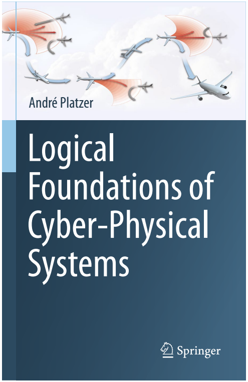
15: Winning Strategies & Regions

Logical Foundations of Cyber-Physical Systems



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Computer Science Department

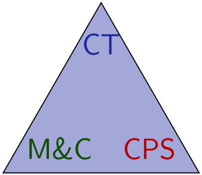
Outline

Outline

1. Learning Objectives
2. Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics
3. Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly
4. Summary

Learning Objectives

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics



adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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All Reals | Some Reals

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Differential Game Logic: Syntax

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"Angel has Wings $\langle \alpha \rangle$"

All Reals | Some Reals | Angel Wins | Demon Wins

Outline

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[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^{\complement} \\
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[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Only for HPs. No interactive play!

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Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

$\varsigma_{x:=e}(X) =$

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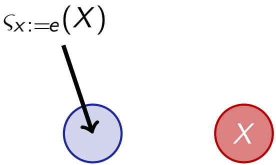
$$\varsigma_{x:=e}(X) = \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\}$$



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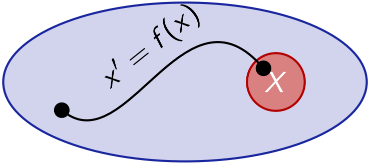

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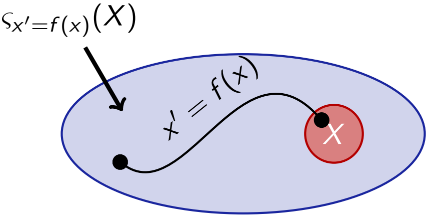

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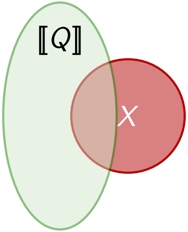

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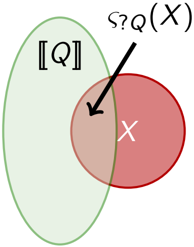

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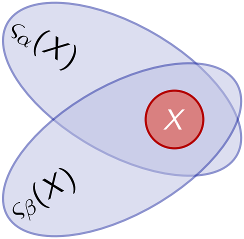

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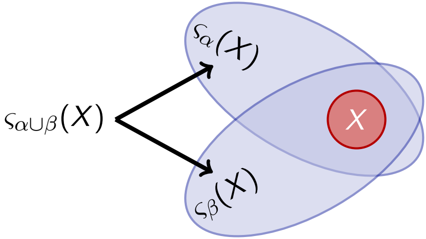

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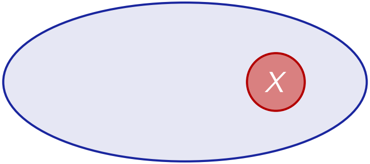

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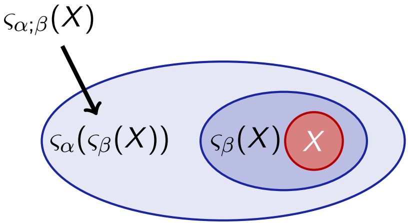

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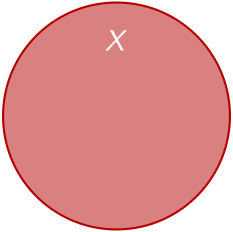

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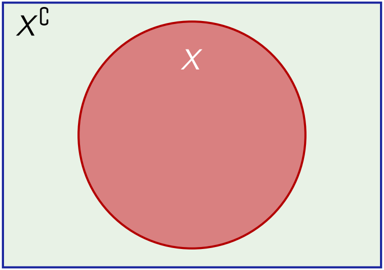

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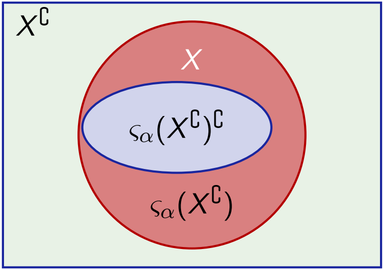

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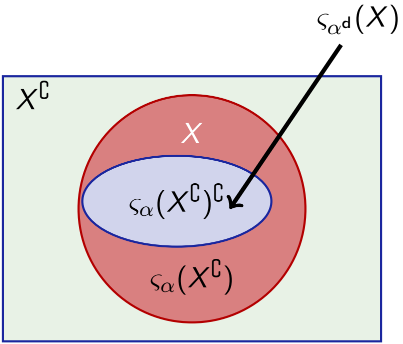

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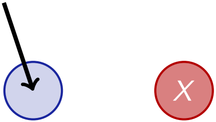

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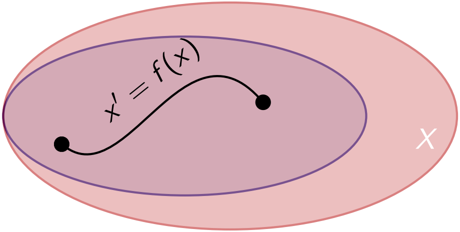

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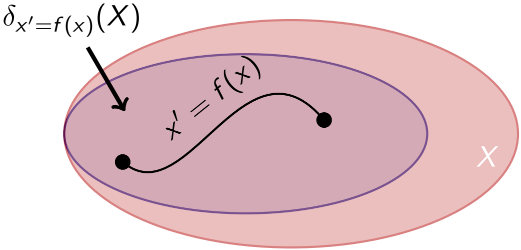

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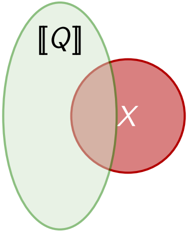

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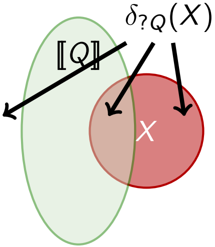
$$\delta_{?Q}(X) = [\![Q]\!]^\complement \cup X$$



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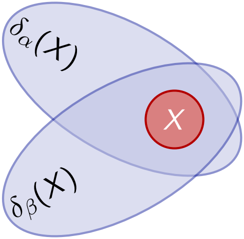

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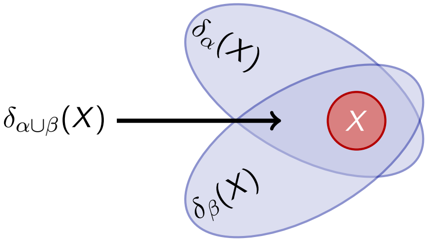

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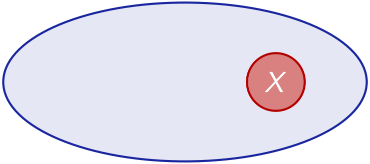

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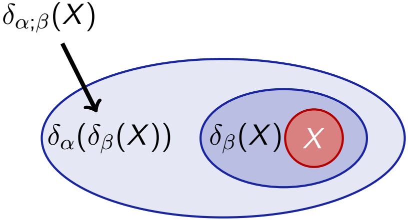

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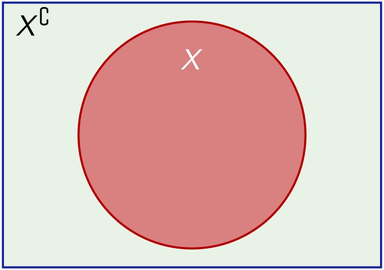

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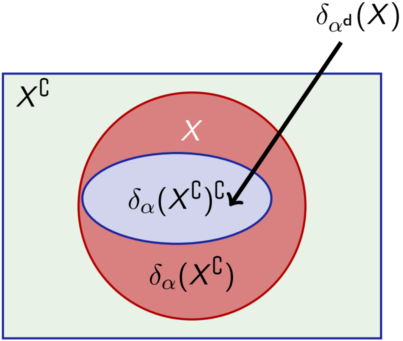

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[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
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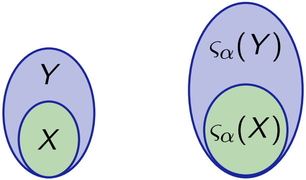
Monotonicity

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$



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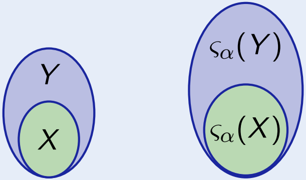
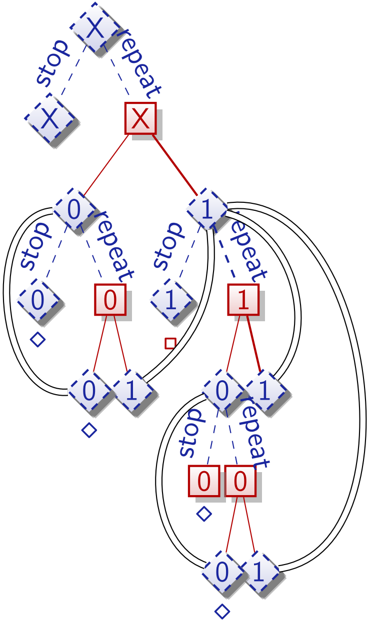
\end{aligned}$$



Outline

$$\langle(x := 0 \cap x := 1)^*\rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) =$

Semantics of Repetition

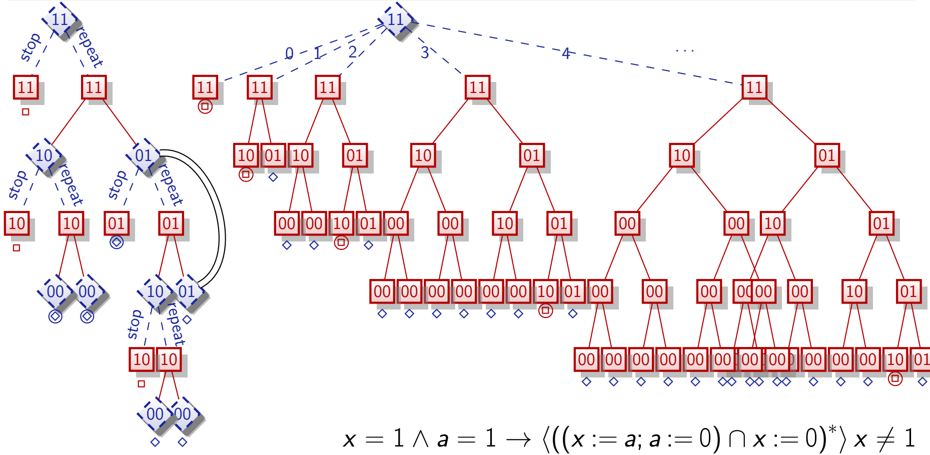
Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$

$[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!]$ where $\alpha^{n+1} \equiv \alpha^n;\alpha$ $\alpha^0 \equiv ?true$ for HP α

Definition (Hybrid game α)

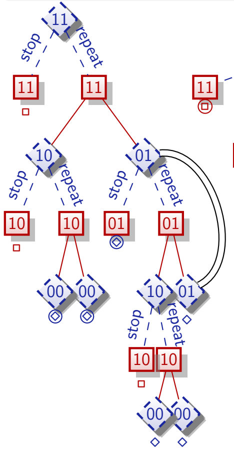
$$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$$

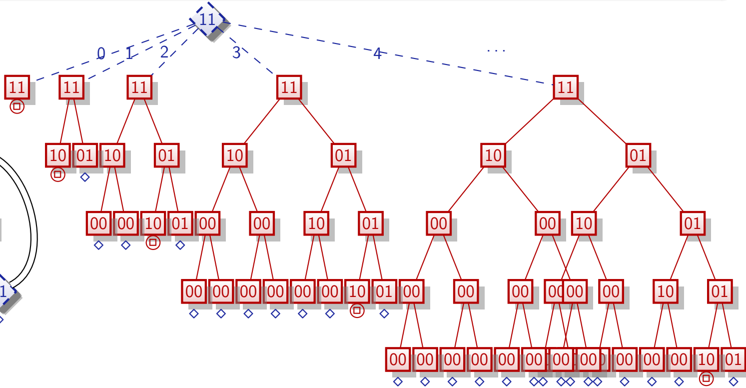


$$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle\, x \neq 1$$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ advance notice semantics?

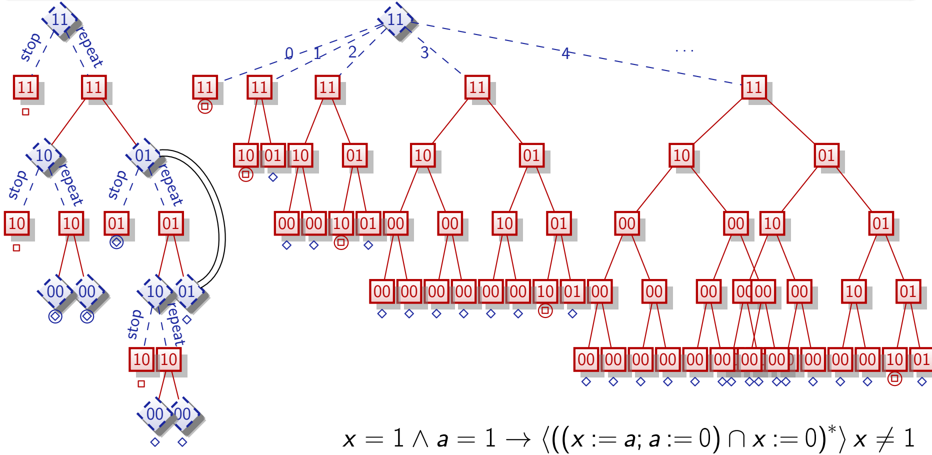




$$x = 1 \land a = 1 \rightarrow \langle((x := a; a := 0) \cap x := 0)^*\rangle\, x \neq 1$$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ too hard to predict all iterations!

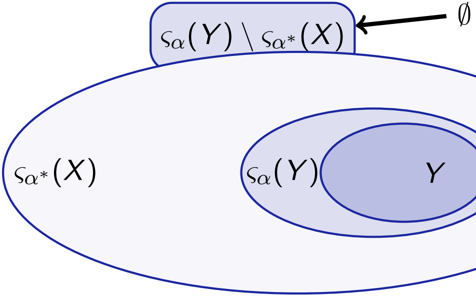


$$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle\, x \neq 1$$

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one more round away from Y.



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$$\varsigma_\alpha^0(X) \stackrel{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$

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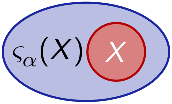

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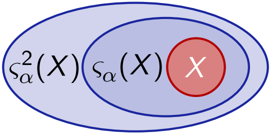

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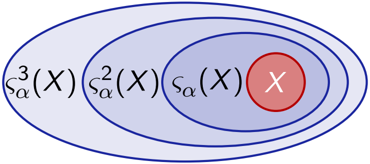

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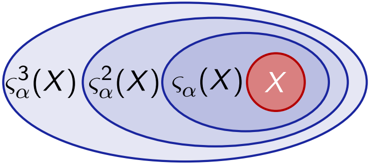

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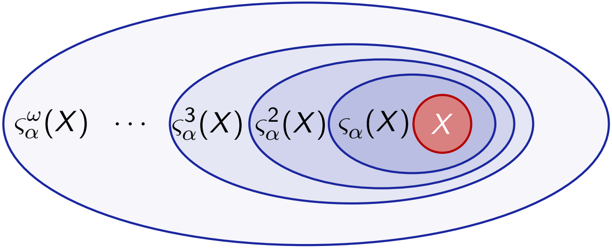

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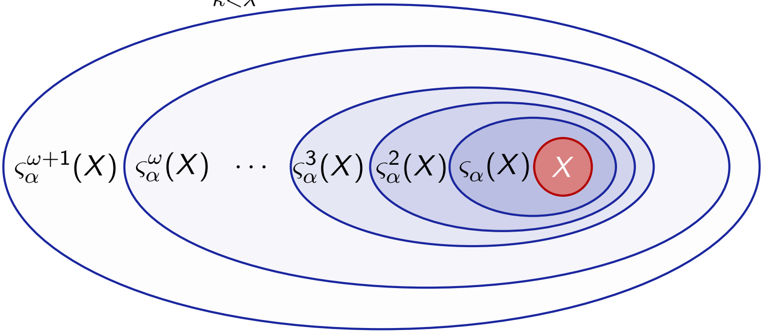
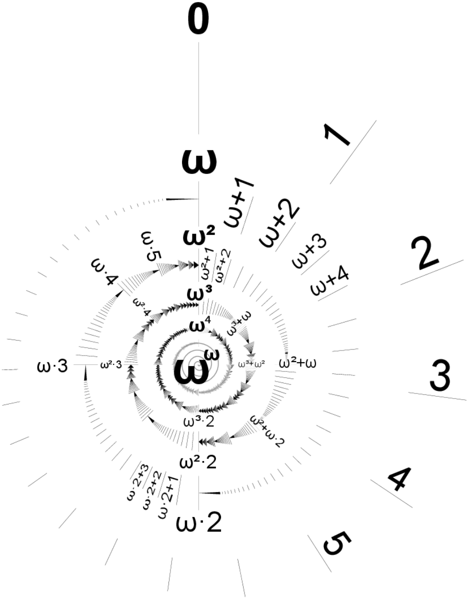


Theorem

Hybrid game closure ordinal $>\omega^\omega$



Expedition: Ordinal Arithmetic

$$\iota + 0 = \iota$$

$$\iota + (\kappa{+}1) = (\iota + \kappa) + 1 \qquad \text{successor } \kappa{+}1$$

$$\iota + \lambda = \bigsqcup_{\kappa<\lambda} \iota + \kappa \qquad \text{limit } \lambda$$

$$\iota \cdot 0 = 0$$

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$$\iota^0 = 1$$

$$\iota^{\kappa+1} = \iota^\kappa \cdot \iota \qquad \text{successor } \kappa{+}1$$

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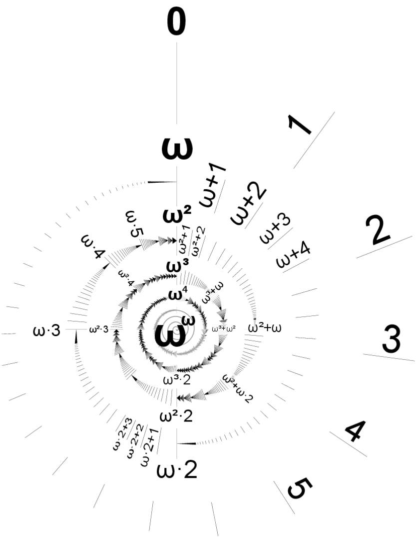
$$2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4$$



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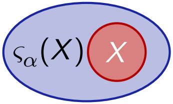

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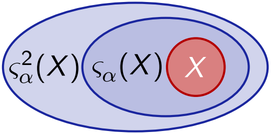

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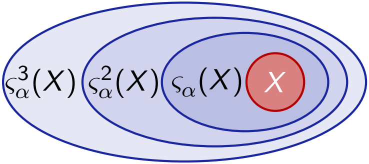

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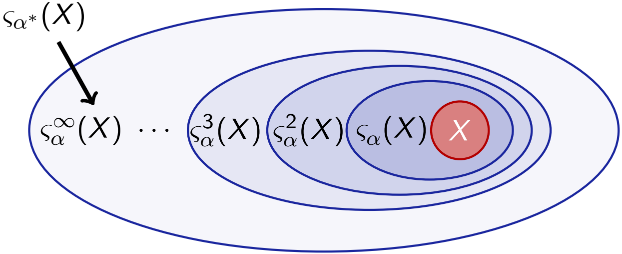
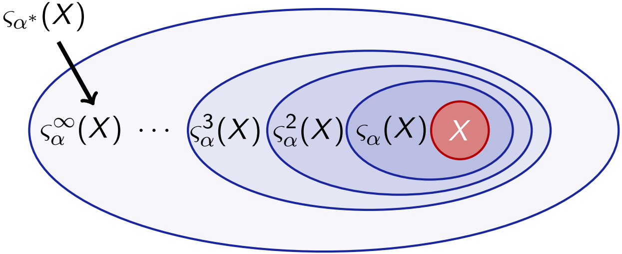


Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ requires transfinite patience



Implicit Definitions

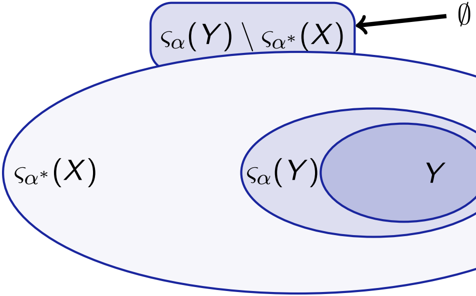
The advantages of implicit definition over construction are roughly those of theft over honest toil.

— Bertrand Russell

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one more round away from Y.



$\mathcal{AP}$ +1 Argument

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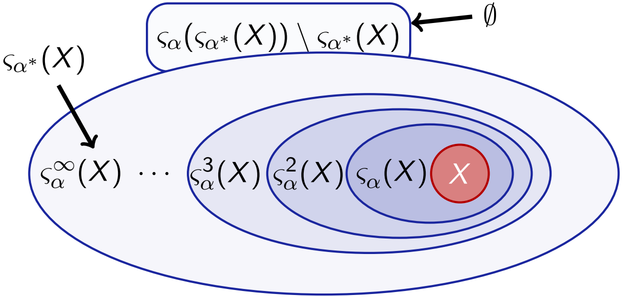
- Still too small: $X \subseteq Z$ since Angel may decide not to repeat

Definition (Pre-fixpoint)

$X \cup \varsigma_\alpha(Z) \subseteq Z$ for the winning region $Z \stackrel{\text{def}}{=} \varsigma_{\alpha^*}(X)$



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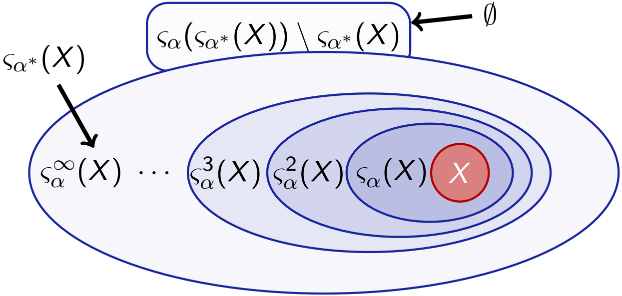

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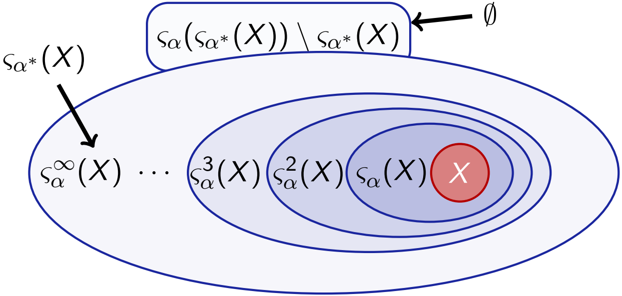

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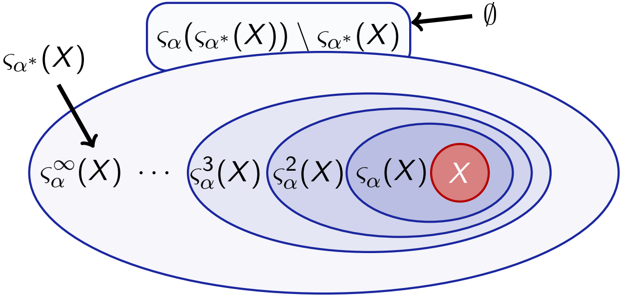

- Which Z is the right one?
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- Existence: $Z = \mathcal{S}$ but that's too big and independent of α

Comparing (Pre-)Fixpoints

Lemma ( )

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
$$X \cup \varsigma_\alpha(Z) \subseteq Z$$

are pre-fixpoints, then

Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
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□

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Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!
So: repetition semantics is the smallest pre-fixpoint (well-founded)

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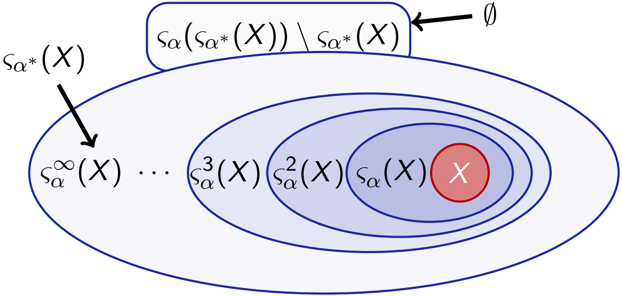

$X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$

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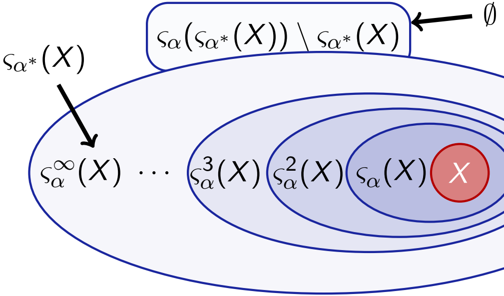

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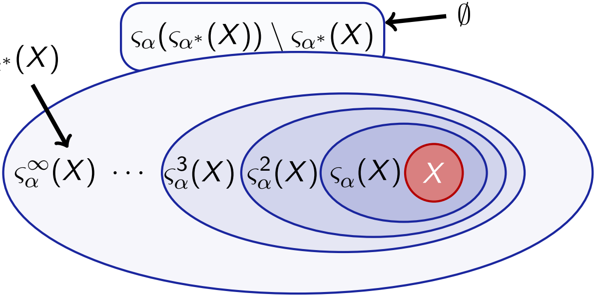

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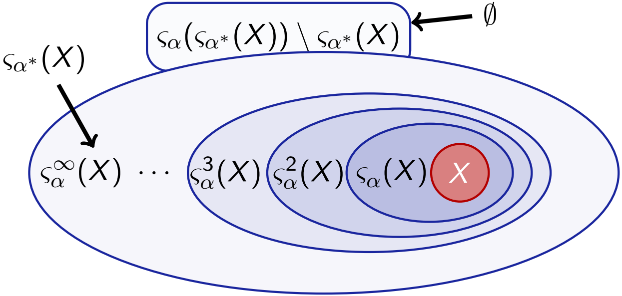

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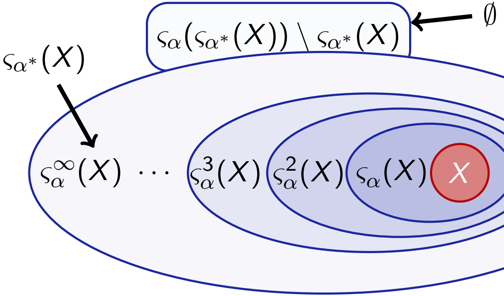

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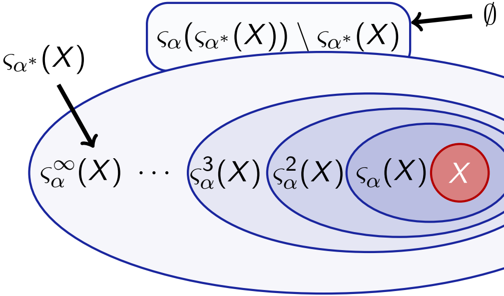

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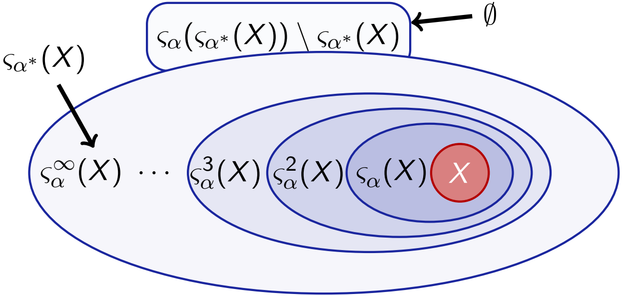


$Z \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$ — $\varsigma_{\alpha^*}(X)$ intersection of solutions

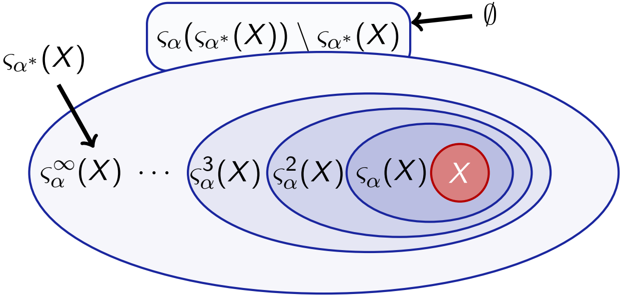
$X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) = Z$ — by mon since $Z \subseteq \varsigma_{\alpha^*}(X)$

$\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) = Z$ — since $\varsigma_{\alpha^*}(X)$ smallest such Z

Semantics of Repetition

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) = Z\} = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X) \quad \text{by Knaster-Tarski}$$



$$
\begin{aligned}
Z \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) &\subseteq \varsigma_{\alpha^*}(X) && \varsigma_{\alpha^*}(X) \text{ intersection of solutions}\\
X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) &= Z && \text{by mon since } Z \subseteq \varsigma_{\alpha^*}(X)\\
\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) &= Z && \text{since } \varsigma_{\alpha^*}(X) \text{ smallest such } Z
\end{aligned}
$$

Outline

Definition (Hybrid game α) $\quad [\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\}\\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x)\}\\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X\\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X)\\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X))\\
\varsigma_{\alpha^*}(X) &= \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)\\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $\quad [\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\}\\
[\![\neg P]\!] &= ([\![P]\!])^\complement\\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!]\\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!])\\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Definition (Hybrid game α) $\quad [\![\cdot]\!] : \text{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
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\varsigma_{\alpha^*}(X) &= \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
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\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
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$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

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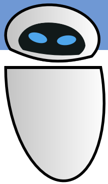
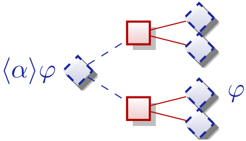
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

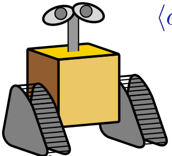
Summary



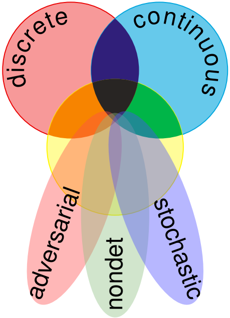


differential game logic

dGL = GL + HG = dL + $^{\mathsf{d}}$



- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint



Next chapter

1. Axiomatics
2. How to win and prove hybrid games

André Platzer.
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ACM Trans. Comput. Log., 17(1):1:1–1:51, 2015.
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