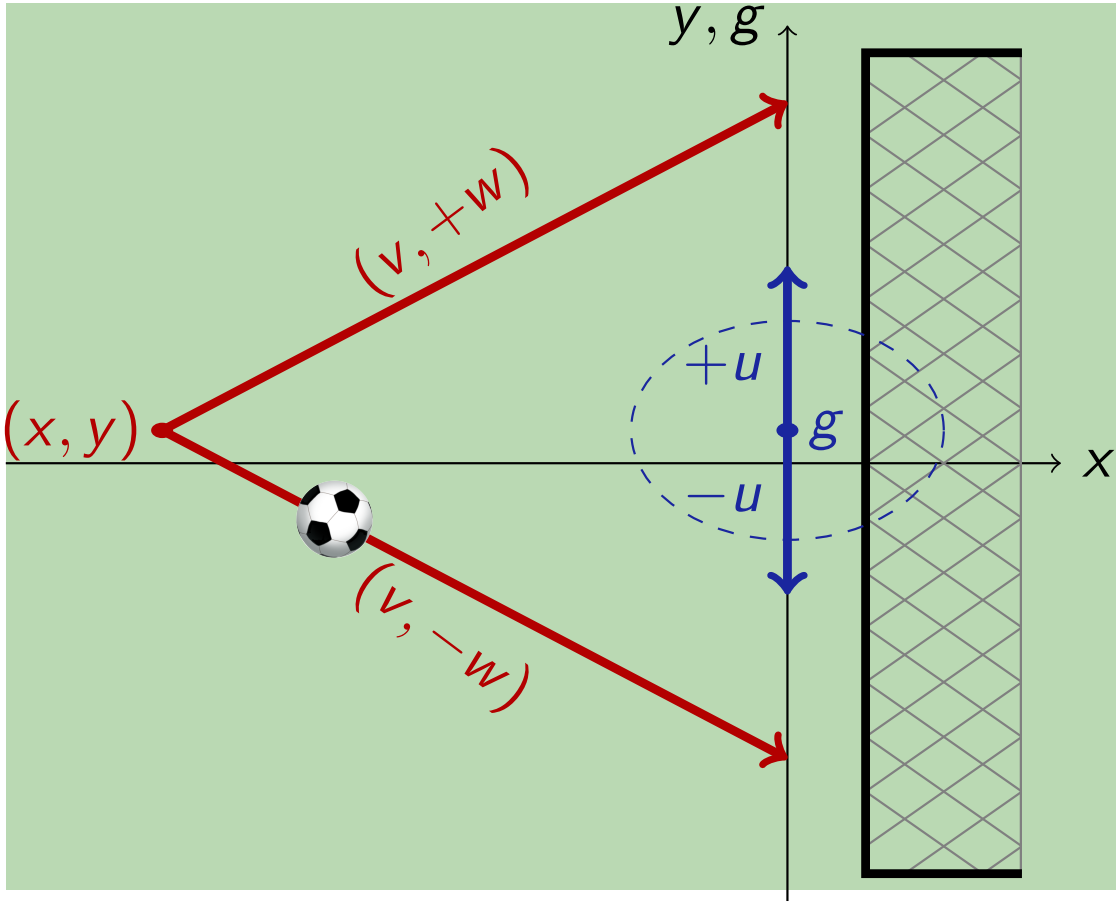
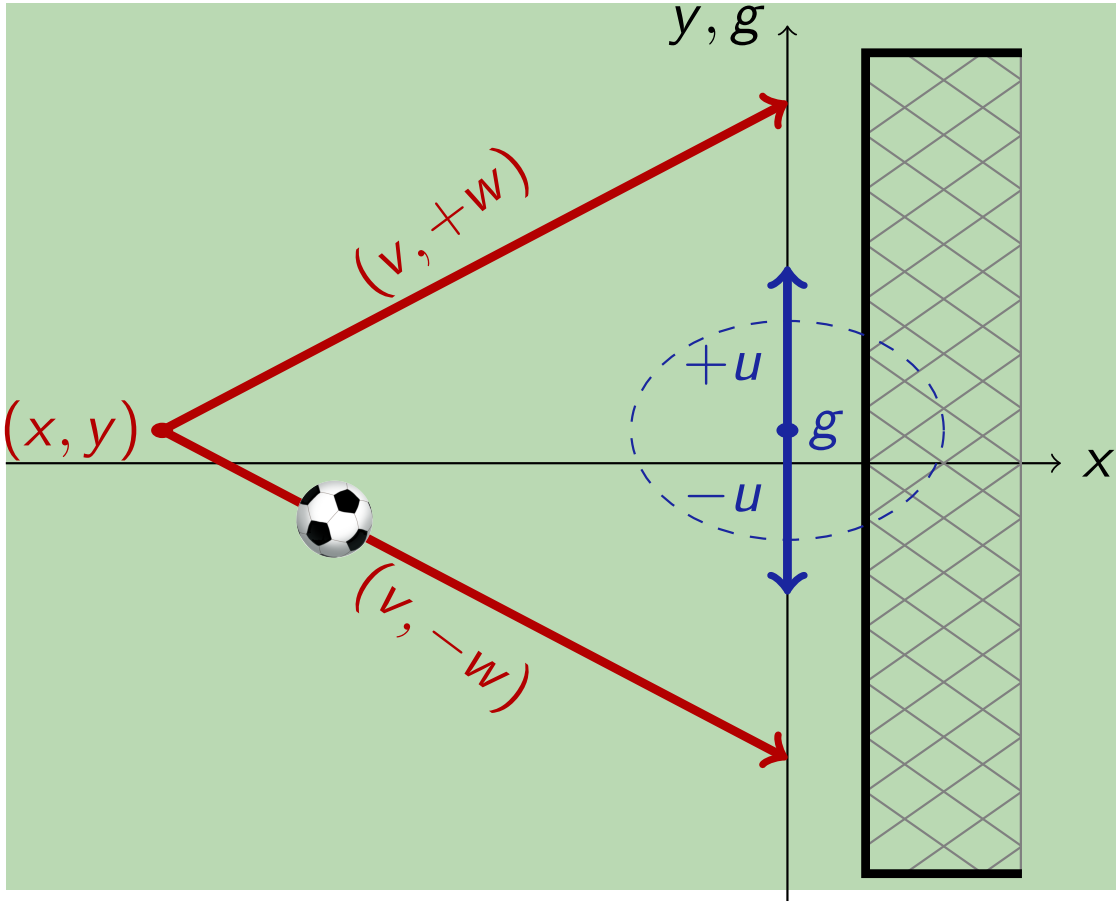
# Example: Goalie in Robot Soccer




$$x < 0 \wedge v > 0 \wedge y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y - g)^2 \leq 1$$

# Example: Goalie in Robot Soccer




Goalie's Secret

$$\left(\frac{x}{v}\right)^2 (u-w)^2 \leq 1 \wedge$$

$$x < 0 \wedge v > 0 \wedge y = g \rightarrow$$

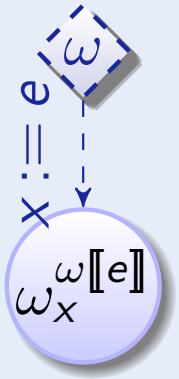
$$\langle (w := +w \cap w := -w);$$

$$((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y-g)^2 \leq 1$$

# Outline

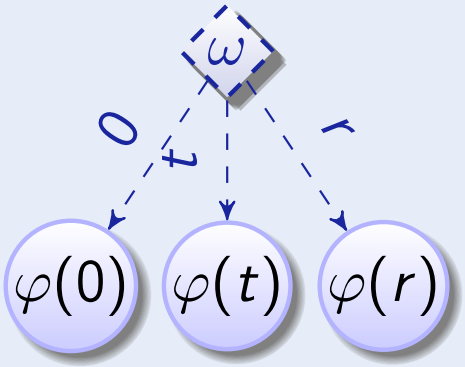
## Definition (Hybrid game $\alpha$: operational semantics)

$x := e$



## Definition (Hybrid game $\alpha$: operational semantics)

$$x' = f(x) \,\&\, Q$$



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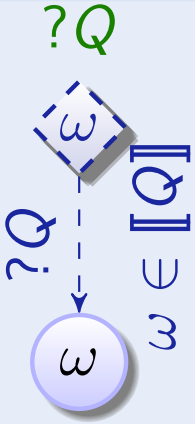

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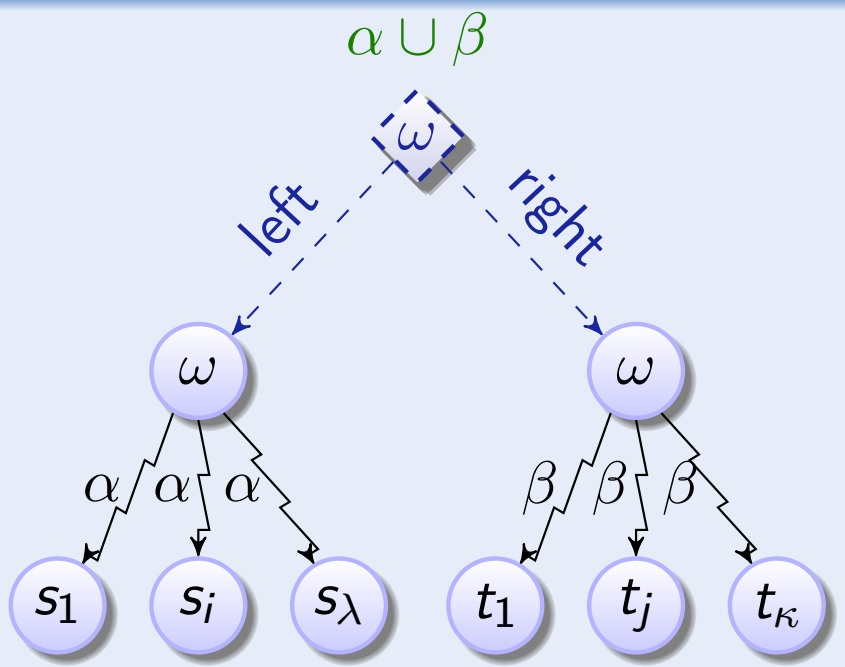

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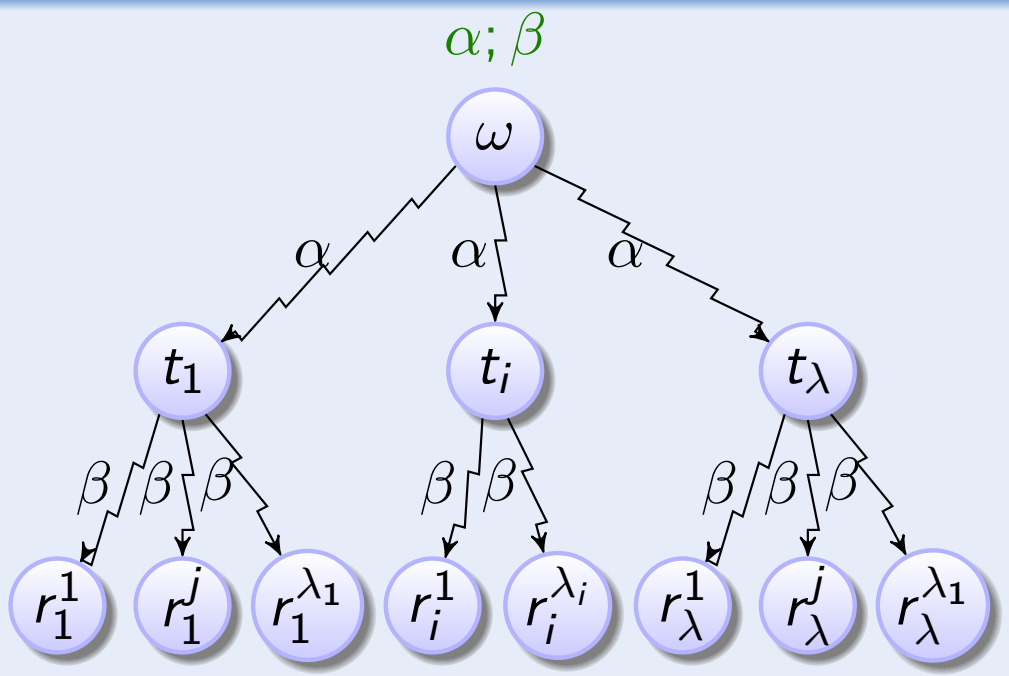

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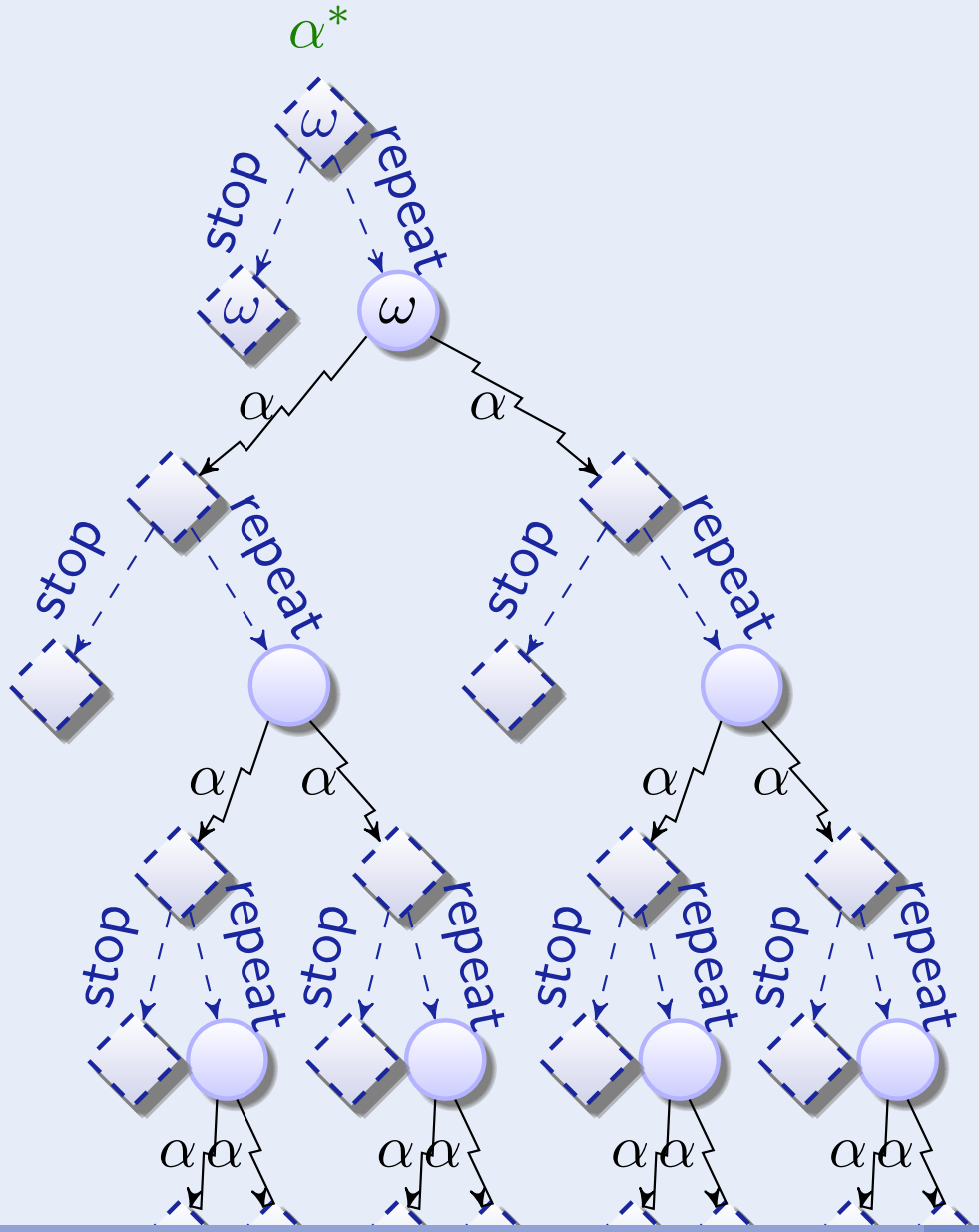
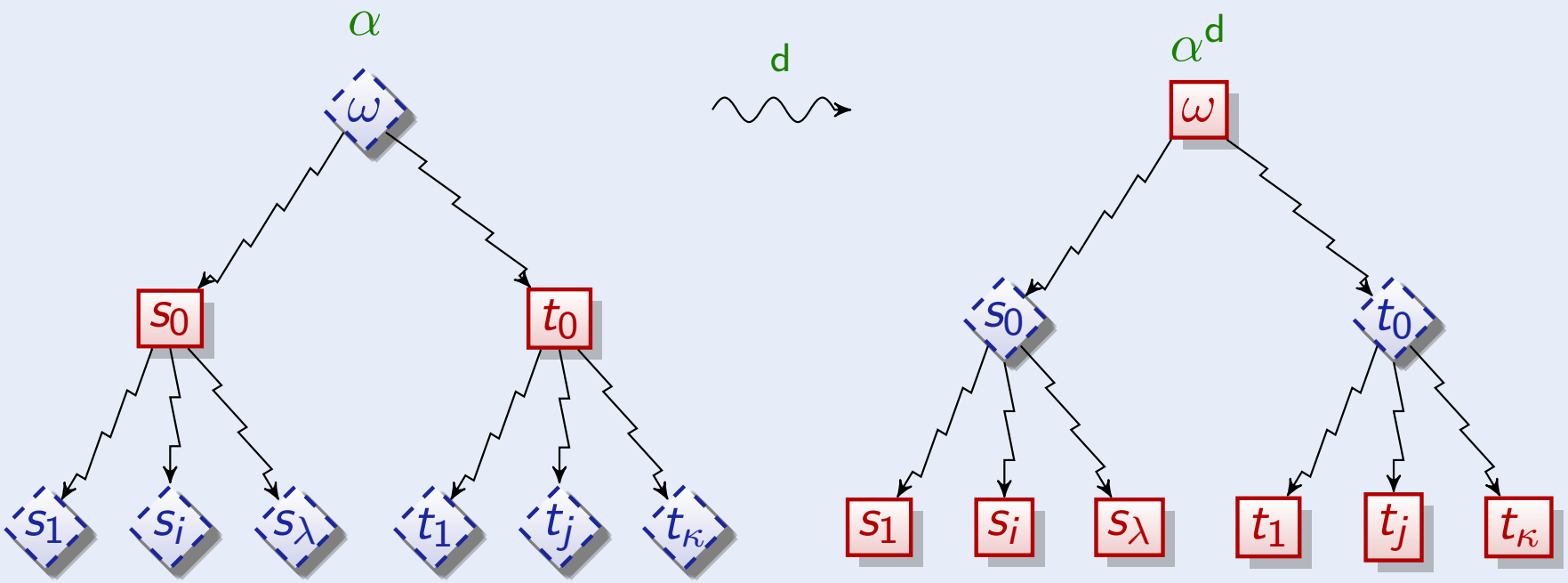
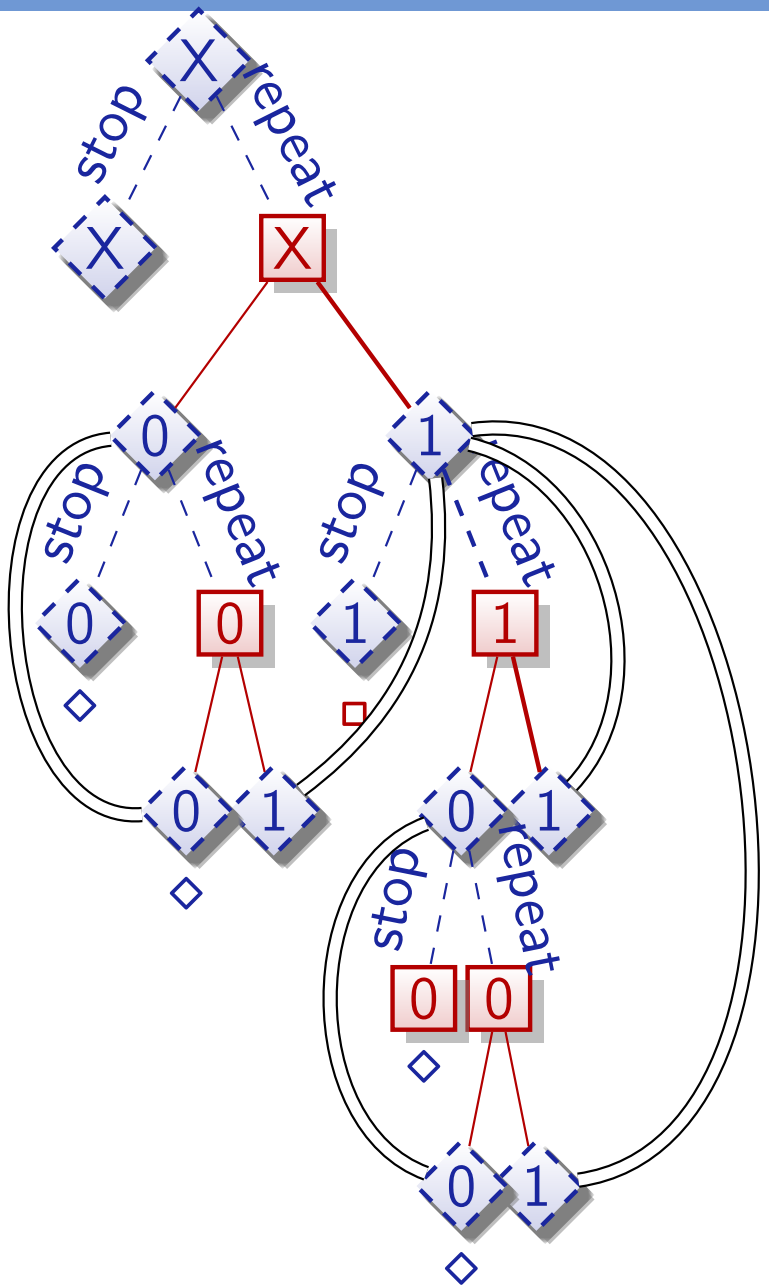


## Definition (Hybrid game $\alpha$: operational semantics)



# Filibusters

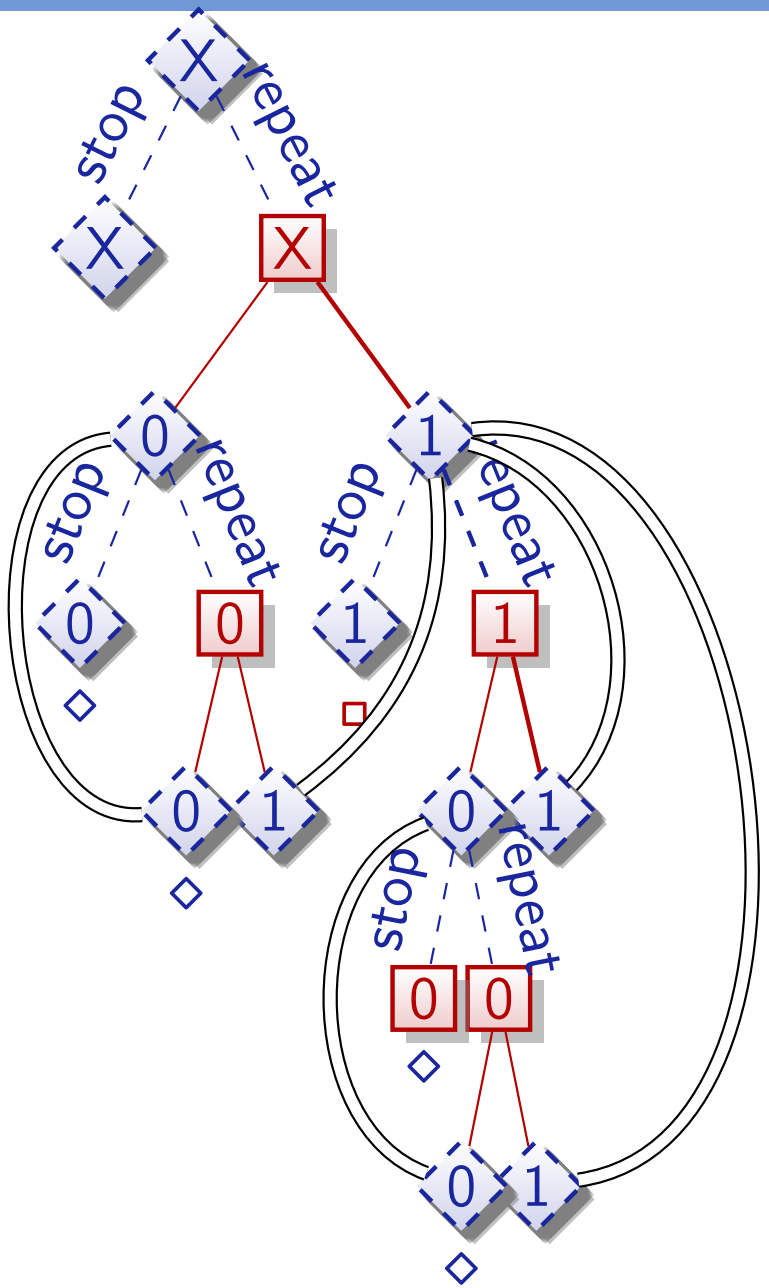
$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$



# Filibusters & The Significance of Finitude

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



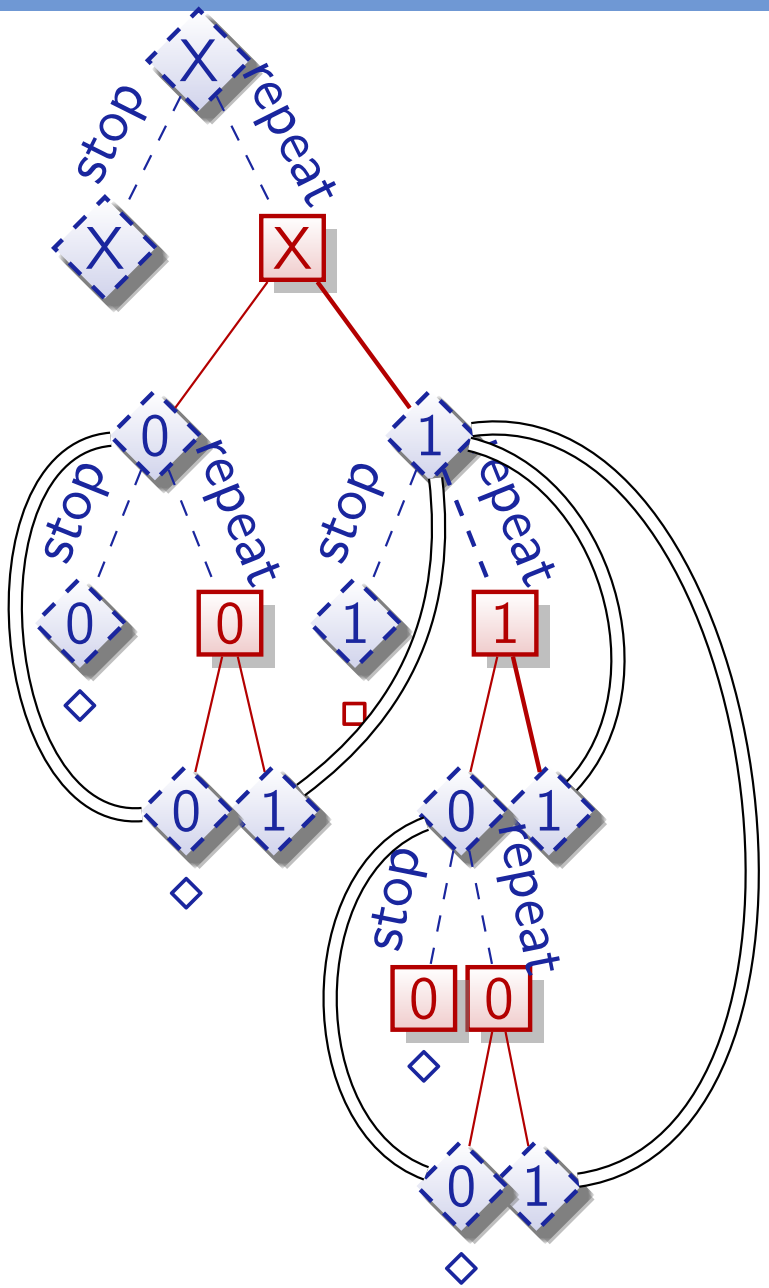
# Filibusters & The Significance of Finitude

$\langle (x' = 1^{\mathsf{d}}; x := 0)^* \rangle x = 0$

$\langle (x := 0; x' = 1^{\mathsf{d}})^* \rangle x = 0$

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



# Filibusters & The Significance of Finitude

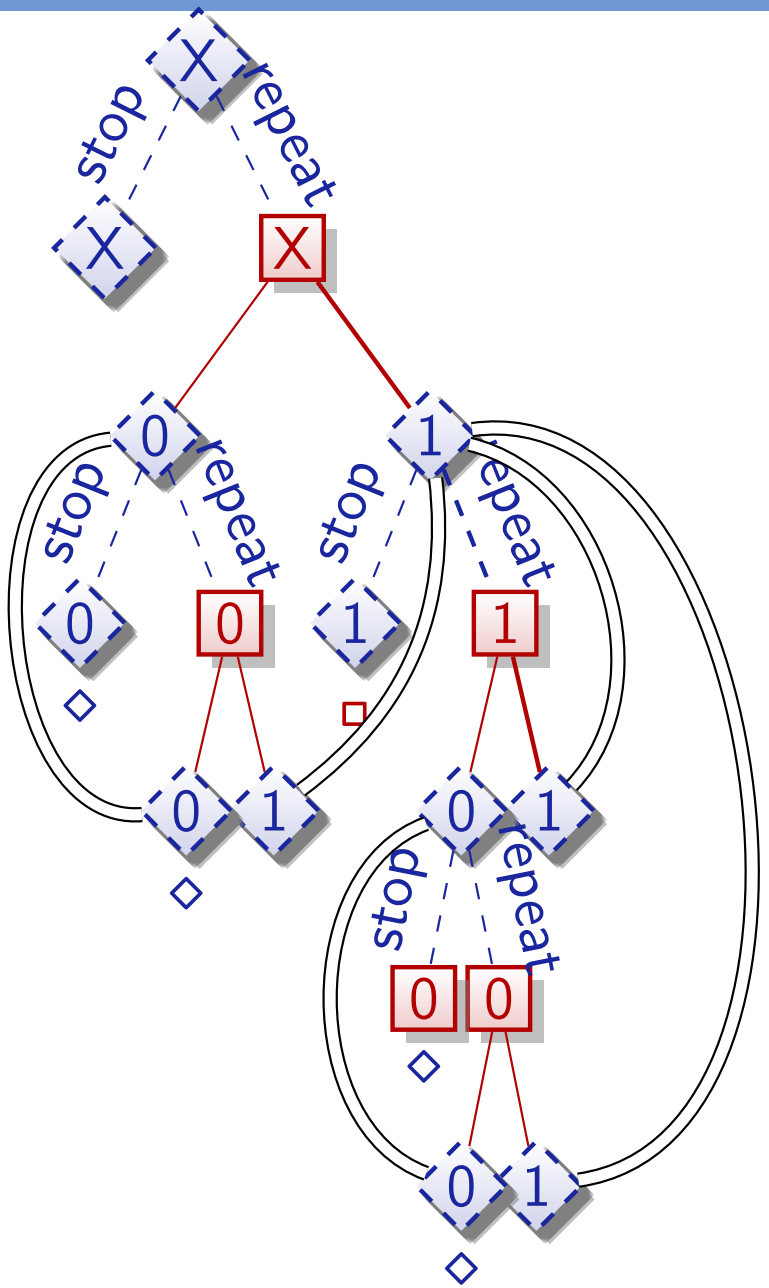
$\overset{<\infty}{\rightsquigarrow}$ true

$\langle (x' = 1^{\mathsf{d}}; x := 0)^* \rangle x = 0$

$\langle (x := 0; x' = 1^{\mathsf{d}})^* \rangle x = 0$

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



# Filibusters & The Significance of Finitude

$\overset{<\infty}{\rightsquigarrow}$ true

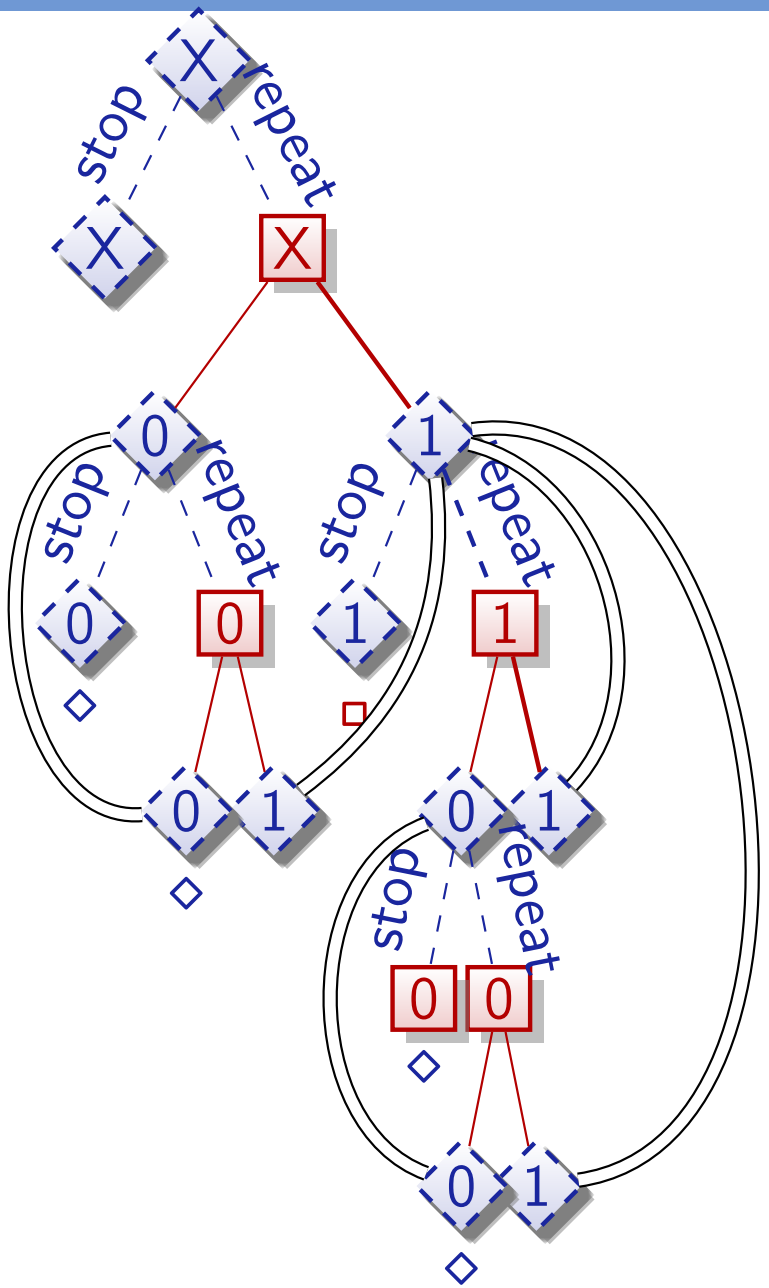
$\langle (x' = 1^{\mathsf{d}}; x := 0)^* \rangle x = 0$

$\langle (x := 0; x' = 1^{\mathsf{d}})^* \rangle x = 0$

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

Well-defined games
can't be postponed forever



# Outline

# Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

**Definition (Hybrid game $\alpha$)**

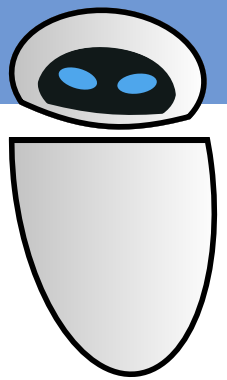
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^{\mathsf{d}}$$

**Definition (dGL Formula $P$)**

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

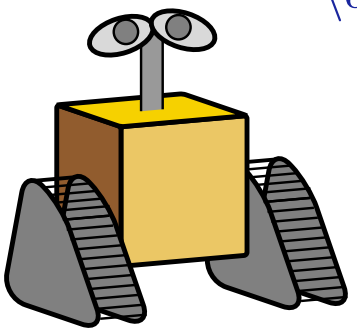
All Reals | Some Reals | Angel Wins | Demon Wins

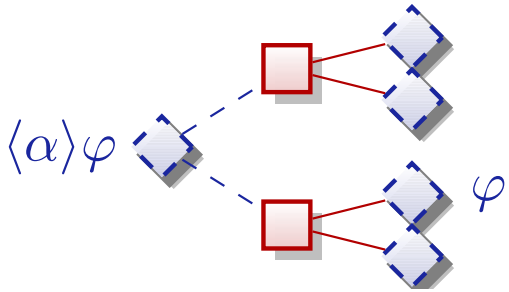
# Summary



**differential game logic**

$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG} = \mathsf{dL} + {}^{\mathsf{d}}$

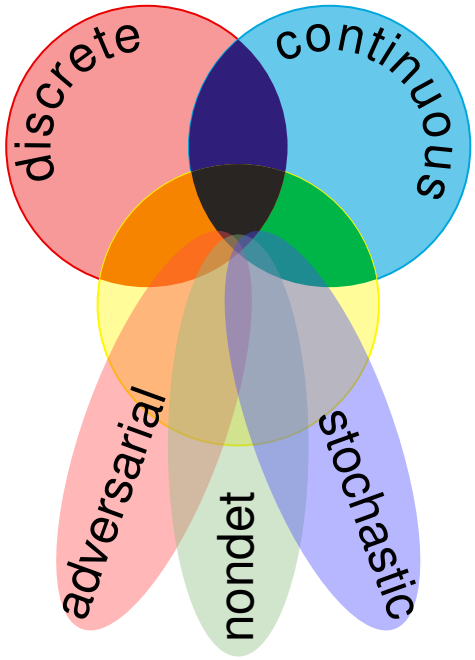





- Differential game logic
- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Operational semantics (informally)

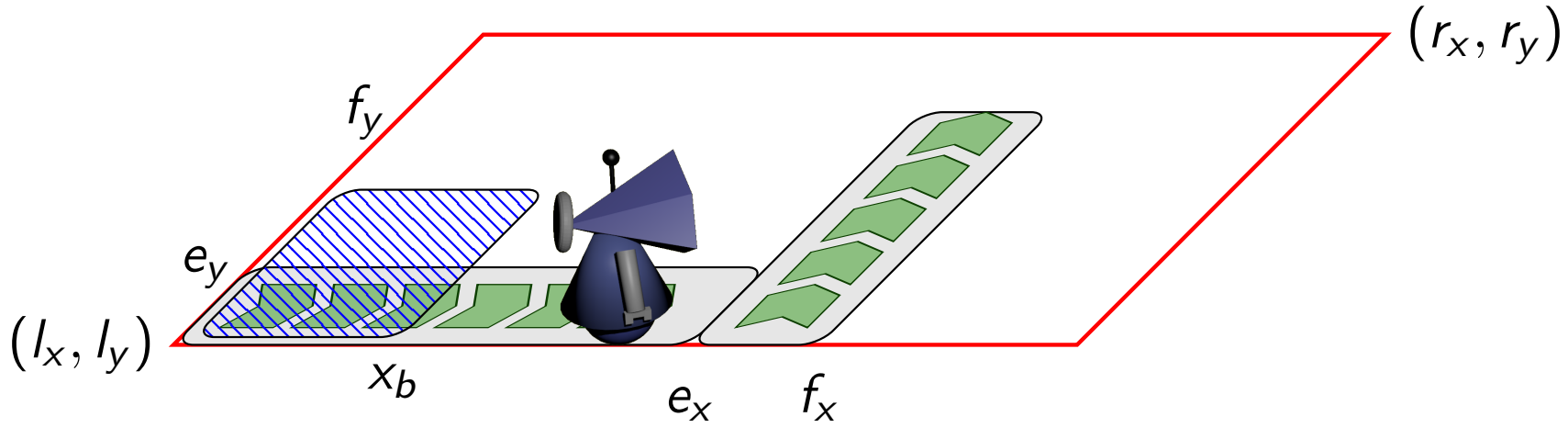
Next chapter

1. Formal semantics



# Outline

# Example: Robot Factory Decentralized Automation




## Model

- $(x, y)$ robot coordinates
- $(v_x, v_y)$ velocities
- conveyor belts may instantaneously increase robot's velocity by $(c_x, c_y)$

## Primary objectives of the robot

- Leave within time $\varepsilon$
- Never leave outer

## Challenges

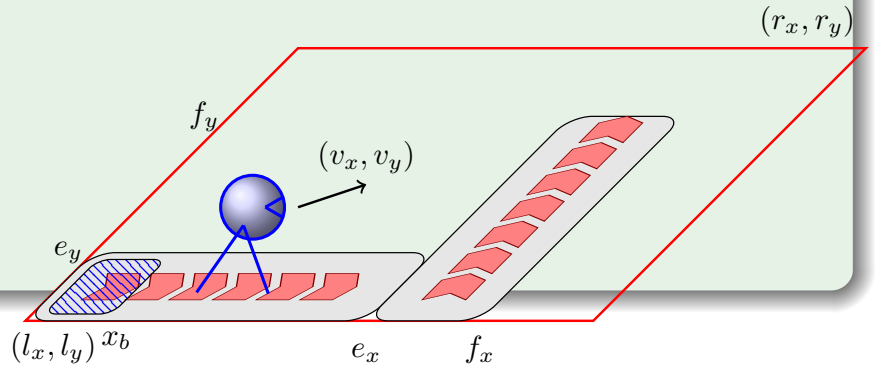
- Distributed, physical environment
- Possibly conflicting secondary objectives

## Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big( ( ?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \quad // \text{ belt}$$

$$\cup (?(e_x \leq x \wedge y \leq f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ )\ ;$$

$$\Big)^*$$



## Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big( ( ?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \quad \text{// belt}$$

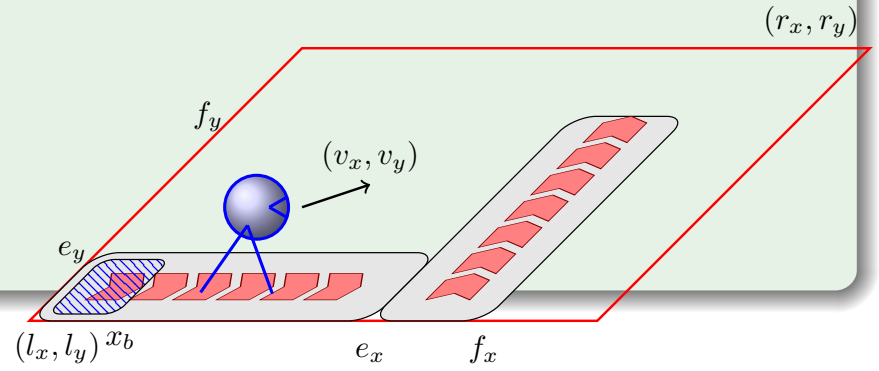
$$\cup (?(e_x \leq x \wedge y \leq f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ );$$

$$(\ a_x := *;\ ?(-A \leq a_x \leq A);$$

$$a_y := *;\ ?(-A \leq a_y \leq A); \quad \text{// “independent” robot acceleration}$$

$$t_s := 0\ )^{\mathrm{d}};$$

$$)^*$$



## Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup\, (?(e_x \leq x \wedge y \leq f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big);$$

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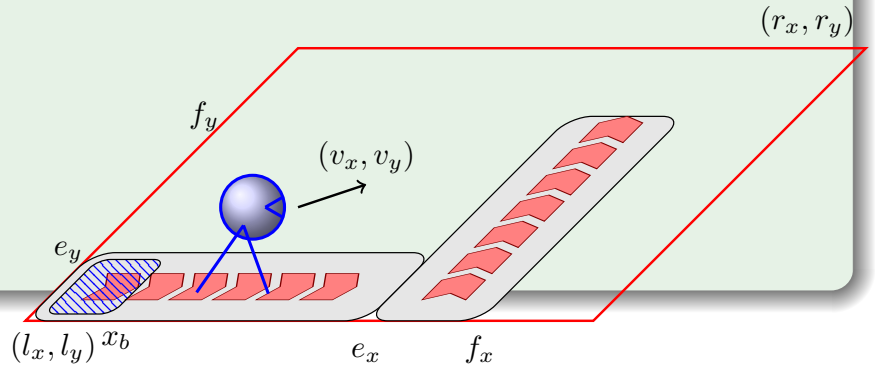
$$t_s := 0\ \big)^{\mathrm{d}};$$

$$(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \leq \varepsilon\ \big);$$

$$\Big)^*$$



# Robot Factory Automation ($RF$)

**Example (Robot-Demon vs. Angel-Factory Environment)**

$$
\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}
$$

$$
\cup\, (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big);
$$

$$
(\, a_x := *;\ ?(-A \le a_x \le A);
$$

$$
a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}
$$

$$
t_s := 0\ )^{\mathrm{d}};
$$

$$
\big(\big(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon\ \big);
$$

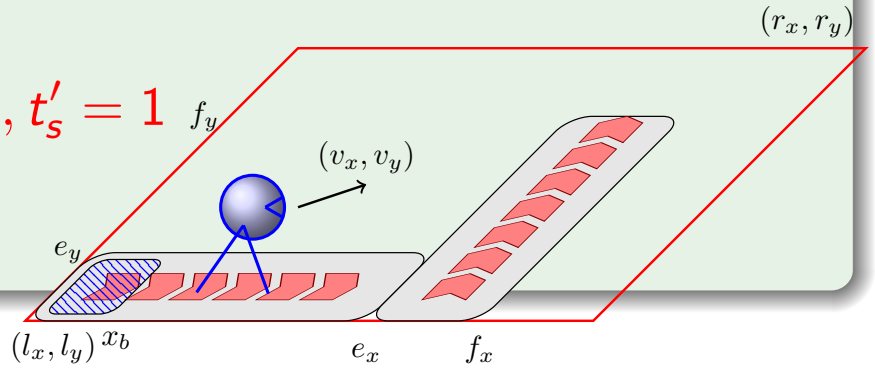
$$
\cap\, (?(a_x v_x \le 0 \wedge a_y v_y \le 0)^{\mathrm{d}}; \qquad \text{// brake}
$$

$$
\text{if } v_x = 0 \text{ then } a_x := 0 \text{ fi}; \qquad \text{// per direction: no time lock}
$$

$$
\text{if } v_y = 0 \text{ then } a_y := 0 \text{ fi};
$$

$$
(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1
$$

$$
\&\ t_s \le \varepsilon \wedge a_x v_x \le 0 \wedge a_y v_y \le 0)))\Big)^*
$$



# Robot Factory Automation ($RF$)

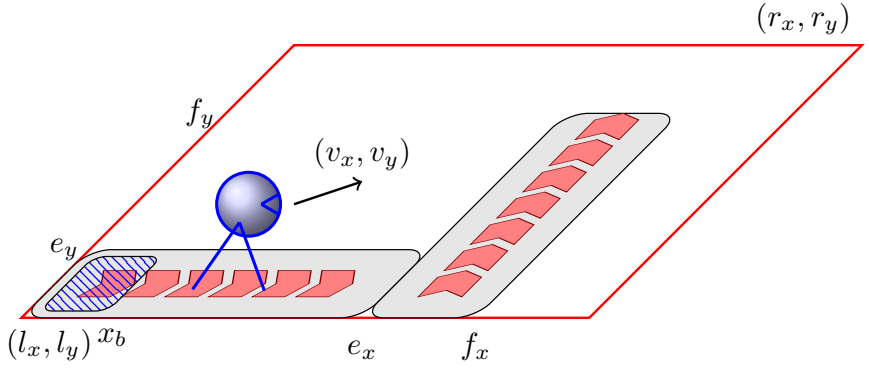
## Proposition (Robot stays in □)

$$\models (x = y = 0 \land v_x = v_y = 0 \land \text{Controllability Assumptions}) \to [RF](x \in [l_x, r_x] \land y \in [l_y, r_y])$$

## Proposition (Stays in □ and leaves ▨ on time)

*$RF|_x$: RF projected to the x-axis*

$$\models (x = 0 \land v_x = 0 \land \text{Controllability Assumptions}) \to [RF|_x](x \in [l_x, r_x] \land (t \geq \varepsilon \to x \geq x_b))$$



André Platzer.
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