15-819/18-879: Logical Analysis of Hybrid Systems

16: Logical Proof Rules

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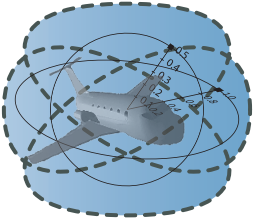

Outline

1. Propositional & First-Order Proof Rules
   - Simple Example Proof
   - Propositional Sequent Calculus
   - Context-free Short Notation
   - Classical First-Order Logic Sequent Calculus

2. Quantifier Proof Rules in d$\mathcal{L}$
   - Deduction Modulo by Side Deduction
   - Deduction Modulo with Free Variables & Skolemization

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A Very Simple Example Proof (propositional)

How do we prove formulas? What about $\land, \lor, \rightarrow$...?

$$\frac{}{\vdash v^2 \leq 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10}$$

A Very Simple Example Proof (propositional)

How do we prove formulas? What about $\wedge, \vee, \rightarrow$...?

$$\dfrac{v^2 \leq 10 \wedge b > 0 \vdash b > 0 \wedge (\neg(v \geq 0) \vee v^2 \leq 10)}{\vdash v^2 \leq 10 \wedge b > 0 \rightarrow b > 0 \wedge (\neg(v \geq 0) \vee v^2 \leq 10}$$

How do we prove formulas? What about $\land, \lor, \rightarrow$...?

$$\dfrac{\dfrac{\dfrac{}{v^2 \le 10 \land b > 0 \vdash b > 0} \qquad \dfrac{}{v^2 \le 10 \land b > 0 \vdash \neg(v \ge 0) \lor v^2 \le 10}}{v^2 \le 10 \land b > 0 \vdash b > 0 \land (\neg(v \ge 0) \lor v^2 \le 10)}}{\vdash v^2 \le 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \ge 0) \lor v^2 \le 10}$$

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How do we prove formulas? What about $\land, \lor, \rightarrow$...?

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v^2 \leq 10, b > 0 \vdash b > 0}}{v^2 \leq 10 \land b > 0 \vdash b > 0} \qquad \dfrac{}{v^2 \leq 10 \land b > 0 \vdash \neg(v \geq 0) \lor v^2 \leq 10}}{v^2 \leq 10 \land b > 0 \vdash b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}}{\vdash v^2 \leq 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10}$$

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Sequent Calculus: Motivation

- How did we do this proof?
- What is the general principle?
- Successively unpack formula and flatten operators.
- Sequents: A “general” notation for proof calculi
- Introduced by Gerhard Gentzen 1935 as a tool for studying natural deduction.
- Sequent calculus intimately related to Tableau calculus.

Sequents

A normal form for formulas: $\approx$ assume all left, show one right

Definition (Sequent)

A *sequent* is a pair of finite sets of formulas, with the notation

$$\Gamma \vdash \Delta$$

Γ is called antecedent and Δ succedent. Empty sets are allowed for Γ, Δ and denoted without set brackets.

Definition (Semantics of sequents)

$$[\![\Gamma \vdash \Delta]\!]_{I,v,\eta} = [\![\bigwedge_{G\in\Gamma} G \to \bigvee_{F\in\Delta} F]\!]_{I,v,\eta}$$

where empty conjunctions are *true* and empty disjunctions *false*

Sequent Calculus: Propositional Rules

$$\overline{\Gamma, \phi \land \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \land \psi \vdash \Delta}$$

Sequent Calculus: Propositional Rules

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$$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

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Sequent Calculus: "Context-free" Short Notation

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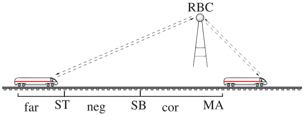
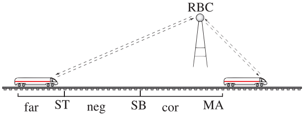
Deduction Modulo Real Arithmetic



$$\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b \rangle \; z > MA$$

Deduction Modulo Real Arithmetic



$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z \rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b \rangle z > MA}}{\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b \rangle\, z > MA}$$

Sequent Calculus: Classical First-Order Logic

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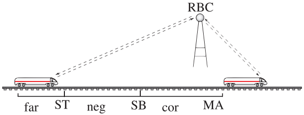
$X \notin \Gamma, \Delta$ new variable

Outline

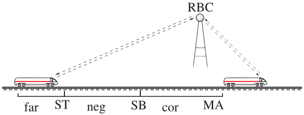
1. Propositional & First-Order Proof Rules
   - Simple Example Proof
   - Propositional Sequent Calculus
   - Context-free Short Notation
   - Classical First-Order Logic Sequent Calculus
2. Quantifier Proof Rules in d$\mathcal{L}$
   - Deduction Modulo by Side Deduction
   - Deduction Modulo with Free Variables & Skolemization

Deduction Modulo Real Arithmetic



$$\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b \rangle\ z > MA$$

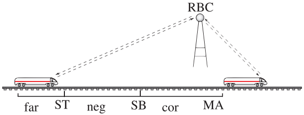
Deduction Modulo Real Arithmetic



$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle\, z > MA}$$

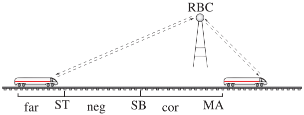
$\exists x \;\; x^2 + 1 < 0$

quantifiers by FOL rules?



$$\dfrac{\dfrac{\overline{v \geq 0, z < MA \vdash \exists t \geq 0 \, \langle z := -\frac{b}{2}t^2 + vt + z \rangle z > MA}}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b \rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b \rangle \; z > MA}$$

FOL: $\exists x \; l(a(m(x,x),1),0)$



uninterpreted FOL

$$\cfrac{\cfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle\; z > MA}$$

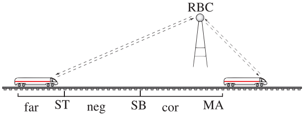
$\text{FOL}_{\mathbb{R}}$: $\exists x\ l(a(m(x,x),1),0)$

$\downarrow\downarrow\downarrow\quad\downarrow\ \downarrow$

$<+\cdot\quad 1\ \ 0$

interpreted $\text{FOL}_{\mathbb{R}}$

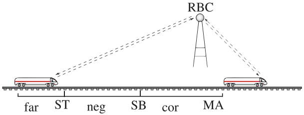
??



$$\cfrac{\cfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle\ z > MA}$$

Deduction Modulo Real Arithmetic

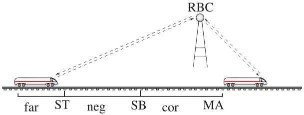
$\text{FOL}_{\mathbb{R}}$: $\exists x\ x^2+1<0$ false ✓



interpreted $\text{FOL}_{\mathbb{R}}$ by QE in RCF

$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle\ z > MA}$$

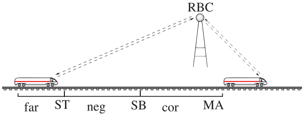
Deduction Modulo Real Arithmetic



Collins/Tarski QE not applicable!

$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle\ z > MA}$$

Deduction Modulo (Side Deduction)

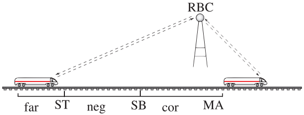


$$\dfrac{}{v \geq 0, z < MA \vdash t \geq 0 \land \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}$$

start side

$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b\rangle\, z > MA}$$

Deduction Modulo (Side Deduction)

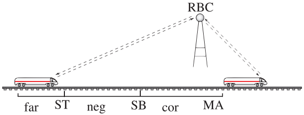


$$\dfrac{v \geq 0, z < MA \vdash t \geq 0 \qquad \dfrac{v \geq 0, z < MA \vdash -\frac{b}{2}t^2 + vt + z > MA}{v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}}{v \geq 0, z < MA \vdash t \geq 0 \land \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}$$

start
side

$$\dfrac{\dfrac{\dfrac{}{v \geq 0, z < MA \vdash \exists t \geq 0 \, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle \, z > MA}$$

Deduction Modulo (Side Deduction)



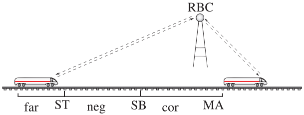
$$\dfrac{v \geq 0, z < MA \vdash t{\geq}0 \qquad \dfrac{v \geq 0, z < MA \vdash -\frac{b}{2}t^2 + vt + z > MA}{v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA}}{v \geq 0, z < MA \vdash t \geq 0 \land \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}$$

QE

start side

$$\dfrac{\dfrac{\dfrac{v \geq 0, z < MA \vdash \mathrm{QE}\big(\exists t\,(\ldots t{\geq}0 \land -\frac{b}{2}t^2 + vt + z > MA)\big)}{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b\rangle\, z > MA}$$

Deduction Modulo (Side Deduction)

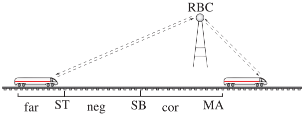


$$\dfrac{v \geq 0, z < MA \vdash {\color{red}t}{\geq}0 \qquad \dfrac{v \geq 0, z < MA \vdash -\frac{b}{2}{\color{red}t}^2 + v{\color{red}t} + z > MA}{v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA}}{v \geq 0, z < MA \vdash t \geq 0 \wedge \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}$$

QE

start side

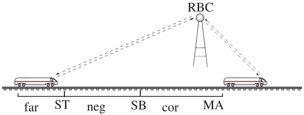
$$\dfrac{\dfrac{\dfrac{v \geq 0, z < MA \vdash {\color{red}v^2 > 2b(MA - z)}}{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA}}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \wedge z < MA \to \langle z' = v, v' = -b\rangle\, z > MA}$$



$$v \geq 0, z < MA \vdash \forall t{\geq}0\, [z := -\tfrac{b}{2}t^2 + vt + z]z{<}MA$$

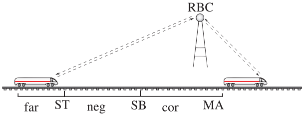
$$v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA$$

$$\vdash v \geq 0 \land z < MA \rightarrow [z' = v, v' = -b]z < MA$$



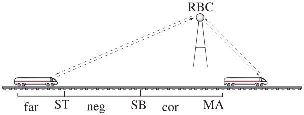
$$\dfrac{\dfrac{\dfrac{\dfrac{}{v \geq 0, z < MA \vdash T \geq 0 \to [z := -\frac{b}{2}T^2 + vT + z]z < MA}}{v \geq 0, z < MA \vdash \forall t{\geq}0\, [z := -\frac{b}{2}t^2 + vt + z]z{<}MA}}{v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA}}{\vdash v \geq 0 \land z < MA \to [z' = v, v' = -b]z < MA}$$

Deduction Modulo (Free Variables for Automation): ∀

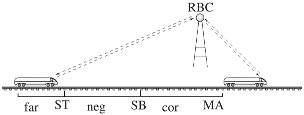


$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v \geq 0, z < MA, T{\geq}0 \vdash -\frac{b}{2}T^2 + vT + z < MA}{v \geq 0, z < MA, T{\geq}0 \vdash [z := -\frac{b}{2}T^2 + vT + z]z < MA}}{v \geq 0, z < MA \vdash T \geq 0 \to [z := -\frac{b}{2}T^2 + vT + z]z < MA}}{v \geq 0, z < MA \vdash \forall t{\geq}0\,[z := -\frac{b}{2}t^2 + vt + z]z{<}MA}}{v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA}}{\vdash v \geq 0 \land z < MA \to [z' = v, v' = -b]z < MA}$$

Deduction Modulo (Free Variables for Automation): ∀

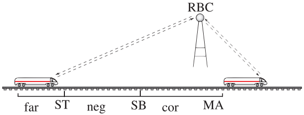


$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{v \geq 0, z < MA \vdash \quad \forall T\,(\ldots T{\geq}0 \to -\frac{b}{2}T^2 + v\,T + z < MA)}{v \geq 0, z < MA, T{\geq}0 \vdash -\frac{b}{2}T^2 + v\,T + z < MA}}{v \geq 0, z < MA, T{\geq}0 \vdash [z := -\frac{b}{2}T^2 + v\,T + z]z < MA}}{v \geq 0, z < MA \vdash T \geq 0 \to [z := -\frac{b}{2}T^2 + v\,T + z]z < MA}}{v \geq 0, z < MA \vdash \forall t{\geq}0\,[z := -\frac{b}{2}t^2 + vt + z]z{<}MA}}{v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA}}{\vdash v \geq 0 \land z < MA \to [z' = v, v' = -b]z < MA}
$$



$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{v \geq 0, z < MA \vdash \mathrm{QE}\big(\forall T\,(\ldots T{\geq}0 \to -\frac{b}{2}T^2 + vT + z < MA)\big)}{v \geq 0, z < MA, T{\geq}0 \vdash -\frac{b}{2}T^2 + vT + z < MA}}{v \geq 0, z < MA, T{\geq}0 \vdash [z := -\frac{b}{2}T^2 + vT + z]z < MA}}{v \geq 0, z < MA \vdash T \geq 0 \to [z := -\frac{b}{2}T^2 + vT + z]z < MA}}{v \geq 0, z < MA \vdash \forall t{\geq}0\,[z := -\frac{b}{2}t^2 + vt + z]z{<}MA}}{v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA}}{\vdash v \geq 0 \land z < MA \to [z' = v, v' = -b]z < MA}$$

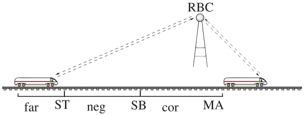
Deduction Modulo (Free Variables for Automation): $\forall$



$$
\begin{array}{c}
v \geq 0, z < MA \vdash v^2 < 2b(MA - z) \\
\hline
v \geq 0, z < MA, T{\geq}0 \vdash -\frac{b}{2} T^2 + vT + z < MA \\
\hline
v \geq 0, z < MA, T{\geq}0 \vdash [z := -\frac{b}{2} T^2 + vT + z] z < MA \\
\hline
v \geq 0, z < MA \vdash T \geq 0 \rightarrow [z := -\frac{b}{2} T^2 + vT + z] z < MA \\
\hline
v \geq 0, z < MA \vdash \forall t{\geq}0\, [z := -\frac{b}{2} t^2 + vt + z] z{<}MA \\
\hline
v \geq 0, z < MA \vdash [z' = v, v' = -b] z > MA \\
\hline
\vdash v \geq 0 \land z < MA \rightarrow [z' = v, v' = -b] z < MA
\end{array}
$$

Deduction Modulo (Free Variables for Automation): $\forall$

- For requantification, not for unification



$$v \geq 0, z < MA \vdash \mathrm{QE}\left(\forall T\,(\ldots T \geq 0 \rightarrow -\frac{b}{2}T^2 + vT + z < MA)\right)$$

$$v \geq 0, z < MA, T \geq 0 \vdash -\frac{b}{2}T^2 + vT + z < MA$$

$$v \geq 0, z < MA, T \geq 0 \vdash [z := -\frac{b}{2}T^2 + vT + z]z < MA$$

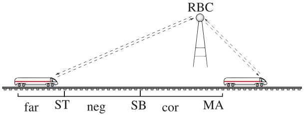
$$v \geq 0, z < MA \vdash T \geq 0 \rightarrow [z := -\frac{b}{2}T^2 + vT + z]z < MA$$

$$v \geq 0, z < MA \vdash \forall t \geq 0\,[z := -\frac{b}{2}t^2 + vt + z]z < MA$$

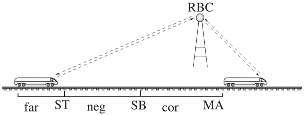
$$v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA$$

$$\vdash v \geq 0 \land z < MA \rightarrow [z' = v, v' = -b]z < MA$$

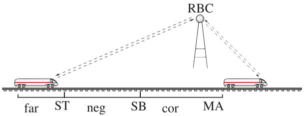
Deduction Modulo (Free Variables for Automation): ∀|∀



$$\dfrac{\dfrac{\dfrac{}{v \geq 0, z < MA \vdash \forall {\color{red}t}{\geq}0\, [z := -\frac{b}{2}t^2 + vt + z] z{<}MA \ldots}}{v \geq 0, z < MA \vdash [z' = v, v' = -b] z > MA}}{\vdash v \geq 0 \wedge z < MA \rightarrow [z' = v, v' = -b] z < MA\,.}$$



$$\dfrac{\dfrac{\dfrac{v \geq 0, z < MA \vdash T^2 \geq 0 \wedge [z := -\frac{b}{2} T^2 + v T + z] z < MA}{v \geq 0, z < MA \vdash \forall t {\geq} 0\, [z := -\frac{b}{2} t^2 + vt + z] z {<} MA \ldots}}{v \geq 0, z < MA \vdash [z' = v, v' = -b] z > MA}}{\vdash v \geq 0 \wedge z < MA \rightarrow [z' = v, v' = -b] z < MA\,.}$$



$$v \geq 0, z < MA \vdash -\tfrac{b}{2}T^2 + vT + z < MA$$

$$v \geq 0, z < MA \vdash T^2 \geq 0 \qquad v \geq 0, z < MA \vdash [z := -\tfrac{b}{2}T^2 + vT + z]z < M$$

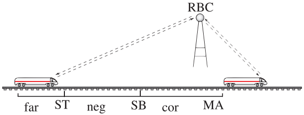
$$v \geq 0, z < MA \vdash T^2 \geq 0 \land [z := -\tfrac{b}{2}T^2 + vT + z]z < MA$$

$$v \geq 0, z < MA \vdash \forall t{\geq}0\, [z := -\tfrac{b}{2}t^2 + vt + z]z{<}MA \ldots$$

$$v \geq 0, z < MA \vdash [z' = v, v' = -b]z > MA$$

$$\vdash v \geq 0 \land z < MA \rightarrow [z' = v, v' = -b]z < MA\,.$$

Deduction Modulo (Free Variables for Automation): $\forall|\forall$



$$v \geq 0, z < MA \vdash \mathrm{QE}\big(\forall T\, (-\tfrac{b}{2} T^2 + v T + z <$$

$$v \geq 0, z < MA \vdash -\tfrac{b}{2} T^2 + v T + z < MA$$

$$v \geq 0, z < MA \vdash [z := -\tfrac{b}{2} T^2 + v T + z] z <$$

$$v \geq 0, z < MA \vdash \mathrm{QE}(\dots)$$

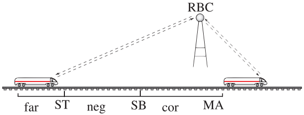
$$v \geq 0, z < MA \vdash T^2 \geq 0$$

$$v \geq 0, z < MA \vdash T^2 \geq 0 \land [z := -\tfrac{b}{2} T^2 + v T + z] z < MA$$

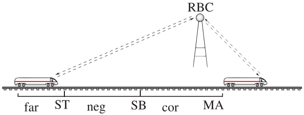
$$v \geq 0, z < MA \vdash \forall t \geq 0\, [z := -\tfrac{b}{2} t^2 + vt + z] z < MA \dots$$

$$v \geq 0, z < MA \vdash [z' = v, v' = -b] z > MA$$

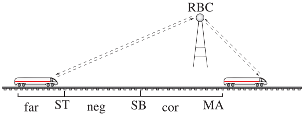
$$\vdash v \geq 0 \land z < MA \to [z' = v, v' = -b] z < MA\,.$$



$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA \ldots}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle z > MA\,.}$$



$$
\dfrac{\dfrac{\dfrac{v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\frac{b}{2} T^2 + vT + z \rangle z > MA}{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2} t^2 + vt + z \rangle z{>}MA \ldots}}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b \rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b \rangle z > MA\,.}
$$



$$v \geq 0, z < MA \vdash -\frac{b}{2}T^2 + vT + z > MA$$

$$v \geq 0, z < MA \vdash T \geq 0 \qquad v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}T^2 + vT + z\rangle z >$$

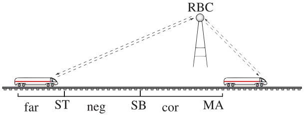
$$v \geq 0, z < MA \vdash T \geq 0 \wedge \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA \ldots$$

$$v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA$$

$$\vdash v \geq 0 \wedge z < MA \to \langle z' = v, v' = -b\rangle z > MA$$

Deduction Modulo (Free Variables for Automation): $\exists | \exists$



$$v \geq 0, z < MA \vdash \mathrm{QE}\big(\exists T\, (-\tfrac{b}{2} T^2 + vT + z >$$

$$v \geq 0, z < MA \vdash \mathrm{QE}(\dots) \qquad v \geq 0, z < MA \vdash -\tfrac{b}{2} T^2 + vT + z > MA$$

$$v \geq 0, z < MA \vdash T \geq 0 \qquad v \geq 0, z < MA \vdash \langle z := -\tfrac{b}{2} T^2 + vT + z\rangle z >$$

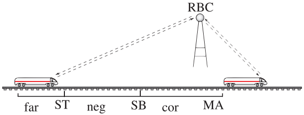
$$v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\tfrac{b}{2} T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\tfrac{b}{2} t^2 + vt + z\rangle z{>}MA \dots$$

$$v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA$$

$$\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle z > MA .$$

Deduction Modulo (Free Variables for Automation): $\exists|\exists$



$$v \geq 0, z < MA \vdash \mathrm{QE}(\exists T\,(-\tfrac{b}{2}T^2 + vT + z >$$

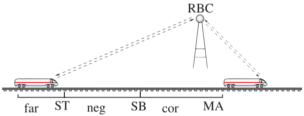
$$\frac{v \geq 0, z < MA \vdash \cancel{\mathrm{QE}}(\ldots)}{v \geq 0, z < MA \vdash T \geq 0} \qquad \frac{v \geq 0, z < MA \vdash -\tfrac{b}{2}T^2 + vT + z > MA}{v \geq 0, z < MA \vdash \langle z := -\tfrac{b}{2}T^2 + vT + z\rangle z >}$$

$$v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\tfrac{b}{2}T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\tfrac{b}{2}t^2 + vt + z\rangle z{>}MA \ldots$$

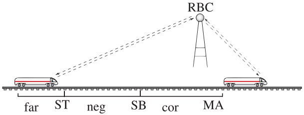
$$v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA$$

$$\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b\rangle z > MA\,.$$

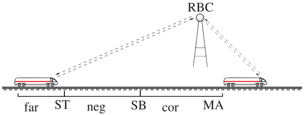


$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle z > MA}$$

Deduction Modulo (Free Variables for Automation): $\exists$



$$\dfrac{\dfrac{\dfrac{\dfrac{}{v \geq 0, z < MA \vdash {\color{red}T} \geq 0 \land \langle z := -\frac{b}{2}{\color{red}T}^2 + v{\color{red}T} + z\rangle z > MA}}{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA}}{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}}{\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b\rangle z > MA}$$



$$v \geq 0, z < MA \vdash -\frac{b}{2}T^2 + vT + z > MA$$

$$v \geq 0, z < MA \vdash T \geq 0 \qquad v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > MA$$

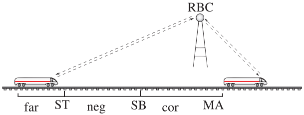
$$v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA$$

$$\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle z > MA$$

Deduction Modulo (Free Variables for Automation): $\exists$



$$v \geq 0, z < MA \vdash \quad \exists T\,(\ldots T{\geq}0 \land -\tfrac{b}{2}T^2 + vT + z > MA)$$

$$v \geq 0, z < MA \vdash -\tfrac{b}{2}T^2 + vT + z > MA$$

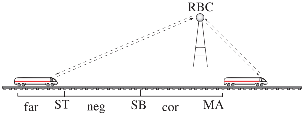
$$v \geq 0, z < MA \vdash T{\geq}0 \qquad v \geq 0, z < MA \vdash \langle z := -\tfrac{b}{2}T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\tfrac{b}{2}T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\tfrac{b}{2}t^2 + vt + z\rangle z{>}MA$$

$$v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA$$

$$\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b\rangle z > MA$$



$$v \geq 0, z < MA \vdash \mathrm{QE}\Big(\exists T\, (\ldots T{\geq}0 \land -\tfrac{b}{2}T^2 + vT + z > MA)\Big)$$

$$v \geq 0, z < MA \vdash -\tfrac{b}{2}T^2 + vT + z > MA$$

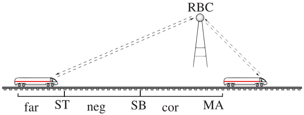
$$v \geq 0, z < MA \vdash T{\geq}0 \qquad v \geq 0, z < MA \vdash \langle z := -\tfrac{b}{2}T^2 + vT + z\rangle z > M$$

$$v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\tfrac{b}{2}T^2 + vT + z\rangle z > MA$$

$$v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\tfrac{b}{2}t^2 + vt + z\rangle z{>}MA$$

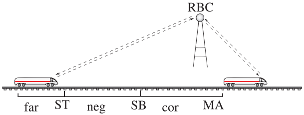
$$v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA$$

$$\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle z > MA$$



$$\dfrac{\dfrac{\dfrac{v \geq 0, z < MA \vdash \textcolor{red}{v^2 > 2b(MA - z)}}{v \geq 0, z < MA \vdash \textcolor{red}{T} \geq 0 \qquad \dfrac{v \geq 0, z < MA \vdash -\frac{b}{2}\textcolor{red}{T}^2 + v\textcolor{red}{T} + z > MA}{v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}\textcolor{red}{T}^2 + v\textcolor{red}{T} + z\rangle z > MA}}}{v \geq 0, z < MA \vdash \textcolor{red}{T} \geq 0 \land \langle z := -\frac{b}{2}\textcolor{red}{T}^2 + v\textcolor{red}{T} + z\rangle z > MA}}{\dfrac{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA}{\dfrac{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}{\vdash v \geq 0 \land z < MA \to \langle z' = v, v' = -b\rangle z > MA}}}$$

Deduction Modulo (Free Variables for Automation): $\exists$



- For requantification, not for unification

$$\dfrac{\dfrac{v \geq 0, z < MA \vdash \mathrm{QE}\left(\exists T\, (\ldots T{\geq}0 \land -\frac{b}{2}T^2 + vT + z > MA)\right)}{v \geq 0, z < MA \vdash -\frac{b}{2}T^2 + vT + z > MA}}{}$$

$$\dfrac{\dfrac{\dfrac{v \geq 0, z < MA \vdash T{\geq}0 \qquad \dfrac{v \geq 0, z < MA \vdash -\frac{b}{2}T^2 + vT + z > MA}{v \geq 0, z < MA \vdash \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > MA}}{v \geq 0, z < MA \vdash T \geq 0 \land \langle z := -\frac{b}{2}T^2 + vT + z\rangle z > MA}}{v \geq 0, z < MA \vdash \exists t{\geq}0\, \langle z := -\frac{b}{2}t^2 + vt + z\rangle z{>}MA}}{\dfrac{v \geq 0, z < MA \vdash \langle z' = v, v' = -b\rangle z > MA}{\vdash v \geq 0 \land z < MA \rightarrow \langle z' = v, v' = -b\rangle z > MA}}$$

Deduction Modulo (Free Variables for Automation)

$$\dfrac{\dfrac{\dfrac{}{\vdash (X < S)}}{\vdash \forall s\, (X < s)}}{\vdash \exists x\, \forall s\, (x < s)}$$

Deduction Modulo (Free Variables for Automation)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathrm{QE}(\forall S \exists X (X < S))}}{\vdash (X < S)}}{\vdash \forall s\, (X < s)}}{\vdash \exists x\, \forall s\, (x < s)}$$

Deduction Modulo (Free Variables for Automation)

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathrm{QE}(\forall S \exists X (X < S)) \qquad \vdash \mathrm{QE}(\exists X \forall S (X < S))}{\vdash (X < S)}}{\vdash \forall s\, (X < s)}}{\vdash \exists x\, \forall s\, (x < s)}}{}$$

Deduction Modulo (Free Variables for Automation)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{true}{\vdash \mathrm{QE}(\forall S \exists X (X < S))} \qquad \dfrac{false}{\vdash \mathrm{QE}(\exists X \forall S (X < S))}}{\vdash (X < S)}}{\vdash \forall s\,(X < s)}}{\vdash \exists x\, \forall s\,(x < s)}}{false!}
$$

Deduction Modulo (Free Variables for Automation)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{true}}{\vdash \mathrm{QE}(\cancel{\forall S \exists X}(X < S))} \qquad \dfrac{\textit{false}}{\vdash \mathrm{QE}(\exists X \forall S (X < S))}}{\vdash (X < S)}}{\vdash \forall s\,(X < s)}}{\vdash \exists x\,\forall s\,(x < s)}}{\textit{false}!}
$$

$\boxed{\text{Skolemisation } S(X)}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathit{false}}{\vdash \mathrm{QE}(\exists X \forall S\,(X < S))}}{\vdash (X < S(X))}}{\vdash \forall s\,(X < s)}}{\vdash \exists x\,\forall s\,(x < s)}
$$

false!