15: Winning Strategies & Regions
Logical Foundations of Cyber-Physical Systems

André Platzer
Outline

1 Learning Objectives

2 Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics

3 Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly

4 Summary
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4 Summary
Learning Objectives

Winning Strategies & Regions

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics

adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems
Definition (Hybrid game $\alpha$)

$$
\alpha, \beta ::= x := e | ?Q | x' = f(x) \& Q | \alpha \cup \beta | \alpha; \beta | \alpha^* | \alpha^d
$$

Definition (dGL Formula $P$)

$$
P, Q ::= e \geq \tilde{e} | \neg P | P \land Q | \forall x \ P | \exists x \ P | \langle \alpha \rangle P | [\alpha]P
$$
Differential Game Logic: Syntax

**Definition (Hybrid game \( \alpha \))**

\[
\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \land Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d
\]

**Definition (dGL Formula \( P \))**

\[
P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha] P
\]
Differential Game Logic: Syntax

Definition (Hybrid game $\alpha$)

$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) & Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$

Definition (dGL Formula $P$)

$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha]P$

All Reals

Some Reals
Differential Game Logic: Syntax

Definition (Hybrid game $\alpha$)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula $P$)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha]P$$
**Differential Game Logic: Syntax**

### Definition (Hybrid game $\alpha$)

\[
\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d
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### Definition (dGL Formula $P$)

\[
P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha]P
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Differential Game Logic: Syntax

Definition (Hybrid game $\alpha$)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula $P$)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \; P \mid \exists x \; P \mid \langle \alpha \rangle P \mid [\alpha] P$$

“Angel has Wings $\langle \alpha \rangle$”

All Reals

Some Reals

Angel Wins

Demon Wins
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4 Summary
Definition (dGL Formula $P$)

$[[e_1 \geq e_2]] = \{ \omega \in S : \omega[e_1] \geq \omega[e_2] \}$

$[[\neg P]] = ([[P]])^c$

$[[P \land Q]] = [[P]] \cap [[Q]]$

$[[\langle \alpha \rangle P]] = \varsigma_\alpha([[P]]) \{ \omega : \nu \in [[P]] \text{ for some } \nu \text{ with } (\omega, \nu) \in [[\alpha]] \}$

$[[[\alpha] P]] = \delta_\alpha([[P]])$

$[[\rho(S)]] : Fml \rightarrow \wp(S)$

Only for HPs. No interactive play!
**Definition (dGL Formula P)**

\[
\begin{align*}
[e_1 \geq e_2] &= \{ \omega \in S : \omega[e_1] \geq \omega[e_2] \} \\
[\neg P] &= ([P])^C \\
[P \land Q] &= [P] \cap [Q] \\
[\langle \alpha \rangle P] &= s_\alpha([P]) \quad \{ \omega: \nu \in [P] \text{ for some } \nu \text{ with } (\omega, \nu) \in [\alpha] \} \quad ??? \\
[[\alpha]P] &= \delta_\alpha([P])
\end{align*}
\]

Only for HPs. No interactive play!
Definition (Hybrid game $\alpha$: denotational semantics)

$$s_x := e(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

\[ \varsigma_{x:=e}(X) = \{ \omega \in S : \omega_x[e] \in X \} \]

\[ \varsigma_{x:=e}(X) \rightarrow X \]
Definition (Hybrid game $\alpha$: denotational semantics)

$\mathcal{X} \vdash x' = f(x) \land Q(X) =$

\[
\mathcal{X} \vdash x' = f(x) \land Q(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for an } r \text{ and } \varphi| = x' = f(x) \land Q(X) \}
\]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\forall x' = f(x) \& Q(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for an } r \text{ and } \varphi \models x' = f(x) \wedge Q \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$\mathcal{S}_Q(X) = \mathcal{Q}(X) \cap X$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_Q(X) = \mathbb{[Q]} \cap X$$
Definition (Hybrid game $\alpha$: denotational semantics)

$s_{\alpha \cup \beta}(X) = s_\alpha(X) \cup s_\beta(X)$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\mathcal{S}_{\alpha \cup \beta}(X) = \mathcal{S}_\alpha(X) \cup \mathcal{S}_\beta(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{\alpha;\beta}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{\alpha;\beta}(X) = \varsigma_{\alpha}(\varsigma_{\beta}(X))$$
Definition (Hybrid game $\alpha$: denotational semantics)

$\varsigma_{\alpha^d}(X) =$

![Diagram](image.png)
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{\alpha^d}(X) =$$

[Diagram showing $X$ and $X^c$]
\[ s_{\alpha_d}(X) = \]
Definition (Hybrid game $\alpha$: denotational semantics)

\[ s_{\alpha_d}(X) = (s_{\alpha}(X^C))^C \]
Definition (Hybrid game $\alpha$: denotational semantics)

\[ \delta_{x:=e}(X) = \{ \omega \in S : \omega \in [e] \} \]

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Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{x: e}(X) = \{\omega \in S : \omega_x^{[e]} \in X\}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{x'} = f(x) \& Q(X) = \{ \phi(0) \in S : \phi(r) \in X \text{ for all } r \text{ with } \phi|_x = x' = f(x) \land Q \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

\[
\delta_{x'=f(x)} \land Q(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for all } r \text{ with } \varphi \models x' = f(x) \land Q \}
\]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta?Q(X) = \left[Q\right] \setminus X$$
Definition (Hybrid game \( \alpha \): denotational semantics)

\[
\delta_?=Q(X) = [Q] \cup X
\]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha \cup \beta}(X) = \delta_{\alpha}(X) \cap \delta_{\beta}(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha \cup \beta}(X) = \delta_{\alpha}(X) \cap \delta_{\beta}(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha;\beta}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha;\beta}(X) = \delta_{\alpha}(\delta_{\beta}(X))$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha_d}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$
\delta_{\alpha^d}(X) = (\delta_\alpha(X^C))^C
$$
Differential Game Logic: Denotational Semantics

**Definition (Hybrid game $\alpha$)**

$[\cdot ] : HG \rightarrow (\wp(S) \rightarrow \wp(S))$

$\varsigma x := e(X) = \{ \omega \in S : \omega^{\omega[e]} \in X \}$

$\varsigma x' = f(x)(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \}$

$\varsigma ?Q(X) = [Q] \cap X$

$\varsigma \alpha \cup \beta(X) = \varsigma \alpha(X) \cup \varsigma \beta(X)$

$\varsigma \alpha ; \beta(X) = \varsigma \alpha(\varsigma \beta(X))$

$\varsigma \alpha^*(X) =$

$\varsigma \alpha^d(X) = (\varsigma \alpha(X^c))^c$

---

**Definition (dGL Formula $P$)**

$[\cdot ] : Fml \rightarrow \wp(S)$

$[e_1 \geq e_2] = \{ \omega \in S : \omega^{e_1} \geq \omega^{e_2} \}$

$[\neg P] = (\lbrack P\rbrack)^c$

$[P \land Q] = \lbrack P \rbrack \cap \lbrack Q \rbrack$

$[\langle \alpha \rangle P] = \varsigma \alpha(\lbrack P \rbrack)$

$[[\alpha]P] = \delta \alpha(\lbrack P \rbrack)$
Lemma (Monotonicity)

\[ s_\alpha(X) \subseteq s_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y \]
Lemma (Monotonicity)

\[ s_\alpha(X) \subseteq s_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y \]

Definition (Hybrid game \( \alpha \))

\[
\begin{align*}
\varsigma_x : e(X) &= \{ \omega \in S : \omega_x^e \in X \} \\
\varsigma_x' : f(x)(X) &= \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \} \\
\varsigma?Q(X) &= [Q] \cap X \\
\varsigma_\alpha \cup \beta(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_\alpha ; \beta(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_\alpha^*(X) &= \varsigma_\alpha(X^c) \\
\varsigma_\alpha^d(X) &= (\varsigma_\alpha(X^c))^c
\end{align*}
\]
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4. Summary
\[ \langle (x := 0 \cap x := 1)^* \rangle x = 0 \]

\[ \text{wfd} \leadsto \text{false unless } x = 0 \]
Definition (Hybrid game $\alpha$)

$s_{\alpha^*}(X) =$
Definition (Hybrid game $\alpha$)

\[ s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X) \]

\[ [\alpha^*] = \bigcup_{n \in \mathbb{N}} [\alpha^n] \quad \text{where} \quad \alpha^{n+1} \equiv \alpha^n; \quad \alpha^0 \equiv \text{true} \quad \text{for HP } \alpha \]
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$

\[ x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle x \neq 1 \]
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$

Defining $x = 1 \land a = 1 \rightarrow \langle((x := a; a := 0) \cap x := 0)^*\rangle x \neq 1$
Definition (Hybrid game $\alpha$)

\[ s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X) \]

too hard to predict all iterations!

\[ x = 1 \land a = 1 \rightarrow \langle((x := a; a := 0) \cap x := 0)^*\rangle x \neq 1 \]
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

Since \( s_\alpha(Y) \) is just one more round away from \( Y \).
Semantics of Repetition

**Definition (Hybrid game \( \alpha \))**

\[
\mathcal{s}_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{s}_{\alpha}^n(X)
\]

\[
\mathcal{s}_{\alpha}^0(X) \overset{\text{def}}{=} X
\]

\[
\mathcal{s}_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \mathcal{s}_{\alpha}(\mathcal{s}_{\alpha}^\kappa(X))
\]
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\mathcal{s}_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \mathcal{s}_\alpha^n(X)$$

$$\mathcal{s}_\alpha^0(X) \overset{\text{def}}{=} X$$

$$\mathcal{s}_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \mathcal{s}_\alpha(\mathcal{s}_\alpha^\kappa(X))$$

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Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n(X)$$

$$\varsigma_\alpha^0(X) \overset{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$
Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n(X)$$

$$\varsigma_\alpha^0(X) \overset{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$
Definition (Hybrid game $\alpha$)

$$\mathcal{s}_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \mathcal{s}_\alpha^n(X)$$

$n$ outside the game so Demon won't know

$$\mathcal{s}_\alpha^0(X) \overset{\text{def}}{=} X$$
$$\mathcal{s}_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \mathcal{s}_\alpha(\mathcal{s}_\alpha^\kappa(X))$$
### Definition (Hybrid game $\alpha$)

\[ s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha}^n(X) \]

\[ s_{\alpha}^0(X) \overset{\text{def}}{=} X \]

\[ s_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^\kappa(X)) \]

### Example

\[ \langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1) \]
**Definition (Hybrid game $\alpha$)**

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha}^n(X)$$

$$s_{\alpha}^0(X) \overset{\text{def}}{=} X$$

$$s_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^\kappa(X))$$

**Example**

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1) \quad s_{\alpha}^n([0, 1)) = [0, n+1) \neq \mathbb{R}$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s^n_{\alpha}(X)$$

$\omega$-semantics

- $s^0_{\alpha}(X) \overset{\text{def}}{=} X$
- $s^{\kappa+1}_{\alpha}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s^\kappa_{\alpha}(X))$
- $s^\lambda_{\alpha}(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s^\kappa_{\alpha}(X) \quad \lambda \neq 0 \text{ a limit ordinal}$

Example

$$\langle (x := 1; \ x' = 1^{d} \cup x := x - 1)^* \rangle (0 \leq x < 1) \quad s^0_{\alpha}([0, 1)) = [0, n+1) \neq \mathbb{R}$$

$$s^\omega_{\alpha}([0, 1)) = \bigcup_{n \in \mathbb{N}} s^n_{\alpha}([0, 1)) = [0, \infty) \neq \mathbb{R}$$
Semantics of Repetition

**Definition (Hybrid game $\alpha$)**

$$\varsigma_\alpha^* (X) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n (X)$$

$$\varsigma_\alpha^0 (X) \overset{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1} (X) \overset{\text{def}}{=} X \cup \varsigma_\alpha (\varsigma_\alpha^\kappa (X))$$

$$\varsigma_\alpha^\lambda (X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} \varsigma_\alpha^\kappa (X) \quad \lambda \neq 0 \text{ a limit ordinal}$$

**Example**

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$$

$$\varsigma_\alpha^\omega ([0, 1)) = \mathbb{R}$$

$$\varsigma_\alpha^\omega ([0, 1)) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n ([0, 1)) = [0, \infty) \neq \mathbb{R}$$
Definition (Hybrid game $\alpha$)

$$s_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} s_\alpha^n(X)$$

$$s_\alpha^0(X) \overset{\text{def}}{=} X$$

$$s_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^\kappa(X))$$

$$s_\alpha^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s_\alpha^\kappa(X)$$

$\lambda \neq 0$ a limit ordinal
Definition (Hybrid game $\alpha$)

$\varsigma_{\alpha}^\ast(X) = \bigcup_{n \in \mathbb{N}} \varsigma_{\alpha}^n(X)$

missing winning strategies

$\varsigma_{\alpha}^0(X) \overset{\text{def}}{=} X$

$\varsigma_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_{\alpha}(\varsigma_{\alpha}^{\kappa}(X))$

$\varsigma_{\alpha}^{\lambda}(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} \varsigma_{\alpha}^{\kappa}(X)$ \hspace{1cm} $\lambda \neq 0$ a limit ordinal

$\varsigma_{\alpha}^{\omega+1}(X) \quad \varsigma_{\alpha}^{\omega}(X) \quad \cdots \quad \varsigma_{\alpha}^{3}(X) \quad \varsigma_{\alpha}^{2}(X) \quad \varsigma_{\alpha}(X) \quad X$
Theorem

Hybrid game closure ordinal $\geq \omega^\omega$
Expedition: Ordinal Arithmetic

\[ \begin{align*}
\iota + 0 &= \iota \\
\iota + (\kappa + 1) &= (\iota + \kappa) + 1 \quad \text{successor } \kappa + 1 \\
\iota + \lambda &= \bigsqcup_{\kappa < \lambda} \iota + \kappa \quad \text{limit } \lambda \\
\iota \cdot 0 &= 0 \\
\iota \cdot (\kappa + 1) &= (\iota \cdot \kappa) + \iota \quad \text{successor } \kappa + 1 \\
\iota \cdot \lambda &= \bigsqcup_{\kappa < \lambda} \iota \cdot \kappa \quad \text{limit } \lambda \\
\iota^0 &= 1 \\
\iota^{\kappa + 1} &= \iota^\kappa \cdot \iota \quad \text{successor } \kappa + 1 \\
\iota^\lambda &= \bigsqcup_{\kappa < \lambda} \iota^\kappa \quad \text{limit } \lambda \\
2 \cdot \omega &= 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4
\end{align*} \]
Definition (Hybrid game $\alpha$)

$$s_{\alpha}^*(X) = \bigcup_{\kappa < \infty} s_{\alpha}^\kappa(X)$$

$$s_{\alpha}^0(X) \overset{\text{def}}{=} X$$

$$s_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^\kappa(X))$$

$$s_{\alpha}^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s_{\alpha}^\kappa(X) \quad \lambda \neq 0 \text{ a limit ordinal}$$
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_{\alpha}^{\kappa}(X)$$
Definition (Hybrid game $\alpha$)

$$\varsigma_{\alpha}^*(X) = \bigcup_{\kappa < \infty} \varsigma_{\alpha}^\kappa(X)$$
Definition (Hybrid game $\alpha$)

$$\ς^{\alpha*}(X) = \bigcup_{\kappa < \infty} \ς^{\kappa}_{\alpha}(X)$$
Definition (Hybrid game $\alpha$)

$$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa < \infty} \varsigma_{\alpha}^{\kappa}(X)$$
Definition (Hybrid game $\alpha$)

$$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa<\infty} \varsigma_{\alpha}^{\kappa}(X)$$
Definition (Hybrid game $\alpha$)

\[ s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_{\alpha}^{\kappa}(X) \]

requires transfinite patience
The advantages of implicit definition over construction are roughly those of theft over honest toil.

— Bertrand Russell
$Y \subseteq s_\alpha^*(X)$ then $s_\alpha(Y) \subseteq s_\alpha^*(X)$

Since $s_\alpha(Y)$ is just one more round away from $Y$. 

Note (+1 argument)
Note (+1 argument)

\[ Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X) \]

\[ Z \overset{\text{def}}{=} \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Z) \subseteq \varsigma_{\alpha^*}(X) = Z \]
$Y \subseteq s_{\alpha^*}(X) \text{ then } s_{\alpha}(Y) \subseteq s_{\alpha^*}(X)$

$Z \overset{\text{def}}{=} s_{\alpha^*}(X) \text{ then } s_{\alpha}(Z) \subseteq s_{\alpha^*}(X) = Z$

- Which $Z$ with $s_{\alpha}(Z) \subseteq Z$ is the right one?
- Are there multiple such $Z$?
- Does such a $Z$ exist?
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
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- No wait, dual tests: \( s_?Q^d(\emptyset) = s_?Q(\emptyset^C)^C = ([Q] \cap S)^C = [Q]^C \not\subseteq \emptyset \)
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( s_?Q^d(\emptyset) = s_?Q(\emptyset^C)^C = ([Q] \cap S)^C = [Q]^C \not\subseteq \emptyset \)
- Then: \( s_?Q^d([\neg Q]) = s_?Q([\neg Q]^C)^C = ([Q] \cap [Q])^C = [\neg Q] \subseteq [\neg Q] \)
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
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- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( s_?Q_d(\emptyset) = s_?Q(\emptyset^C)^C = ([Q] \cap S)^C = [Q]^C \not\subseteq \emptyset \)
- Then: \( s_?Q_d([\neg Q]) = s_?Q([\neg Q]^C)^C = ([Q] \cap [Q])^C = [\neg Q] \subseteq [\neg Q] \)
- Still too small: \( X \subseteq Z \) since Angel may decide not to repeat
Definition (Pre-fixpoint)

\[ X \cup \mathcal{S}_\alpha(Z) \subseteq Z \]

for the winning region \( Z \overset{\text{def}}{=} \mathcal{S}_{\alpha^*}(X) \)
Fixpoints and Pre-Fixpoints

Definition (Pre-fixpoint)

\[ X \cup s_\alpha(Z) \subseteq Z \]

for the winning region \( Z \overset{\text{def}}{=} s_\alpha^*(X) \)

- Which \( Z \) is the right one?
- Are there multiple such \( Z \)? Does such a \( Z \) exist?
Fixpoints and Pre-Fixpoints

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for the winning region \( Z \defeq \mathcal{S}_\alpha^*(X) \)

- Which \( Z \) is the right one?
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- Existence: \( Z = S \)
Definition (Pre-fixpoint)

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for the winning region \( Z \overset{\text{def}}{=} \mathcal{S}_\alpha^*(X) \)

- Which \( Z \) is the right one?
- Are there multiple such \( Z \)? Does such a \( Z \) exist?
- Existence: \( Z = S \) but that’s too big and independent of \( \alpha \)
Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

\[ X \cup \varsigma_\alpha(Y) \subseteq Y \]
\[ X \cup \varsigma_\alpha(Z) \subseteq Z \]

are pre-fixpoints, then

\[ Y \cap Z \text{ is a smaller pre-fixpoint.} \]
Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

\[
X \cup \varsigma_\alpha(Y) \subseteq Y \\
X \cup \varsigma_\alpha(Z) \subseteq Z
\]

are pre-fixpoints, then \(Y \cap Z\) is a smaller pre-fixpoint.

Proof. 

\[
X \cup \varsigma_\alpha(Y \cap Z) \subseteq X \cup \varsigma_\alpha(Y) \cap \varsigma_\alpha(Z)
\]

above \(\subseteq Y \cap Z\)

Even: The intersection of any family of pre-fixpoints is a pre-fixpoint!

So: repetition semantics is the smallest pre-fixpoint (well-founded)

André Platzer (CMU)
Lemma (Intersection closure)

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are pre-fixpoints, then \( Y \cap Z \) is a smaller pre-fixpoint.

Proof.

\[ X \cup \mathcal{s}_\alpha(Y \cap Z) \mathsf{mon} \subseteq X \cup (\mathcal{s}_\alpha(Y) \cap \mathcal{s}_\alpha(Z)) \mathsf{above} \subseteq Y \cap Z \]
Lemma (Intersection closure)

\[
\begin{align*}
X \cup s_\alpha(Y) & \subseteq Y \\
X \cup s_\alpha(Z) & \subseteq Z
\end{align*}
\]

are pre-fixpoints, then \(Y \cap Z\) is a smaller pre-fixpoint.

Proof.

\[
\begin{align*}
X \cup s_\alpha(Y \cap Z) & \subseteq X \cup (s_\alpha(Y) \cap s_\alpha(Z)) \subseteq Y \cap Z
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Even: The intersection of any family of pre-fixpoints is a pre-fixpoint!
So: repetition semantics is the smallest pre-fixpoint (well-founded)
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup \varsigma_\alpha(Z) \subseteq Z \}$$

\begin{align*}
\varsigma_\alpha^*(X) = X \\
\varsigma_\alpha^*(X) \subseteq \varsigma_\alpha^{\infty}(X) \\
\varsigma_\alpha^*(X) \subseteq \varsigma_\alpha^{3}(X) \\
\vdots \\
\varsigma_\alpha^*(X) \subseteq \varsigma_\alpha^{2}(X) \\
\varsigma_\alpha^*(X) \subseteq \varsigma_\alpha(X) \\
\varsigma_\alpha^{\infty}(X) \subseteq \varsigma_\alpha^*(X) \\
\end{align*}

$$X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$$

$\varsigma_\alpha^*(X)$ intersection of solutions
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z \}$$

$$s_\alpha^* \left( s_\alpha^*(X) \right) \setminus s_\alpha^*(X) = \emptyset$$

$Z \overset{\text{def}}{=} X \cup s_\alpha(\left( s_\alpha^*(X) \right)) \subseteq s_\alpha^*(X)$

$s_\alpha(Z) \subseteq s_\alpha(s_\alpha^*(X))$ by mon since $Z \subseteq s_\alpha^*(X)$
Definition (Hybrid game $\alpha$)

$$s_\alpha^*(X) = \bigcap\{Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z\}$$

**Z** $\overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^*(X)) \subseteq s_\alpha^*(X)$

$X \cup s_\alpha(Z) \subseteq X \cup s_\alpha(s_\alpha^*(X)) = Z$ by mon since $Z \subseteq s_\alpha^*(X)$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha}^*(X) = \bigcap \{Z \subseteq S : X \cup s_{\alpha}(Z) \subseteq Z\}$$

$Z \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^*(X)) \subseteq s_{\alpha}^*(X)$

$X \cup s_{\alpha}(Z) \subseteq X \cup s_{\alpha}(s_{\alpha}^*(X)) = Z$

$s_{\alpha}^*(X) \subseteq X \cup s_{\alpha}(s_{\alpha}^*(X)) = Z$

$s_{\alpha}^*(X)$ intersection of solutions by mon since $Z \subseteq s_{\alpha}^*(X)$

$s_{\alpha}^*(X)$ smallest such $Z$
Semantics of Repetition

**Definition (Hybrid game $\alpha$)**

$$s_{\alpha^*}(X) = \bigcap\{Z \subseteq S : X \cup s_{\alpha}(Z) \subseteq Z\}$$

$Z \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha^*}(X)) \subseteq s_{\alpha^*}(X)$

$X \cup s_{\alpha}(Z) \subseteq X \cup s_{\alpha}(s_{\alpha^*}(X)) = Z$ by mon since $Z \subseteq s_{\alpha^*}(X)$

$s_{\alpha^*}(X) \subseteq X \cup s_{\alpha}(s_{\alpha^*}(X)) = Z$ since $s_{\alpha^*}(X)$ smallest such $Z$
Definition (Hybrid game $\alpha$)

\[ s_{\alpha^*}(X) = \bigcap\{Z \subseteq S : X \cup s_{\alpha}(Z) \subseteq Z\} \]

\[ Z \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha^*}(X)) \subseteq s_{\alpha^*}(X) \]

\[ X \cup s_{\alpha}(Z) \subseteq X \cup s_{\alpha}(s_{\alpha^*}(X)) = Z \] by mon since $Z \subseteq s_{\alpha^*}(X)$

\[ s_{\alpha^*}(X) = X \cup s_{\alpha}(s_{\alpha^*}(X)) = Z \] since $s_{\alpha^*}(X)$ smallest such $Z$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$\varsigma_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup \varsigma_\alpha(Z) = Z \}$

$\varsigma_\alpha^*(X)$ intersection of solutions

$X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z$ by mon since $Z \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha^*(X) = X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z$ since $\varsigma_\alpha^*(X)$ smallest such $Z$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$\varsigma_\alpha^*(X) = \bigcap\{Z \subseteq S : X \cup \varsigma_\alpha(Z) = Z \} = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ by Knaster-Tarski

$Z \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$  
$X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z$ by mon since $Z \subseteq \varsigma_\alpha^*(X)$  
$\varsigma_\alpha^*(X) = X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z$ since $\varsigma_\alpha^*(X)$ smallest such $Z$
Outline

1 Learning Objectives

2 Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics

3 Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly

4 Summary
Definition (Hybrid game $\alpha$) 

$[\cdot] : \text{HG} \rightarrow (\wp(S) \rightarrow \wp(S))$

$\varsigma_x := e(X) = \{ \omega \in S : \omega^e_x \in X \}$

$\varsigma_x' = f(x)(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \}$

$\varsigma？Q(X) = [Q] \cap X$

$\varsigma_\alpha \cup \beta(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X)$

$\varsigma_\alpha ; \beta(X) = \varsigma_\alpha(\varsigma_\beta(X))$

$\varsigma_\alpha^*(X) = \bigcup_{\kappa < \infty} \varsigma_\alpha^\kappa(X)$

$\varsigma_\alpha^d(X) = (\varsigma_\alpha(X^C))^C$

Definition (dGL Formula $P$) 

$[\cdot] : \text{Fml} \rightarrow \wp(S)$

$[e_1 \geq e_2] = \{ \omega \in S : \omega[e_1] \geq \omega[e_2] \}$

$[\neg P] = ([P])^C$

$[P \land Q] = [P] \cap [Q]$ 

$[\langle \alpha \rangle P] = \varsigma_\alpha([P])$

$[[\alpha]P] = \delta_\alpha([P])$
### Definition (Hybrid game $\alpha$)

$$\begin{align*}
\s_x &:= e(X) = \{ \omega \in S : \omega_x^{\llbracket e \rrbracket} \in X \} \\
\s_{x'} &:= f(x)(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \} \\
\s?Q(X) &:= \llbracket Q \rrbracket \cap X \\
\s_{\alpha \cup \beta}(X) &:= \s_{\alpha}(X) \cup \s_{\beta}(X) \\
\s_{\alpha ; \beta}(X) &:= \s_{\alpha}(\s_{\beta}(X)) \\
\s_{\alpha^*}(X) &:= \bigcup_{\kappa < \infty} \s_{\alpha}^{\kappa}(X) = \bigcap \{ Z \subseteq S : X \cup \s_{\alpha}(Z) \subseteq Z \} \\
\s_{\alpha^d}(X) &:= (\s_{\alpha}(X^C))^C
\end{align*}$$

### Definition (dGL Formula $P$)

$$\begin{align*}
\llbracket e_1 \geq e_2 \rrbracket &:= \{ \omega \in S : \omega^{\llbracket e_1 \rrbracket} \geq \omega^{\llbracket e_2 \rrbracket} \} \\
\llbracket \neg P \rrbracket &:= (\llbracket P \rrbracket)^C \\
\llbracket P \land Q \rrbracket &:= \llbracket P \rrbracket \cap \llbracket Q \rrbracket \\
\llbracket \langle \alpha \rangle P \rrbracket &:= \s_{\alpha}(\llbracket P \rrbracket) \\
\llbracket [\alpha] P \rrbracket &:= \delta_{\alpha}(\llbracket P \rrbracket)
\end{align*}$$
Definition (Hybrid game $\alpha$) \([\cdot] : HG \rightarrow (\wp(S) \rightarrow \wp(S))\)

\[
\begin{align*}
\varsigma_x := e(X) &= \{\omega \in S : \omega^{[e]}_x \in X\} \\
\varsigma_{x'} = f(x)(X) &= \{\varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x)\} \\
\varsigma?Q(X) &= \Box [Q] \cap X \\
\varsigma_{\alpha \cup \beta}(X) &= \varsigma_{\alpha}(X) \cup \varsigma_{\beta}(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_{\alpha}(\varsigma_{\beta}(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq S : X \cup \varsigma_{\alpha}(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_{\alpha}(X^c))^c
\end{align*}
\]

Definition (dGL Formula $P$) \([\cdot] : Fml \rightarrow \wp(S)\)

\[
\begin{align*}
[\mathbf{e}_1 \geq \mathbf{e}_2] &= \{\omega \in S : \omega^{[\mathbf{e}_1]} \geq \omega^{[\mathbf{e}_2]}\} \\
[\neg P] &= (\Box [P])^c \\
[P \land Q] &= [P] \cap [Q] \\
[\langle \alpha \rangle P] &= \varsigma_{\alpha}(\Box [P]) \\
[\lceil \alpha \rceil P] &= \delta_{\alpha}(\Box [P])
\end{align*}
\]
Summary

- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint

Next chapter

1. Axiomatics
2. How to win and prove hybrid games
André Platzer.
*Logical Foundations of Cyber-Physical Systems.*
URL: http://www.springer.com/978-3-319-63587-3,
doi:10.1007/978-3-319-63588-0.

André Platzer.
Differential game logic.