15-424/15-624 Background Quiz Solutions

1. **First-Order Real Arithmetic**

   Recall that a logical formula is

   - *valid* if it is true for all possible assignments of free variables,
   - *satisfiable* if it is true for at least one assignment of free variables, and
   - *unsatisfiable* if it is not true for any assignment of free variables.

   In the following, determine if the statements are *valid*, *satisfiable*, **and/or** *unsatisfiable*.

   (a) $\frac{5}{2} < x \land x < 2$
   unsatisfiable

   (b) $2 < x \land x < \frac{5}{2}$
   satisfiable, but not valid for all x

   (c) $(x < y \land y < z) \rightarrow x < z$
   valid and therefore satisfiable

   (d) $x < z \land \exists y(x < y \land y < z)$
   satisfiable, but not valid for all x and z

   (e) $\exists y(x < y)$
   valid and satisfiable, no matter the value of x, there exists a y that is bigger

   (f) $\forall y(x < y)$
   unsatisfiable, there is no x such that it is smaller than all y

   (g) $(x > y \rightarrow x > z) \lor x > y$
   valid and satisfiable

   (h) $x > y \leftrightarrow x^2 > y^2$
   satisfiable

2. **Differential Equations**

Solve the following IVPs. All derivatives are taken with respect to implicit variable t.

(a)

$$\begin{bmatrix} x' & = & v \\ v' & = & a \\ x(0) & = & x_0 \\ v(0) & = & v_0 \end{bmatrix}$$

$x(t) = \frac{a}{2}t^2 + v_0 t + x_0$
$v(t) = at + v_0$

(b)

$$\begin{bmatrix} x' & = & -y \\ y' & = & x \\ x(0) & = & 0 \\ y(0) & = & 1 \end{bmatrix}$$

$x(t) = -\sin(t)$
$y(t) = \cos(t)$

Note: This solution can be written in multiple ways.

(c)

$$\begin{bmatrix} x' & = & x\cos t \\ x(0) & = & x_0 \end{bmatrix}$$

You can also write x' as $\frac{dx}{dt}$, and then use the following trick:

$$\frac{dx}{dt} = x\cos t$$
$$\frac{1}{x}dx = \cos t dt$$

$$\int \frac{1}{x}dx = \int \cos t dt$$
$$\ln x = \sin t + c$$
$$x = e^{\sin t + c} = ce^{\sin t}$$

But be careful with the constants; the solution to the initial value problem is:

$$x = x_0 e^{\sin t}$$

A common mistake:

$$x = e^{\sin t} - 1 + x_0$$
$$x' = \cos t \cdot e^{\sin t} \neq x\cos t$$