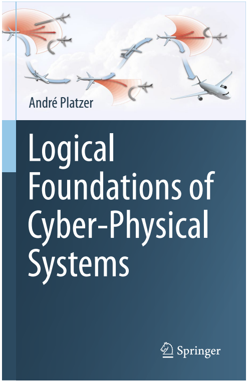
16: Winning & Proving Hybrid Games

Logical Foundations of Cyber-Physical Systems



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Logical Foundations of Cyber-Physical Systems

Springer

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Computer Science Department

Outline

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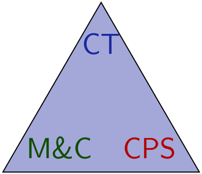
1. Learning Objectives
2. Semantical Considerations
3. Dynamic Axioms for Hybrid Games
    - Assignments
    - Differential Equations
    - Challenge Games
    - Choice Games
    - Sequential Games
    - Dual Games
    - Example Proof: Demon's Choice
4. Repetitions
    - Proofs for Loops
    - Example Proof: Dual Filibuster
    - Example Proof: Push-around Cart
5. Axiomatization
6. Summary

Learning Objectives

Winning & Proving Hybrid Games

rigorous reasoning for adversarial dynamics
compositional reasoning from compositional semantics
modular addition of adversarial dynamics
axiomatization of dGL



analytical&semantical interaction
discrete+continuous+adversarial
fixpoints

CPS semantics
align semantics&reasoning
operational CPS effects

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Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Differential Game Logic: Syntax

Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Definition (Hybrid game α)

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All Reals

Some Reals

Differential Game Logic: Syntax

Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Dual Game

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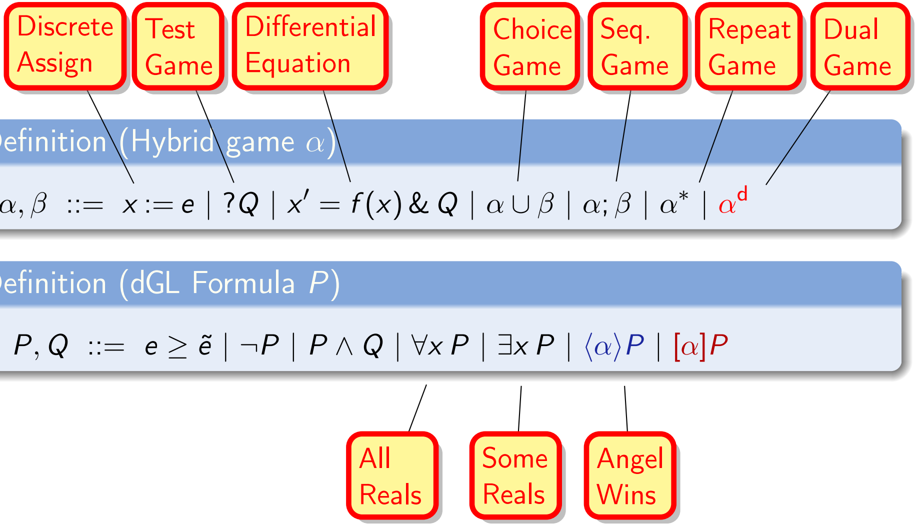
$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals

Some Reals

Differential Game Logic: Syntax



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Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

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All Reals | Some Reals | Angel Wins | Demon Wins

Differential Game Logic: Syntax

Discrete Assign

Test Game

Differential Equation

Choice Game

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Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

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$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

“Angel has Wings $\langle \alpha \rangle$”

All Reals

Some Reals

Angel Wins

Demon Wins

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} \;:\; \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } \varphi{:}[0,r]{\to}\mathcal{S},\ \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathcal{S} \;:\; X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

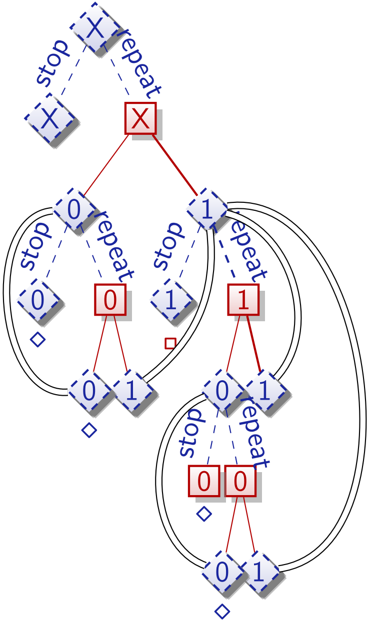
$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} \;:\; \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Differential Game Logic: Axiomatization

Filibusters & The Significance of Finitude

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



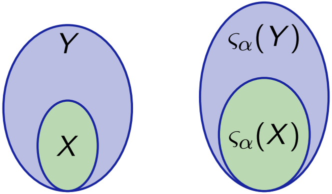
Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$



Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

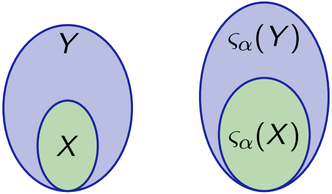
Corollary

Determined: At least one player wins: $\neg\langle\alpha\rangle\neg P \rightarrow [\alpha]P$ *so* $\langle\alpha\rangle\neg P \lor [\alpha]P$

Consistent: At most one player wins: $[\alpha]P \rightarrow \neg\langle\alpha\rangle\neg P$ *so* $\neg([\alpha]P \land \langle\alpha\rangle\neg P)$

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$



Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$*.*

Proof Sketch.

$\varsigma_{\alpha\cup\beta}(X^{\complement})^{\complement} = (\varsigma_\alpha(X^{\complement}) \cup \varsigma_\beta(X^{\complement}))^{\complement} = \varsigma_\alpha(X^{\complement})^{\complement} \cap \varsigma_\beta(X^{\complement})^{\complement} = \delta_\alpha(X) \cap \delta_\beta(X) = \delta_{\alpha\cup\beta}(X)$ □

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$

Proof Sketch.

- $X \subseteq Y$ so $X^{\complement} \supseteq Y^{\complement}$ so $\varsigma_\alpha(X^{\complement}) \supseteq \varsigma_\alpha(Y^{\complement})$ so $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^{\complement}))^{\complement} \subseteq (\varsigma_\alpha(Y^{\complement}))^{\complement} = \varsigma_{\alpha^d}(Y)$.
- $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ :\ X \cup \varsigma_\alpha(Z) \subseteq Z\} \subseteq$
  $\varsigma_{\alpha^*}(Y) = \bigcap\{Z \subseteq \mathcal{S} \ :\ Y \cup \varsigma_\alpha(Z) \subseteq Z\}$ because $X \subseteq Y$ □

Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y$$

Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Corollary (Axiom: Determinacy)

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y$$

Corollary (Rule: Monotonicity)

$$M\ \frac{P \to Q}{\langle\alpha\rangle P \to \langle\alpha\rangle Q} \qquad M[\cdot]\ \frac{P \to Q}{[\alpha]P \to [\alpha]Q}$$

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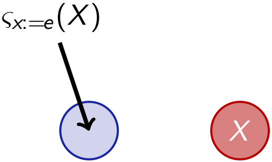
Hybrid Game Axioms

Axiom (Assignment) TOCL'15

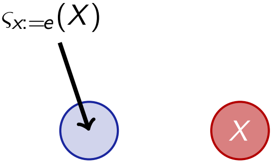
$\langle{:=}\rangle \quad \langle x := e\rangle p(x) \leftrightarrow$



Hybrid Game Axioms

Axiom (Assignment) TOCL'15

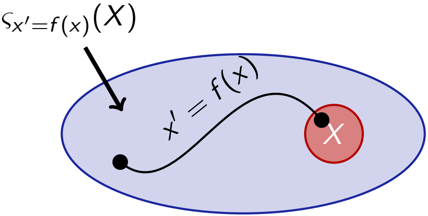
$\langle{:=}\rangle \quad \langle x := e\rangle p(x) \leftrightarrow p(e)$



Hybrid Game Axioms

Axiom (Differential Equation) TOCL'15

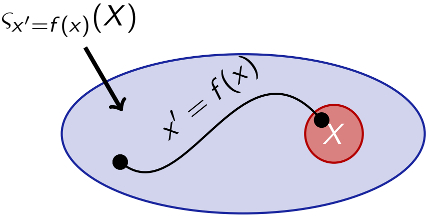
$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \qquad (y'(t) = f(y))$



Hybrid Game Axioms

Axiom (Differential Equation) TOCL'15

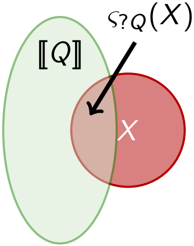
$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t) \rangle p(x) \qquad (y'(t) = f(y))$



Hybrid Game Axioms

Axiom (Test / Challenge) TOCL'15

$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow$

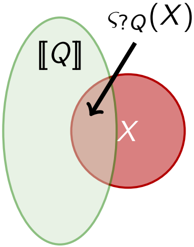


Hybrid Game Axioms

Axiom (Test / Challenge)

TOCL'15

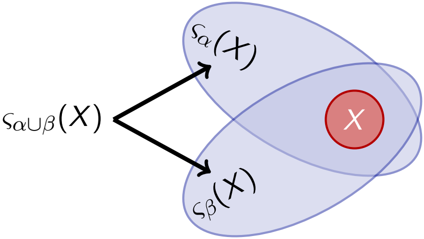
$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow Q \land P$



Hybrid Game Axioms

Axiom (Choice Game) TOCL'15

$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle P \leftrightarrow$

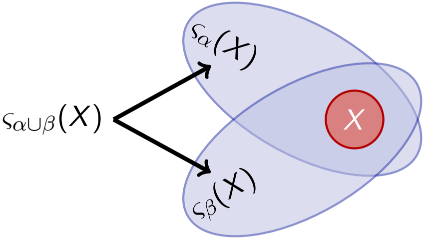


Hybrid Game Axioms

Axiom (Choice Game)

TOCL'15

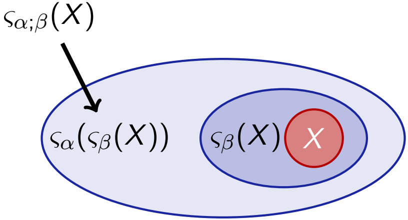
$\langle\cup\rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \lor \langle \beta \rangle P$



Hybrid Game Axioms

Axiom (Sequential Game) TOCL'15

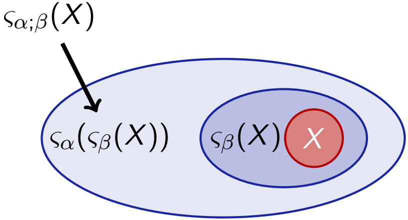
$\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow$



Hybrid Game Axioms

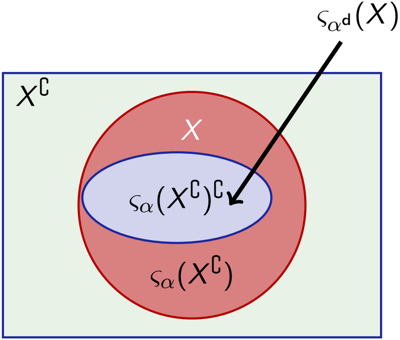
Axiom (Sequential Game) TOCL'15

$\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$



Axiom (Dual Game) TOCL'15

$\langle{}^d\rangle \quad \langle\alpha^{\mathsf{d}}\rangle P \leftrightarrow$

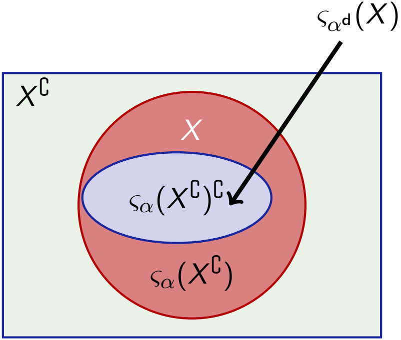


Hybrid Game Axioms

Axiom (Dual Game)

TOCL'15

$\langle{}^{d}\rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$



Example: Demon's Choice Derives by Duality

$$\frac{}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow}$$

Example: Demon's Choice Derives by Duality

$$\frac{}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \land \langle \beta \rangle P}$$

Example: Demon's Choice Derives by Duality

$$\alpha \cap \beta \equiv (\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}$$

$$\dfrac{{}^{\langle d\rangle}\,\overline{\vdash \langle(\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}$$

Example: Demon's Choice Derives by Duality

$\langle{}^{d}\rangle \quad \langle\alpha^{\mathsf{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$$\dfrac{\dfrac{\dfrac{}{\vdash \neg\langle\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle\cup\rangle}{\vdash \langle(\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle{}^{d}\rangle}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}$$

Example: Demon's Choice Derives by Duality

$$\langle\cup\rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \lor \langle \beta \rangle P$$

$$\begin{array}{rl}
{\scriptstyle \langle {}^{d} \rangle} & \overline{\vdash \neg(\langle \alpha^{\mathsf{d}} \rangle \neg P \lor \langle \beta^{\mathsf{d}} \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \land \langle \beta \rangle P} \\
{\scriptstyle \langle \cup \rangle} & \overline{\vdash \neg \langle \alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}} \rangle \neg P \leftrightarrow \langle \alpha \rangle P \land \langle \beta \rangle P} \\
{\scriptstyle \langle {}^{d} \rangle} & \overline{\vdash \langle (\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}} \rangle P \leftrightarrow \langle \alpha \rangle P \land \langle \beta \rangle P} \\
 & \overline{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \land \langle \beta \rangle P}
\end{array}$$

Example: Demon's Choice Derives by Duality

$$\langle{}^{d}\rangle \quad \langle\alpha^{\mathsf{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\begin{array}{r l}
\hline
 & \vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
\langle{}^{d}\rangle & \vdash \neg(\langle\alpha^{\mathsf{d}}\rangle\neg P \lor \langle\beta^{\mathsf{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
\langle\cup\rangle & \vdash \neg\langle\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
\langle{}^{d}\rangle & \vdash \langle(\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
 & \vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P
\end{array}$$

Example: Demon's Choice Derives by Duality

$$\begin{array}{rl}
\cline{2-2}
 & \vdash \langle\alpha\rangle P \land \langle\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
 & \vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
\langle^{d}\rangle & \vdash \neg(\langle\alpha^{\mathsf{d}}\rangle\neg P \lor \langle\beta^{\mathsf{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
\langle\cup\rangle & \vdash \neg\langle\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
\langle^{d}\rangle & \vdash \langle(\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
 & \vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P
\end{array}$$

Example: Demon's Choice Derives by Duality

$$
\begin{array}{rl}
 & * \\
\cline{2-2}
 & \vdash \langle\alpha\rangle P \land \langle\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
 & \vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
\langle{}^{d}\rangle & \vdash \neg(\langle\alpha^{\mathsf{d}}\rangle\neg P \lor \langle\beta^{\mathsf{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
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\cline{2-2}
\langle{}^{d}\rangle & \vdash \langle(\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\cline{2-2}
 & \vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P
\end{array}
$$

Example: Demon's Choice Derives by Duality

$$
\begin{array}{rl}
 & * \\ \hline
 & \vdash \langle\alpha\rangle P \land \langle\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\ \hline
 & \vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\ \hline
\langle^{d}\rangle & \vdash \neg(\langle\alpha^{\mathsf{d}}\rangle\neg P \lor \langle\beta^{\mathsf{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\ \hline
\langle\cup\rangle & \vdash \neg\langle\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\ \hline
\langle^{d}\rangle & \vdash \langle(\alpha^{\mathsf{d}} \cup \beta^{\mathsf{d}})^{\mathsf{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\ \hline
 & \vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P
\end{array}
$$

Derived axiom:

$$\langle\cap\rangle \quad \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P$$

Example: Demon's Choice Derives by Duality

$$[\cdot]\ \frac{}{\vdash [\alpha \cap \beta]P \leftrightarrow}$$

Example: Demon's Choice Derives by Duality

$$[\cdot]\ \frac{}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P}$$

Example: Demon's Choice Derives by Duality

$[\cdot]\quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$$\dfrac{\overline{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\;\langle\cap\rangle}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P}\;[\cdot]$$

Example: Demon's Choice Derives by Duality

$\langle\cap\rangle \quad \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P$

$$
{\scriptstyle [\cdot]}\,\dfrac{{\scriptstyle \langle\cap\rangle}\,\dfrac{\overline{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P}
$$

Example: Demon's Choice Derives by Duality

$$
{\scriptstyle[\cdot]}\,\dfrac{{\scriptstyle\langle\cap\rangle}\,\dfrac{\dfrac{{\scriptstyle[\cdot]}\,\dfrac{}{\vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P}
$$

Example: Demon's Choice Derives by Duality

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\begin{array}{rl}
\cline{2-2}
 & \vdash [\alpha]P \lor [\beta]P \leftrightarrow [\alpha]P \lor [\beta]P \\
\cline{2-2}
[\cdot] & \vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P \\
\cline{2-2}
 & \vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P \\
\cline{2-2}
\langle\cap\rangle & \vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P \\
\cline{2-2}
[\cdot] & \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P
\end{array}$$

Example: Demon's Choice Derives by Duality

$$
\begin{array}{rl}
 & * \\ \cline{2-2}
 & \vdash [\alpha]P \lor [\beta]P \leftrightarrow [\alpha]P \lor [\beta]P \\ \cline{2-2}
{[\cdot]} & \vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P \\ \cline{2-2}
 & \vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P \\ \cline{2-2}
\langle\cap\rangle & \vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P \\ \cline{2-2}
{[\cdot]} & \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P
\end{array}
$$

Example: Demon's Choice Derives by Duality

$$\begin{array}{rl}
 & * \\ \hline
 & \vdash [\alpha]P \lor [\beta]P \leftrightarrow [\alpha]P \lor [\beta]P \\ \hline
[\cdot] & \vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P \\ \hline
 & \vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P \\ \hline
\langle\cap\rangle & \vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P \\ \hline
[\cdot] & \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P
\end{array}$$

Derived axioms:

$[\cap]$ $[\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P$

$\langle\cap\rangle$ $\langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P$

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle{:=}\rangle$ $\langle x := e\rangle p(x) \leftrightarrow p(e)$

$\langle{}'\rangle$ $\langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$

$\langle ?\rangle$ $\langle ?Q\rangle P \leftrightarrow Q \land P$

$\langle\cup\rangle$ $\langle\alpha \cup \beta\rangle P \leftrightarrow \langle\alpha\rangle P \lor \langle\beta\rangle P$

$\langle ;\rangle$ $\langle\alpha; \beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$\langle^{d}\rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

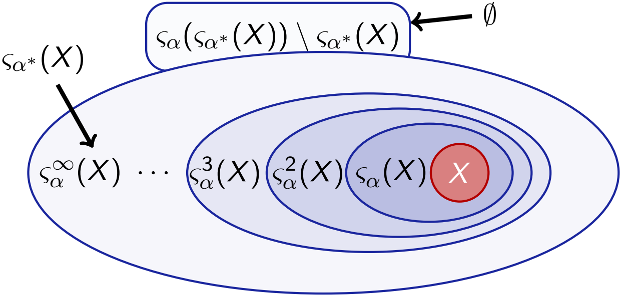
Outline

1. Learning Objectives
2. Semantical Considerations
3. Dynamic Axioms for Hybrid Games
   - Assignments
   - Differential Equations
   - Challenge Games
   - Choice Games
   - Sequential Games
   - Dual Games
   - Example Proof: Demon's Choice
4. Repetitions
   - Proofs for Loops
   - Example Proof: Dual Filibuster
   - Example Proof: Push-around Cart
5. Axiomatization
6. Summary

Semantics of Repetition

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \;:\; X \cup \varsigma_\alpha(Z) = Z\}$$



Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ : \ X \cup \varsigma_\alpha(Z) \subseteq Z\}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ : \ X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ : \ X \cup \varsigma_\alpha(Z) \subseteq Z\} \qquad \varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$$

Lemma (Axiom:)

$\langle{}^*\rangle$ $\quad \langle\alpha^*\rangle P \leftrightarrow$

Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \;:\; X \cup \varsigma_\alpha(Z) \subseteq Z\} \qquad \varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$$

Lemma (Axiom: Iteration)

$$\langle{}^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ : \ X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

$\langle {}^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

Lemma (Rule:)

$$\mathit{FP} \quad \frac{}{\langle\alpha^*\rangle P \rightarrow Q}$$

Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ : \ X \cup \varsigma_\alpha(Z) \subseteq Z\} \qquad \varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$$

Lemma (Axiom: Iteration)

$$\langle{}^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$$

Lemma (Rule: Least Fixpoint)

$$FP \quad \frac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ :\ X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

$\langle{*}\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$

Lemma (Rule: Least Fixpoint)

$$FP \quad \frac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

Corollary (Derived Rule:)

$$loop \quad \frac{}{P \rightarrow [\alpha^*]P}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \ : \ X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

$\langle^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$

Lemma (Rule: Least Fixpoint)

$$FP \quad \frac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

Corollary (Derived Rule: Loop Invariant)

$$loop \quad \frac{P \rightarrow [\alpha]P}{P \rightarrow [\alpha^*]P}$$

Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} \,:\, X \cup \varsigma_\alpha(Z) \subseteq Z\} \qquad \varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$$

Lemma (Axiom: Iteration)

$$\langle{*}\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$$

Lemma (Rule: Least Fixpoint)

$$\mathrm{FP}\ \ \frac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

Corollary (Derived Rule: Loop Invariant)

$$\mathrm{loop}\ \ \frac{P \rightarrow [\alpha]P}{P \rightarrow [\alpha^*]P}$$

Proof

$$\begin{array}{rl}
 & \vdash P \rightarrow [\alpha]P \\ \cline{2-2}
 & \vdash P \rightarrow P \land [\alpha]P \\ \cline{2-2}
{[\cdot]} & \vdash P \rightarrow P \land \neg\langle\alpha\rangle\neg P \\ \cline{2-2}
 & \vdash \neg P \lor \langle\alpha\rangle\neg P \rightarrow \neg P \\ \cline{2-2}
\mathrm{FP} & \vdash \langle\alpha^*\rangle\neg P \rightarrow \neg P \\ \cline{2-2}
 & \vdash P \rightarrow \neg\langle\alpha^*\rangle\neg P \\ \cline{2-2}
{[\cdot]} & \vdash P \rightarrow [\alpha^*]P
\end{array}$$

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle{:=}\rangle$ $\langle x := e\rangle p(x) \leftrightarrow p(e)$

$\langle{'}\rangle$ $\langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$

$\langle ? \rangle$ $\langle ?Q\rangle P \leftrightarrow Q \wedge P$

$\langle\cup\rangle$ $\langle\alpha \cup \beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$

$\langle ; \rangle$ $\langle\alpha; \beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$\langle {}^* \rangle$ $\langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

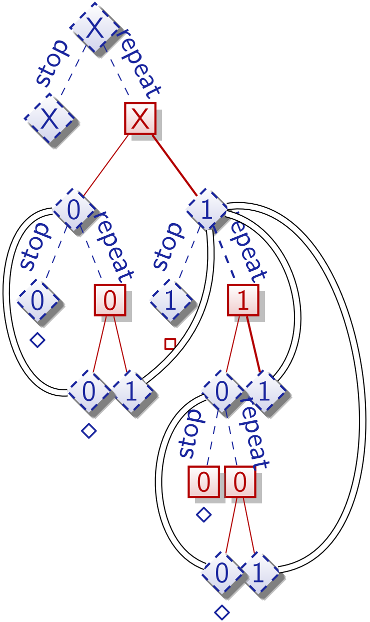
$\langle {}^d \rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$$\text{M}\ \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$$

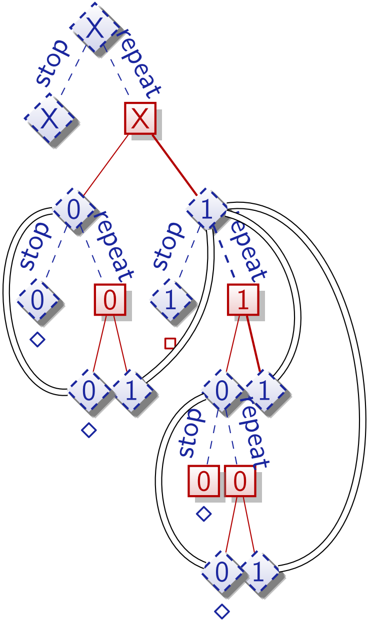
$$\text{FP}\ \frac{P \vee \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

Example Proof: Dual Filibuster

$$\langle{}^d\rangle\ \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^{\times} \rangle x = 0}$$



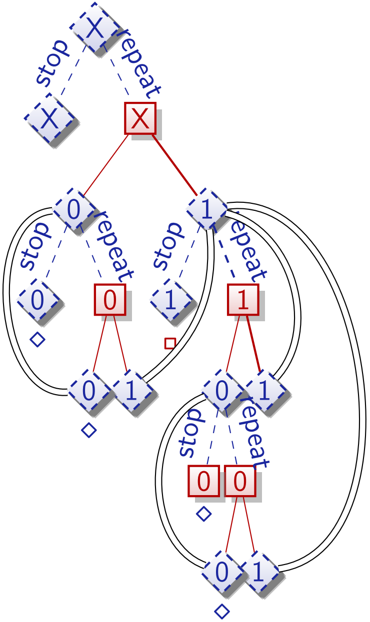
Example Proof: Dual Filibuster



$$\dfrac{\dfrac{}{x = 0 \vdash [(x := 0 \cap x := 1)^*]x = 0}\text{ind}}{x = 0 \vdash \langle(x := 0 \cup x := 1)^\times\rangle x = 0}\langle^d\rangle$$

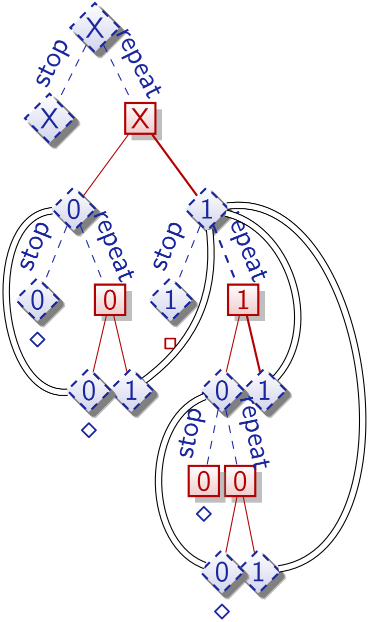
Example Proof: Dual Filibuster

$$\begin{array}{ll}
{\scriptstyle [\cdot]} & \overline{\phantom{x = 0 \vdash \langle (x := 0 \cup x := 1)^{\times} \rangle x = 0}} \\
{\scriptstyle \text{ind}} & \overline{x = 0 \vdash [x := 0 \cap x := 1] x = 0} \\
{\scriptstyle \langle^{d} \rangle} & \overline{x = 0 \vdash [(x := 0 \cap x := 1)^{*}] x = 0} \\
 & x = 0 \vdash \langle (x := 0 \cup x := 1)^{\times} \rangle x = 0
\end{array}$$



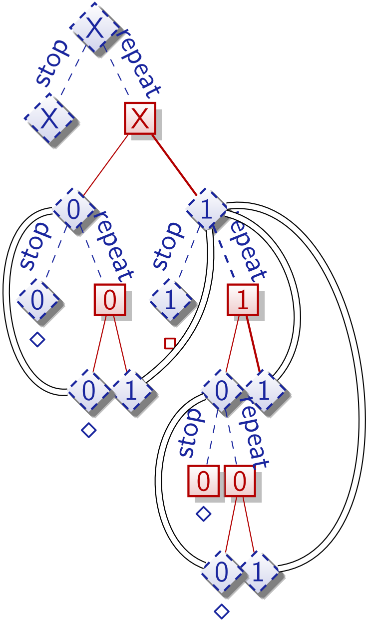
Example Proof: Dual Filibuster

$$\begin{array}{ll} \langle^d\rangle & \overline{x = 0 \vdash \neg\langle x := 0 \cap x := 1\rangle \neg x = 0} \\ [\cdot] & \overline{x = 0 \vdash [x := 0 \cap x := 1]x = 0} \\ \text{ind} & \overline{x = 0 \vdash [(x := 0 \cap x := 1)^*]x = 0} \\ \langle^d\rangle & \overline{x = 0 \vdash \langle(x := 0 \cup x := 1)^\times\rangle x = 0} \end{array}$$



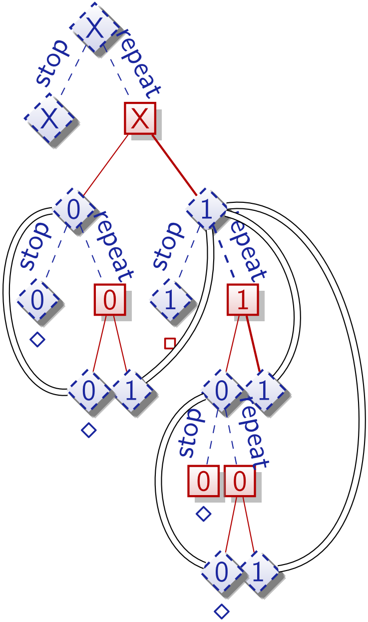
Example Proof: Dual Filibuster

$$\begin{array}{ll}
 & \\ \cline{2-2}
\langle\cup\rangle & x = 0 \vdash \langle x := 0 \cup x := 1\rangle x = 0 \\ \cline{2-2}
\langle^d\rangle & x = 0 \vdash \neg\langle x := 0 \cap x := 1\rangle \neg x = 0 \\ \cline{2-2}
[\cdot] & x = 0 \vdash [x := 0 \cap x := 1] x = 0 \\ \cline{2-2}
\text{ind} & x = 0 \vdash [(x := 0 \cap x := 1)^*] x = 0 \\ \cline{2-2}
\langle^d\rangle & x = 0 \vdash \langle (x := 0 \cup x := 1)^\times \rangle x = 0
\end{array}$$



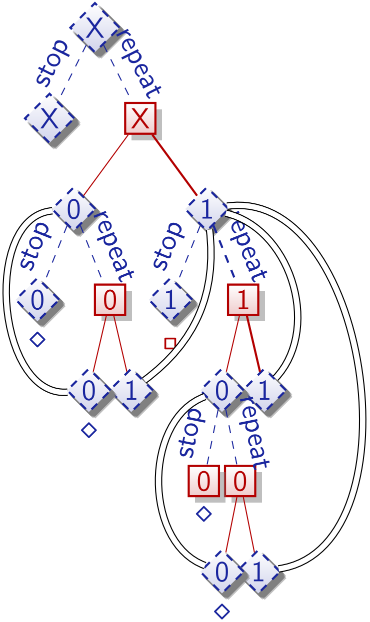
Example Proof: Dual Filibuster

$$
\begin{array}{ll}
 & \\ \hline
\langle:=\rangle & x=0 \vdash \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0 \\ \hline
\langle\cup\rangle & x=0 \vdash \langle x:=0 \cup x:=1\rangle x=0 \\ \hline
\langle^{d}\rangle & x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle \neg x=0 \\ \hline
[\cdot] & x=0 \vdash [x:=0 \cap x:=1]x=0 \\ \hline
\text{ind} & x=0 \vdash [(x:=0 \cap x:=1)^{*}]x=0 \\ \hline
\langle^{d}\rangle & x=0 \vdash \langle (x:=0 \cup x:=1)^{\times}\rangle x=0
\end{array}
$$



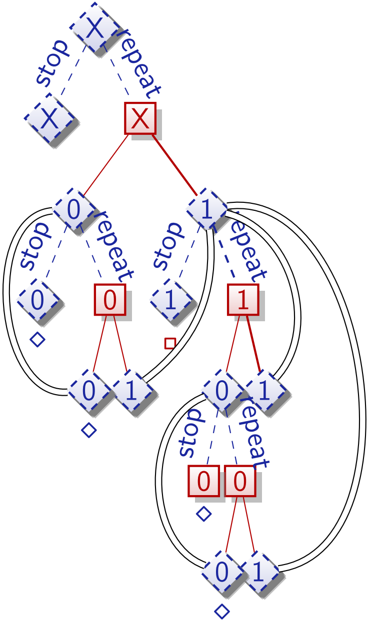
Example Proof: Dual Filibuster

$$\begin{array}{ll}
\mathbb{R} & \\ \hline
 & x=0 \vdash 0=0 \lor 1=0 \\ \hline
\langle:=\rangle & x=0 \vdash \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0 \\ \hline
\langle\cup\rangle & x=0 \vdash \langle x:=0 \cup x:=1\rangle x=0 \\ \hline
\langle^d\rangle & x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle \neg x=0 \\ \hline
[\cdot] & x=0 \vdash [x:=0 \cap x:=1] x=0 \\ \hline
\text{ind} & x=0 \vdash [(x:=0 \cap x:=1)^*] x=0 \\ \hline
\langle^d\rangle & x=0 \vdash \langle (x:=0 \cup x:=1)^\times \rangle x=0
\end{array}$$

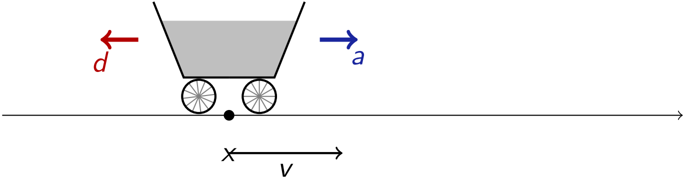


Example Proof: Dual Filibuster

$$
\begin{array}{rl}
 & \qquad\qquad * \\ \hline
\mathbb{R} & x=0 \vdash 0=0 \lor 1=0 \\ \hline
\langle:=\rangle & x=0 \vdash \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0 \\ \hline
\langle\cup\rangle & x=0 \vdash \langle x:=0 \cup x:=1\rangle x=0 \\ \hline
\langle^{d}\rangle & x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle \neg x=0 \\ \hline
[\cdot] & x=0 \vdash [x:=0 \cap x:=1]x=0 \\ \hline
\text{ind} & x=0 \vdash [(x:=0 \cap x:=1)^{*}]x=0 \\ \hline
\langle^{d}\rangle & x=0 \vdash \langle (x:=0 \cup x:=1)^{\times}\rangle x=0
\end{array}
$$



Example Proof: Push-around Cart



$$\text{ind}\ \frac{}{J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\}\big)^*]x\geq 0}$$

Example Proof: Push-around Cart

$$\dfrac{\dfrac{}{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}\,[;]}{J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\}\big)^*]x\geq 0}\,\text{ind}$$

Example Proof: Push-around Cart

$$
\begin{array}{ll}
[\cap] & \overline{J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
[;] & \overline{J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J} \\
\text{ind} & \overline{J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\}\big)^*]x \geq 0}
\end{array}
$$

Example Proof: Push-around Cart

$$
\begin{array}{ll}
\hline
\vee\mathrm{R},\mathrm{WR} & J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J \lor [d:=-1]\ldots \\ \hline
[\cap] & J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J \\ \hline
[;] & J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J \\ \hline
\mathrm{ind} & J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*]x \geq 0
\end{array}
$$

Example Proof: Push-around Cart

$$\begin{array}{rl}
 & \hline
[:=] & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
\lor R, WR & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \lor [d := -1] \ldots \\ \hline
[\cap] & J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
[;] & J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
\text{ind} & J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\}\big)^*]x \geq 0
\end{array}$$

Example Proof: Push-around Cart

$$\begin{array}{rl}
 & \\ \cline{2-2}
\text{[;]} & J \vdash [(a:=1 \cup a:=-1);\{x'=v,v'=a+1\}]J \\ \cline{2-2}
\text{[:=]} & J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \\ \cline{2-2}
\vee\text{R,WR} & J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \vee [d:=-1]\ldots \\ \cline{2-2}
\text{[}\cap\text{]} & J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \\ \cline{2-2}
\text{[;]} & J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \\ \cline{2-2}
\text{ind} & J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*]x \geq 0
\end{array}$$

Example Proof: Push-around Cart

$$
\begin{array}{ll}
\hline
[\cup] & J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J \\ \hline
[;] & J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J \\ \hline
[:=] & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
\lor\mathrm{R,WR} & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \lor [d := -1] \dots \\ \hline
[\cap] & J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
[;] & J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
\mathrm{ind} & J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\}\big)^*]x \geq 0
\end{array}
$$

Example Proof: Push-around Cart

$$
\begin{array}{rl}
\cline{2-2}
\text{[:=]} & J \vdash [a:=1][\{x'=v, v'=a+1\}]J \land [a:=-1][\{x'=v, v'=a+1\}]J \\ \cline{2-2}
\text{[}\cup\text{]} & J \vdash [a:=1 \cup a:=-1][\{x'=v, v'=a+1\}]J \\ \cline{2-2}
\text{[;]} & J \vdash [(a:=1 \cup a:=-1); \{x'=v, v'=a+1\}]J \\ \cline{2-2}
\text{[:=]} & J \vdash [d:=1][(a:=1 \cup a:=-1); \{x'=v, v'=a+d\}]J \\ \cline{2-2}
\lor\text{R,WR} & J \vdash [d:=1][(a:=1 \cup a:=-1); \{x'=v, v'=a+d\}]J \lor [d:=-1]\dots \\ \cline{2-2}
\text{[}\cap\text{]} & J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1); \{x'=v, v'=a+d\}]J \\ \cline{2-2}
\text{[;]} & J \vdash [(d:=1 \cap d:=-1); (a:=1 \cup a:=-1); \{x'=v, v'=a+d\}]J \\ \cline{2-2}
\text{ind} & J \vdash [\big((d:=1 \cap d:=-1); (a:=1 \cup a:=-1); \{x'=v, v'=a+d\}\big)^*]x\geq 0
\end{array}
$$

Example Proof: Push-around Cart

$$\begin{array}{ll}
 & J \vdash [\{x' = v, v' = 1+1\}]J \land [\{x' = v, v' = -1+1\}]J \\
\hline
[:=] & J \vdash [a := 1][\{x' = v, v' = a+1\}]J \land [a := -1][\{x' = v, v' = a+1\}]J \\
\hline
[\cup] & J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a+1\}]J \\
\hline
[;] & J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a+1\}]J \\
\hline
[:=] & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a+d\}]J \\
\hline
\lor R,WR & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a+d\}]J \lor [d := -1]\ldots \\
\hline
[\cap] & J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a+d\}]J \\
\hline
[;] & J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a+d\}]J \\
\hline
\mathrm{ind} & J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\}\big)^*]x \geq 0
\end{array}$$

Example Proof: Push-around Cart

$$\begin{array}{rl}
 & J \vdash [\{x' = v, v' = 1 + 1\}]J \land [\{x' = v, v' = -1 + 1\}]J \\ \hline
[:=] & J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \land [a := -1][\{x' = v, v' = a + 1\}]J \\ \hline
[\cup] & J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J \\ \hline
[;] & J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J \\ \hline
[:=] & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
\lor\mathrm{R,WR} & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \lor [d := -1] \dots \\ \hline
[\cap] & J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
[;] & J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \hline
\mathrm{ind} & J \vdash [\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'{=}v, v'{=}a{+}d\}\big)^*]x{\geq}0
\end{array}$$

$$J \overset{\text{def}}{\equiv} x{\geq}0 \land v{\geq}0$$

Example Proof: Push-around Cart

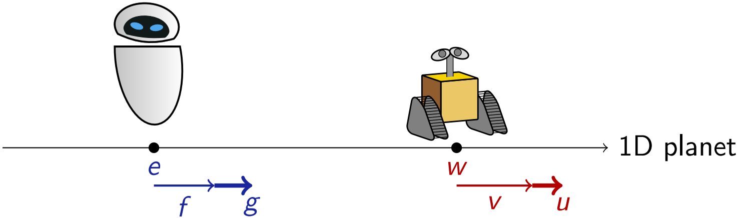
$$
\begin{array}{rl}
 & J \vdash [\{x' = v, v' = 1 + 1\}]J \land [\{x' = v, v' = -1 + 1\}]J \\ \cline{2-2}
[:=] & J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \land [a := -1][\{x' = v, v' = a + 1\}]J \\ \cline{2-2}
[\cup] & J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J \\ \cline{2-2}
[;] & J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J \\ \cline{2-2}
[:=] & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \cline{2-2}
\lor R, WR & J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \lor [d := -1] \ldots \\ \cline{2-2}
[\cap] & J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \cline{2-2}
[;] & J \vdash [(d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \\ \cline{2-2}
\text{ind} & J \vdash [((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x'=v, v'=a+d\})^*]x \geq 0
\end{array}
$$

$$J \overset{\text{def}}{\equiv} x \geq 0 \land v \geq 0$$

$$[{}'],[:=]\,\frac{x \geq 0 \land v \geq 0 \vdash \forall t \geq 0\, (x + vt + t^2 \geq 0 \land v + 2t \geq 0)}{J \vdash [\{x' = v, v' = 1 + 1\}]J}$$

$$[{}'],[:=]\,\frac{x \geq 0 \land v \geq 0 \vdash \forall t \geq 0\, (x + vt \geq 0 \land v \geq 0)}{J \vdash [\{x' = v, v' = 0\}]J}$$

Example Proof: WALL·E and EVE Robot Dance



$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \to \\
&\langle ((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d} \\
&)^{\times} \rangle\, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u

Outline

1. Learning Objectives
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3. Dynamic Axioms for Hybrid Games
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   - Choice Games
   - Sequential Games
   - Dual Games
   - Example Proof: Demon's Choice
4. Repetitions
   - Proofs for Loops
   - Example Proof: Dual Filibuster
   - Example Proof: Push-around Cart
5. **Axiomatization**
6. Summary

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle:=\rangle$ $\langle x := e\rangle p(x) \leftrightarrow p(e)$

$\langle'\rangle$ $\langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$

$\langle?\rangle$ $\langle ?Q\rangle P \leftrightarrow Q \land P$

$\langle\cup\rangle$ $\langle\alpha \cup \beta\rangle P \leftrightarrow \langle\alpha\rangle P \lor \langle\beta\rangle P$

$\langle;\rangle$ $\langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$\langle^*\rangle$ $\langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$

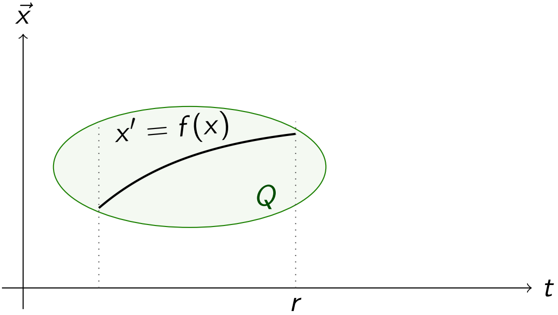
$\langle^d\rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

M $\dfrac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$

FP $\dfrac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$

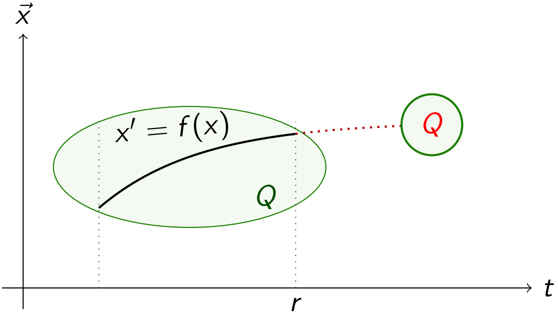
Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$ $\qquad\qquad$ $x' = f(x); ?(Q)$



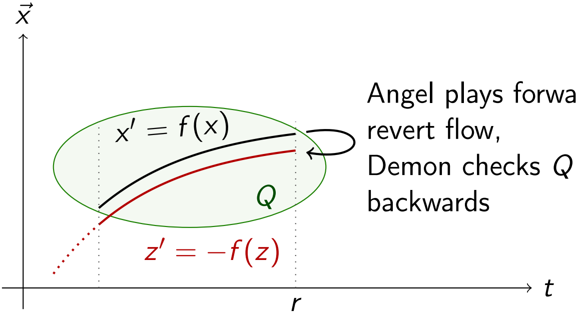
Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$ $x' = f(x); ?(Q)$



Defining Evolution Domain Constraints

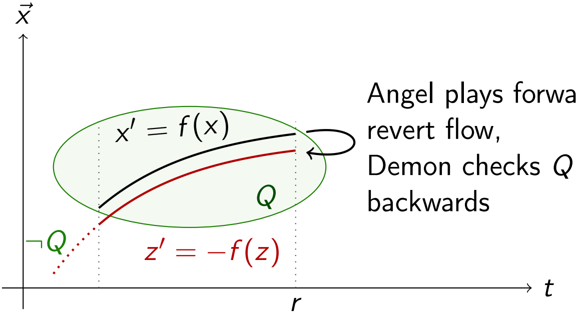
$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$



Angel plays forward
revert flow,
Demon checks Q
backwards

Defining Evolution Domain Constraints

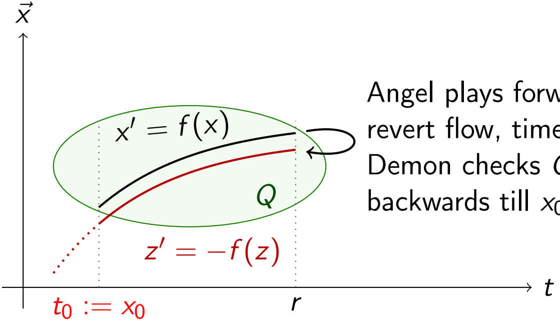
$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); (z := x; z' = -f(z))^{d}; ?(Q(z))$



Angel plays forward
revert flow,
Demon checks Q
backwards

Defining Evolution Domain Constraints $x_0' = 1$

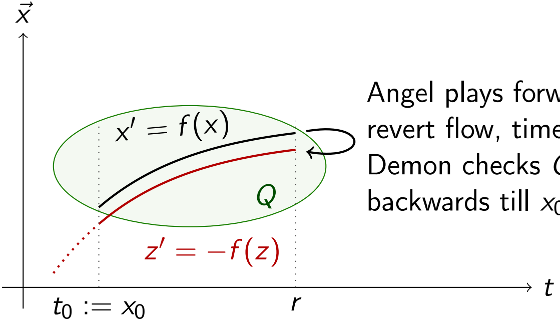
$$x' = f(x) \,\&\, Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^{d}; ?(z_0 \geq t_0 \rightarrow Q(z))$$



Angel plays forward
revert flow, time x_0;
Demon checks Q
backwards till x_0

"There and Back Again" Game

$$x' = f(x)\,\&\,Q \equiv t_0 := x_0;\, x' = f(x);\, (z := x;\, z' = -f(z))^d;\, ?(z_0 \geq t_0 \rightarrow Q(z))$$



Angel plays forward
revert flow, time x_0;
Demon checks Q
backwards till x_0

Lemma

Evolution domains definable by games

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Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} \;:\; \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } \varphi{:}[0,r]{\to}\mathcal{S},\ \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathcal{S} \;:\; X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} \;:\; \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle:=\rangle$ $\langle x := e\rangle p(x) \leftrightarrow p(e)$

$\langle'\rangle$ $\langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$

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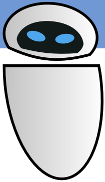
M $\dfrac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$

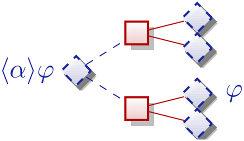
FP $\dfrac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$

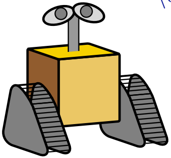
Summary

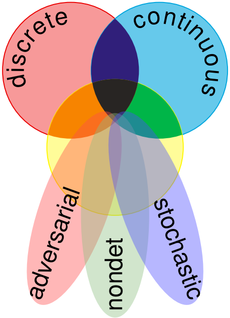
differential game logic

$$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG} = \mathsf{dL} + {}^{\mathsf{d}}$$









- Axiomatics for hybrid games
- Proving winning strategies

Next chapter

1. Soundness
2. Proofs
3. Separations

André Platzer.
Logical Foundations of Cyber-Physical Systems.
Springer, Switzerland, 2018.
URL: `http://www.springer.com/978-3-319-63587-3`,
`doi:10.1007/978-3-319-63588-0`.

André Platzer.
Differential game logic.
ACM Trans. Comput. Log., 17(1):1:1–1:51, 2015.
`doi:10.1145/2817824`.