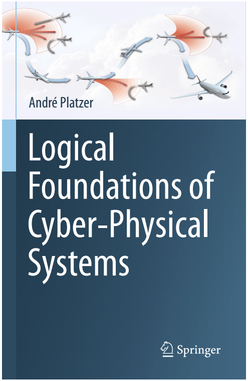
21: Virtual Substitution & Real Arithmetic

Logical Foundations of Cyber-Physical Systems



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Outline

1. Learning Objectives
   - Recap: Quadratic Equations
2. Real Arithmetic
   - Quadratic Weak Inequalities
   - Virtual Substitution of Infinities
   - Expedition: Infinities
   - Quadratic Strict Inequalities
   - Infinitesimals
   - Virtual Substitution of Infinitesimals
3. Quantifier Elimination by Virtual Substitution
4. Summary

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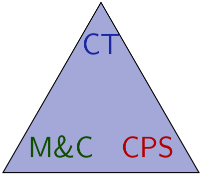
rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles
infinities & infinitesimals



analytic complexity
modeling tradeoffs

verifying CPS at scale

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b}$$
$$\lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

$FOL_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \text{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/\omega[\![d]\!] \in \mathbb{R}$$

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$$
\begin{aligned}
& a \neq 0 \lor b \neq 0 \lor c \neq 0 \to \\
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& \quad a = 0 \land b \neq 0 \land F_{\not{x}}^{-c/b} \\
& \quad \lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\not{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\not{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)
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Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic → $F_x^{(a+b\sqrt{c})/d} \equiv F_{\not{x}}^{(a+b\sqrt{c})/d}$ ← $\mathrm{FOL}_{\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ *iff* $\omega \in [\![F_{\not{x}}^{(a+b\sqrt{c})/d}]\!]$ *where* $r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/\omega[\![d]\!] \in \mathbb{R}$

Quadratic Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x \, (ax^2 + bx + c \leq 0 \land F) \leftrightarrow$$

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$$\lor\ F_{\bar{x}}^{\text{small}}\Big)$$

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$$\lor\ F_{\not{x}}^{-\infty}\Big)$$

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$$\lor\; F_{\bar{x}}^{-\infty}\Big)$$

$-\infty$ the rubber band number that's smaller on any comparison

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$$\lor\ (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty}\Big)$$

$-\infty$ needs to satisfy the quadratic inequality (obvious for roots, not $-\infty$)

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&\quad \lor\, (ax^2 + bx + c \leq 0)_{\not x}^{-\infty} \land F_{\not x}^{-\infty}\Big)
\end{aligned}
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Lemma (Virtual Substitution Lemma for $-\infty$)

$$F_x^{-\infty} \equiv F_{\not x}^{-\infty}$$

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Lemma (Virtual Substitution Lemma for $-\infty$)

Extended logic $\mathrm{FOL}_{\mathbb{R}\cup\{-\infty,\infty\}}$ ▸ $F_x^{-\infty} \equiv F_{\cancel{x}}^{-\infty}$ ◂ $\mathrm{FOL}_{\mathbb{R}}$

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$$\lor\ \ (ax^2 + bx + c \leq 0)_{\cancel{x}}^{-\infty} \land F_{\cancel{x}}^{-\infty}\Big) \ldots$$

Lemma (Virtual Substitution Lemma for $-\infty$)

Extended logic $\mathrm{FOL}_{\mathbb{R}\cup\{-\infty,\infty\}}$ $\quad F_x^{-\infty} \equiv F_{\cancel{x}}^{-\infty}$ $\quad \mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \textit{ iff } \omega \in [\![F_{\cancel{x}}^{-\infty}]\!] \textit{ where } r \rightarrow -\infty$$

Virtual Substitution of $-\infty$ into Comparisons $\quad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\not x}^{-\infty} \equiv$$

$$(p \leq 0)_{\not x}^{-\infty} \equiv$$

$$(p < 0)_{\not x}^{-\infty} \equiv$$

$$(p \neq 0)_{\not x}^{-\infty} \equiv$$

Virtual Substitution of $-\infty$ into Comparisons $\quad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{-\infty} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

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$$(p \leq 0)_{\bar{x}}^{-\infty} \equiv (p < 0)_{\bar{x}}^{-\infty} \lor (p = 0)_{\bar{x}}^{-\infty}$$

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$$(p < 0)_{\not{x}}^{-\infty} \equiv p(\text{-}\infty) < 0$$

$$(p \neq 0)_{\not{x}}^{-\infty} \equiv$$

Ultimately negative at $-\infty$ $\quad \lim_{x \to -\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

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$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} & \text{if } \deg(p) \leq 0 \\ & \text{if} \end{cases}$$

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

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$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ & \text{if} \end{cases}$$

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$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ (-1)^n a_n < 0 & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of Infinities

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$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ (-1)^n a_n < 0 \lor \left(a_n = 0 \land (\sum_{i=0}^{n-1} a_i x^i)(\text{-}\infty) < 0\right) & \text{if } \deg(p) > 0 \end{cases}$$

Example: Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $\quad p = \sum_{i=0}^{n} a_i x^i$

$$(p < 0)_{\bar{x}}^{-\infty} \equiv (p < 0)_{\bar{x}}^{-\infty}$$

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$$\begin{aligned}(ax^2 + bx + c \le 0)_{\bar{x}}^{-\infty} &\equiv (-1)^2 a < 0 \lor a = 0 \land ((-1)b < 0 \lor b = 0 \land c < 0) \\ &\equiv a < 0 \lor a = 0 \land (b > 0 \lor b = 0 \land c < 0)\end{aligned}$$

Example: Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $\quad p = \sum_{i=0}^{n} a_i x^i$

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$$(ax^2 + bx + c \le 0)_{\bar{x}}^{-\infty} \equiv (-1)^2 a < 0 \lor a = 0 \land ((-1)b < 0 \lor b = 0 \land c < 0)$$

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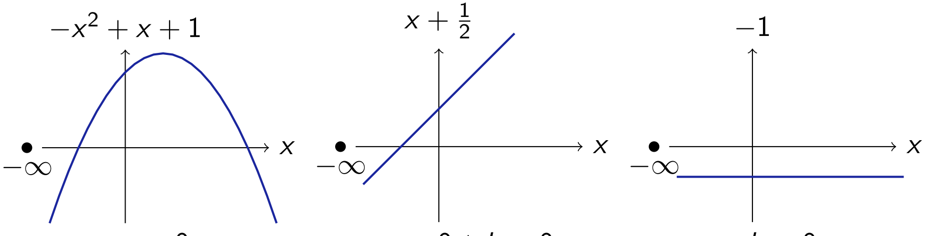

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$\mathbb{R} \cup \{-\infty, \infty\}$ extended reals

- Order: $\forall x\,(-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$?

$$
\begin{aligned}
\infty + x &= \\
-\infty + x &= \\
\infty \cdot x &= \\
\infty \cdot x &= \\
-\infty \cdot x &= \\
-\infty \cdot x &= \\
\infty - \infty &= \\
0 \cdot \infty &= \\
\pm\infty / \pm\infty &= \\
1/0 &=
\end{aligned}
$$

Expedition: Infinite Challenges with Infinities

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\infty + x &= \infty && \text{for all } x \neq -\infty \\
-\infty + x &= -\infty && \text{for all } x \neq \infty \\
\infty \cdot x &= \infty && \text{for all } x > 0 \\
\infty \cdot x &= -\infty && \text{for all } x < 0 \\
-\infty \cdot x &= -\infty && \text{for all } x > 0 \\
-\infty \cdot x &= \infty && \text{for all } x < 0 \\
\infty - \infty &= \text{undefined} && \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1 \\
0 \cdot \infty &= \\
\pm\infty / \pm\infty &= \\
1/0 &=
\end{aligned}
$$

Expedition: Infinite Challenges with Infinities

$\mathbb{R} \cup \{-\infty, \infty\}$ extended reals

- Order: $\forall x \, (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$? $\infty \leq \infty + x$ but $\infty + x \leq \infty$

$$
\begin{aligned}
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$$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x\,(ax^2 + bx + c < 0 \land F) \leftrightarrow$$

Quadratic Strict Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$
\begin{aligned}
& a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow \\
& \Big(\exists x \, (ax^2 + bx + c < 0 \land F) \leftrightarrow \\
& \quad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b} \\
& \quad \lor a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \big) \\
& \quad \lor \; (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty} \Big)
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strict inequality can't be satisfied at the roots but slightly off

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ε the rubber band number that's smaller in magnitude on any comparison

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Lemma (Virtual Substitution Lemma for ε)

$$F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

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Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $FOL_{\mathbb{R}[\varepsilon]}$ → $F_x^{e+\varepsilon} \equiv F_{\not{x}}^{e+\varepsilon}$ ← $FOL_{\mathbb{R}}$

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Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\mathrm{FOL}_{\mathbb{R}[\varepsilon]}$ $\qquad F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$ $\qquad \mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \textit{ iff } \omega \in [\![F_{\bar{x}}^{-\infty}]\!] \textit{ where } r \searrow e$$

Expedition: Infinitesimals

ε in nonstandard field extension $\mathbb{R}[\varepsilon]$ is "always as small as needed"

- Positive: $\varepsilon > 0$
- Smaller: $\forall x \in \mathbb{R}\,(x > 0 \rightarrow \varepsilon < x)$
- Standard $\mathbb{R}$ are Archimedean:
  $\forall x \in \mathbb{R} \setminus \{0\}\ \exists n \in \mathbb{N}\ |\underbrace{x + x + \cdots + x}_{n \text{ times}}| > 1$
- $\mathbb{R}[\varepsilon]$ are non-Archimedean: $\underbrace{\varepsilon + \varepsilon + \cdots + \varepsilon}_{\text{any } n \in \mathbb{N} \text{ times}} < 1$
- Infinitesimals as inverses of infinities?

$$\varepsilon \cdot \infty = 1? \quad -\varepsilon \cdot -\infty = 1? \quad (\varepsilon + 1) \cdot (\underbrace{\infty + 2}_{\infty?}) = \dots$$

- How to order?

$$\varepsilon^2 \qquad \varepsilon \qquad x^2 + \varepsilon \qquad (x + \varepsilon)^2 \qquad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

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$$\varepsilon^2 \quad < \quad \varepsilon \quad < \quad x^2 + \varepsilon \quad < \quad (x + \varepsilon)^2 \quad < \quad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

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Virtual Substitution of Infinitesimals

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Immediately negative at e $\quad \lim_{x \searrow e} p(x) < 0$

$$p^+ < 0 \overset{\text{def}}{\equiv} \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

ordinary virtual substitution into immediate negativity

Virtual Substitution of Infinitesimals

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Virtual Substitution of Infinitesimals

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Successive derivative p' immediately negative at root of p to break ties

$$(ax^2+bx+c)^+ < 0 \equiv ax^2+bx+c < 0$$
$$\lor\ ax^2 + bx + c = 0 \land (2ax + b < 0 \lor 2ax + b = 0 \land 2a<0)$$

$$(ax^2+bx+c<0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \equiv ((ax^2+bx+c)^+ < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}$$

$$(ax^2+bx+c<0 \lor ax^2+bx+c=0 \land (2ax+b<0 \lor 2ax+b=0 \land 2a<0))_{\bar{x}}^{(-b+\sqrt{b^2-}}$$

$$\equiv 0 \cdot 1<0 \lor 0=0 \land \Big(\underbrace{(0<0 \lor 4a^2\leq 0 \land (0<0 \lor -4a^2(b^2-4ac)<0))}_{(2ax+b<0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}}$$
$$\lor \underbrace{0=0}_{(2ax+b=0)_{\bar{x}}^{\cdots}} \land \underbrace{2a1 < 0}_{(2a<0)_{\bar{x}}^{\cdots}}\Big)$$

$$\equiv 4a^2\leq 0 \land -4a^2(b^2-4ac)<0 \lor 2a<0$$

$$\equiv$$

Example: Virtual Substitution of Infinitesimals

$$(ax^2+bx+c)^+ < 0 \equiv ax^2+bx+c < 0$$
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$$\equiv 4a^2\leq 0 \land -4a^2(b^2-4ac)<0 \lor 2a<0$$

$$\equiv a = 0 \land 0(b^2 - 0) < 0 \lor 2a < 0 \equiv 2a < 0$$

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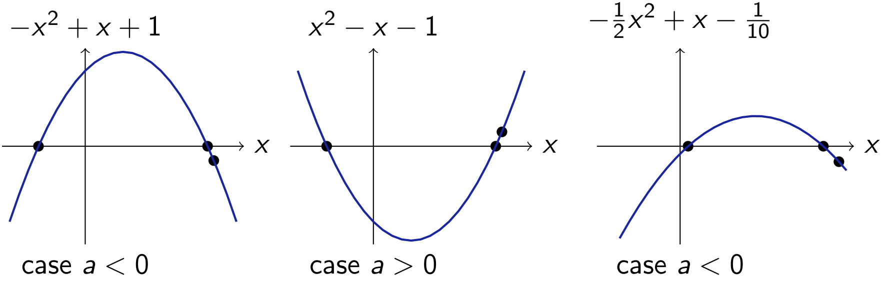
$$\equiv a = 0 \land 0(b^2 - 0) < 0 \lor 2a < 0 \equiv 2a < 0$$



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Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\mathrm{FOL}_{\mathbb{R}[\varepsilon]}$ → $F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$ ← $\mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{-\infty}]\!] \text{ where } r \searrow e$$

Outline

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$$\exists x\, F \leftrightarrow$$

$$F_x^{-\infty}$$

$$\lor \bigvee_{\left(ax^2+bx+c\left\{\substack{= \\ \leq}\right\}0\right)\in F} \left(a{=}0\land b{\neq}0 \land F_x^{-c/b} \lor a{\neq}0\land d{\geq}0 \land \left(F_x^{(-b+\sqrt{d})/(2a)} \lor F_x^{(-b-\sqrt{d})/(2a)}\right)\right)$$

$$\lor \bigvee_{\left(ax^2+bx+c\left\{\substack{\neq \\ <}\right\}0\right)\in F} \left(a{=}0\land b{\neq}0\land F_x^{-c/b+\varepsilon} \lor a{\neq}0\land d{\geq}0\land\left(F_x^{(-b+\sqrt{d})/(2a)+\varepsilon}\lor F_x^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)$$

Need roots and off-roots from all atomic formulas in F

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$$\exists x\, F \leftrightarrow$$

$$F_{\bar{x}}^{-\infty}$$

$$\lor \bigvee_{\left(ax^2+bx+c\left\{\substack{=\\ \leq}\right\}0\right)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\bar{x}}^{-c/b} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}\right)\right)$$

$$\lor \bigvee_{\left(ax^2+bx+c\left\{\substack{\neq\\ <}\right\}0\right)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\bar{x}}^{-c/b+\varepsilon} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)$$

Need roots and off-roots from all atomic formulas in F

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F *be of the form* $ax^2 + bx + c \sim 0$ *with* $x \notin a, b, c$ *and* $\sim \in \{=, \leq, <, \neq\}$ *and its discriminant* $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$$\exists x\, F \leftrightarrow$$

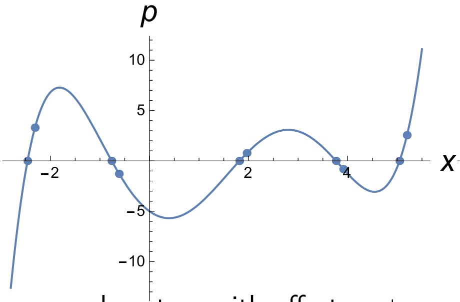
$$F_{\bar{x}}^{-\infty}$$

$$\lor \bigvee_{(ax^2+bx+c\,\{\substack{=\\ \leq}\}\,0)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\bar{x}}^{-c/b} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}\right)\right)$$

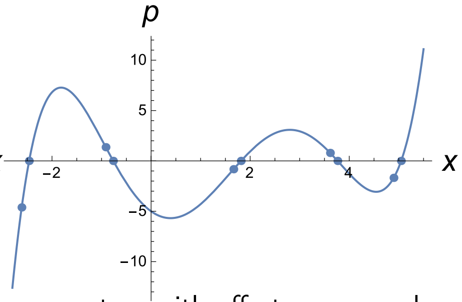
$$\lor \bigvee_{(ax^2+bx+c\,\{\substack{\neq\\ <}\}\,0)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\bar{x}}^{-c/b+\varepsilon} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)$$

Lemma (Virtual Substitution Lemmas)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \qquad F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty} \qquad F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$



$-\infty$ and roots e with offsets $e + \varepsilon$



roots e with offsets $e - \varepsilon$ and ∞

No rejection without mention

Other parts of F not satisfied by the points of p have their own polynomial q that contributes different roots $\tilde{e}$ and off-roots $\tilde{e} + \varepsilon$.

Generalizations of quantifier elimination to higher degrees also place a representative point into every region of interest, but derivatives and relationships of derivatives become crucially relevant.

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

$$
\begin{aligned}
\exists x\, F \leftrightarrow\ & F_{\not x}^{-\infty} \\
\lor & \bigvee_{(ax^2+bx+c \left\{\substack{= \\ \leq}\right\} 0)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\not x}^{-c/b} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\not x}^{(-b+\sqrt{d})/(2a)} \lor F_{\not x}^{(-b-\sqrt{d})/(2a)}\right)\right) \\
\lor & \bigvee_{(ax^2+bx+c \left\{\substack{\neq \\ <}\right\} 0)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\not x}^{-c/b+\varepsilon} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\not x}^{(-b+\sqrt{d})/(2a)+\varepsilon} \lor F_{\not x}^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)
\end{aligned}
$$

"Proof" Sketch.

"$\leftarrow$" simple from virtual substitution lemma with (extended) term witness

"$\rightarrow$" Valid iff true in every state, so all variables have real numeric value

- o-minimal: solutions form finite union of disjoint intervals (univariate)
- WLOG endpoints are the roots since all polynomials quadratic
- All side conditions have to be met otherwise can't be solution
- Non-point intervals contain ε offset since smaller than endpoint □

Outline

1. Learning Objectives
   - Recap: Quadratic Equations
2. Real Arithmetic
   - Quadratic Weak Inequalities
   - Virtual Substitution of Infinities
   - Expedition: Infinities
   - Quadratic Strict Inequalities
   - Infinitesimals
   - Virtual Substitution of Infinitesimals
3. Quantifier Elimination by Virtual Substitution
4. Summary

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$
\begin{aligned}
& a \neq 0 \lor b \neq 0 \lor c \neq 0 \to \\
& \quad \Big(\exists x \, (ax^2 + bx + c = 0 \land F) \leftrightarrow \\
& \qquad a = 0 \land b \neq 0 \land F_{\not x}^{-c/b} \\
& \qquad \lor\, a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\not x}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\not x}^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)
\end{aligned}
$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $F_x^{(a+b\sqrt{c})/d} \equiv F_{\not x}^{(a+b\sqrt{c})/d}$ $\text{FOL}_{\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ *iff* $\omega \in [\![F_{\not x}^{(a+b\sqrt{c})/d}]\!]$ *where* $r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/\omega[\![d]\!] \in \mathbb{R}$

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$$\exists x\, F \leftrightarrow$$

$$F_{\not{x}}^{-\infty}$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{=\\ \leq}\}0)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\not{x}}^{-c/b} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\not{x}}^{(-b+\sqrt{d})/(2a)} \lor F_{\not{x}}^{(-b-\sqrt{d})/(2a)}\right)\right)$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{\neq\\ <}\}0)\in F} \left(a{=}0 \land b{\neq}0 \land F_{\not{x}}^{-c/b+\varepsilon} \lor a{\neq}0 \land d{\geq}0 \land \left(F_{\not{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \lor F_{\not{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)$$

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Summary

- Miracle: $\text{FOL}_{\mathbb{R}}$ is decidable: Tarski'31
  (Complex) Algorithm says whether (closed) formula valid or not.
- Quantifier elimination effectively associates quantifier-free equivalent
- Successive quantifier elimination decides $\text{FOL}_{\mathbb{R}}$ (universal closure)
- QE accepts free variables, giving equivalent that identifies the requirements for truth
- Virtual substitution does QE for degree ≤ 3 by equivalent syntactic rephrasing of semantics Weispfenning'97
- QE proceeds inside out, so degree ≤ 3 needed on *each* iteration
- Important fragments permit many optimizations your research?
- Universally quantified weak inequalities / existentially quantified strict inequalities are easier since infinitesimals/infinities don't satisfy $=$
- Cylindrical algebraic decomposition (CAD) any degree Collins'75
- Simplify arithmetic to relevant parts, transform to fit together

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