Assignment 1: Introduction to Hybrid Programs
15-424/15-624 Foundations of Cyber-Physical Systems
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Total Points: 60

1. **Term, formula, hybrid program, or none of the above?**
For each of the following, determine if the expression is a d$\mathcal{L}$ term, a well-formed d$\mathcal{L}$ formula, a well-formed hybrid program, or none of the above (ie. it is not well-formed). In the case that the expression is none of the above, give a short explanation.
   (a) $?(3 > 2)$
   (b) $[x := 1; \{x'' = x\}](-1 \leq x \wedge x \leq 1) \leftrightarrow [y := \cos z](-1 \leq y \wedge y \leq 1)$
   (c) $x = 42$
   (d) $x := y \cup x := z$
   (e) $(x > y)^*$
   (f) 42
   (g) $L > 0 \wedge U > 0 \rightarrow [(x := *; ?(L < x \wedge x < U))^*](x = y)$
   (h) x
   (i) $y := \pi$
   (j) $(x > y \wedge (y > z \cup y > w))$
   (k) $x := 0 \rightarrow [\{x' = 1, y' = 1 \;\&\; y \leq 10\}]x \leq 10$
   (l) $(x := x + 1)^*$
   (m) $\forall x \exists y\ (x < y)$
   (n) $[x := 42]$
   (o) $x := y^2$
   (p) $y > 1 \wedge z > 1 \rightarrow [x := z^y](x > z)$
   (q) $[x = y; y = z]?(x \geq z)$

2. **Practice writing hybrid programs.**
   (a) The if-then-else construct is not actually in the grammar of hybrid programs; it's just syntactic sugar. Write an equivalent hybrid program which does not depend on if-then-else:

$$\texttt{if}\ (x^2 \geq 5 \wedge x \leq 0)\ x := x - 1\ \ \texttt{else}\ x := x + 1$$

   (b) Non-deterministic Choice: Write a hybrid program that assigns either a or b to the variable x.

3. **Safety and Contracts**

(a) A *safety* property is something that a cyber-physical system should *always* maintain. Describe a cyber-physical system and then name three safety properties that it should never violate.

(b) Suppose you want to prove that a property ϕ is a safety property of hybrid program α, i.e. that ϕ holds under all possible runs of hybrid program α. Write the $\mathsf{d}\mathcal{L}$ formula that expresses this.

(c) Now, suppose you want to prove that, given initial conditions ψ, another property ϕ is a safety property of hybrid program α. Write the $\mathsf{d}\mathcal{L}$ formula that expresses this.

(d) Consider hybrid program α:

$$\alpha \equiv x := w; ((y := 3; z := 20) \cup (z := 6; y := w + 7)); \{x' = 10\}; ?(y \leq 30)$$

For each variable in α, list the set of all values they can reach at the end of a run of α.

(e) Come up with at least two properties that hold at the end of all runs of α (example: $y \leq 30$ is a property that the program always satisfies, but now you can't use upper bounds on y). Write down a $\mathsf{d}\mathcal{L}$ formula which is true iff hybrid program α always satisfies those properties.

4. **Non-Deterministic Evolution**

$$\beta \equiv \quad x := x_0;\ v := v_0;\ t := 0;\ \{x' = v, v' = a, t' = 1 \ \&\ v \geq 0\}; ?(v = 0)$$

(a) Assume that $a < 0 \land v_0 \geq 0$. At the end of a run of hybrid program β, what is the value of t as a function of x_0, v_0, and a?

(b) Suppose we remove the guard $?(v = 0)$ at the end of hybrid program β. Again assuming that $a < 0 \land v_0 \geq 0$, what are the possible values of t at the end of any run of this modified version of β?

(c) Suppose we assume instead that $a < 0 \land v_0 \leq 0$ (v_0 is **less than** or equal to zero). What are the possible values of t at the end of any run of β?