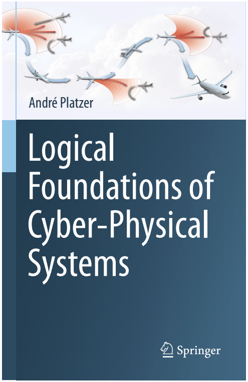
15: Winning Strategies & Regions

Logical Foundations of Cyber-Physical Systems



André Platzer

Karlsruhe Institute of Technology
Department of Informatics

Computer Science Department
Carnegie Mellon University

Outline

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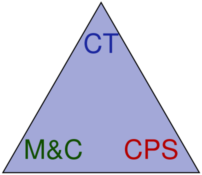
Learning Objectives
Winning Strategies & Regions

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics



adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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All Reals | Some Reals

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"Angel has Wings $\langle \alpha \rangle$"

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Outline

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Differential Game Logic: Denotational Semantics

Only for HPs. No interactive play!

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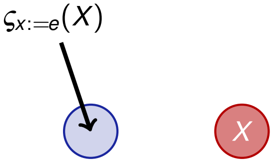

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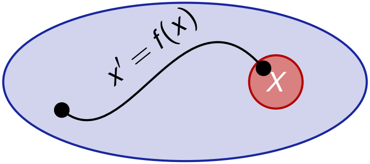

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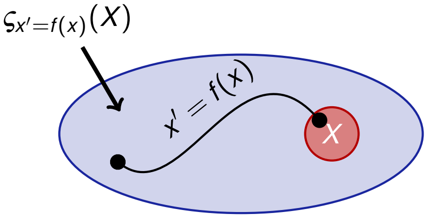

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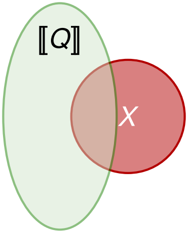

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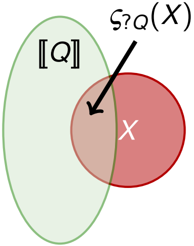

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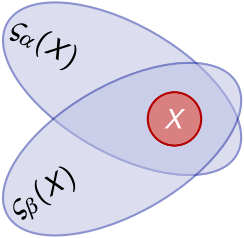

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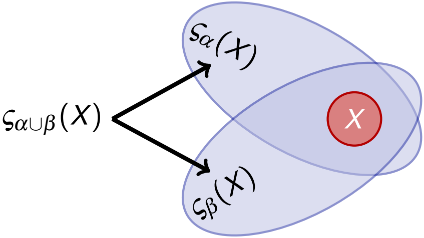

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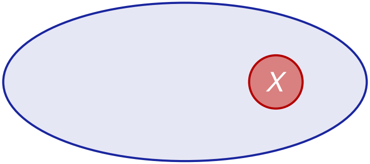

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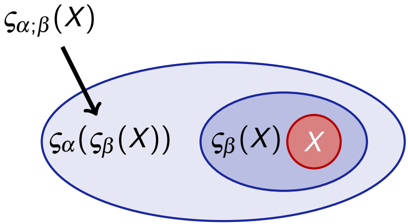

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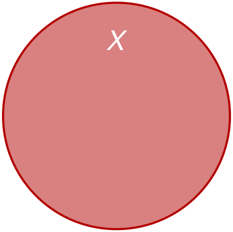

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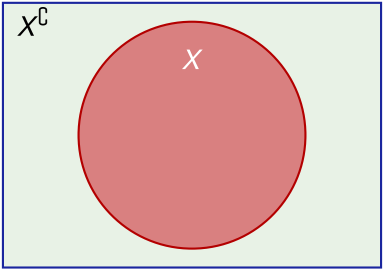

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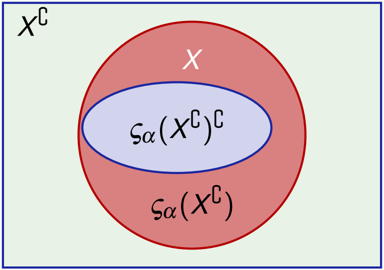

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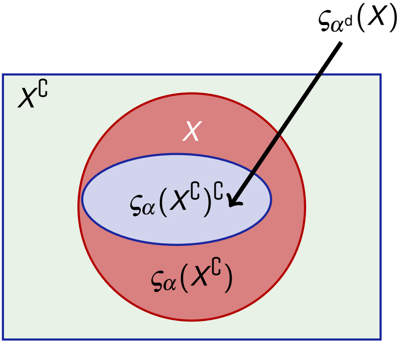

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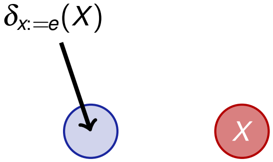

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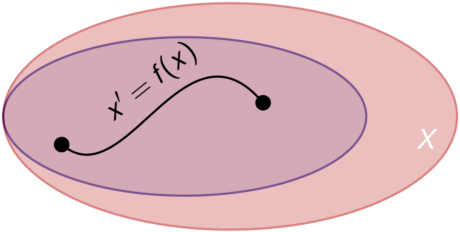

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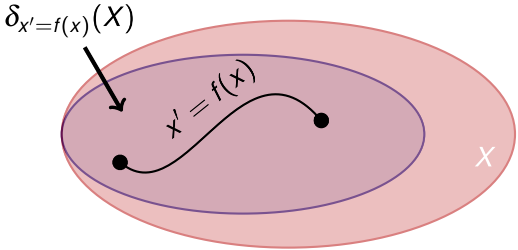

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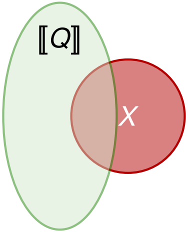

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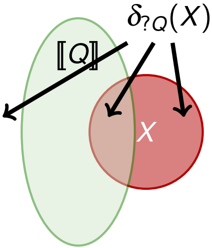

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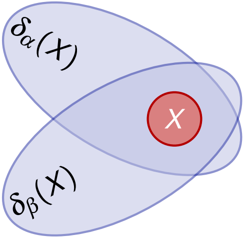

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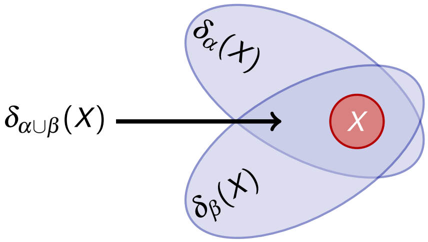

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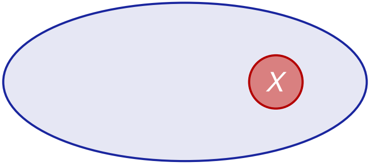

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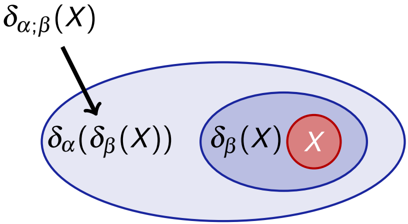

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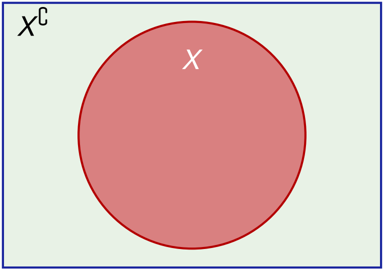

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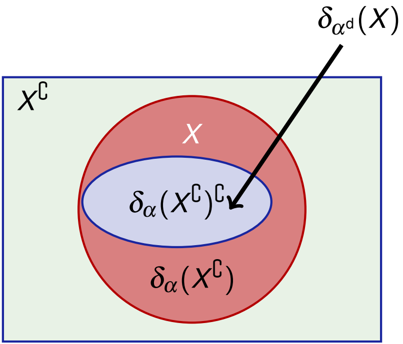

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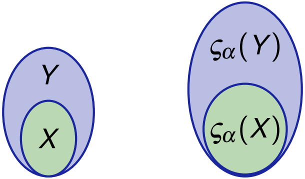
Monotonicity

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y$$



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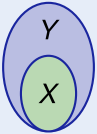
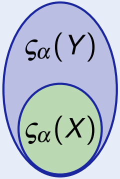
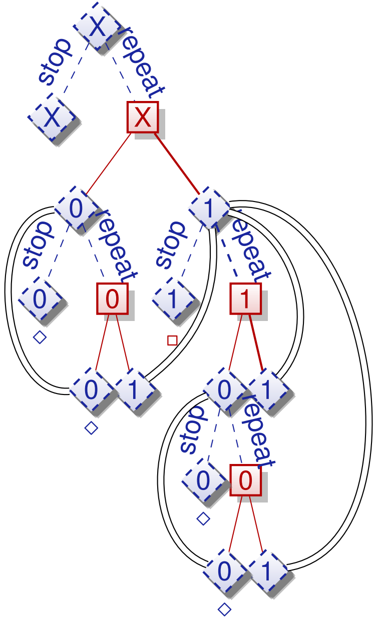




Outline

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) =$

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$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$

$[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!]$ where $\alpha^{n+1} \equiv \alpha^n;\alpha$ $\quad \alpha^0 \equiv ?true$ $\quad$ for HP α

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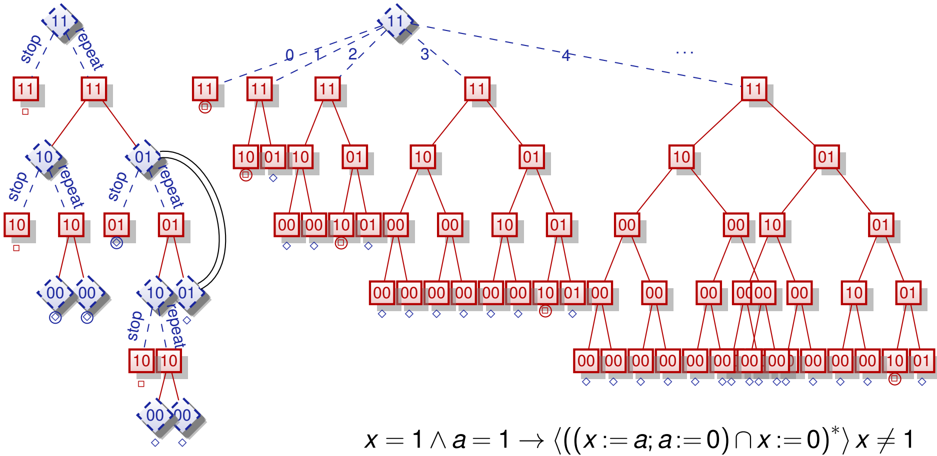
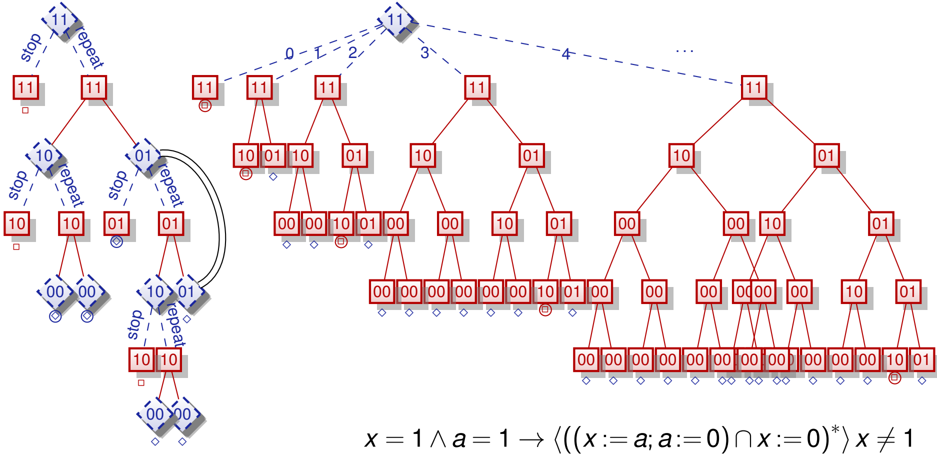


$$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle\, x \neq 1$$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ advance notice semantics?



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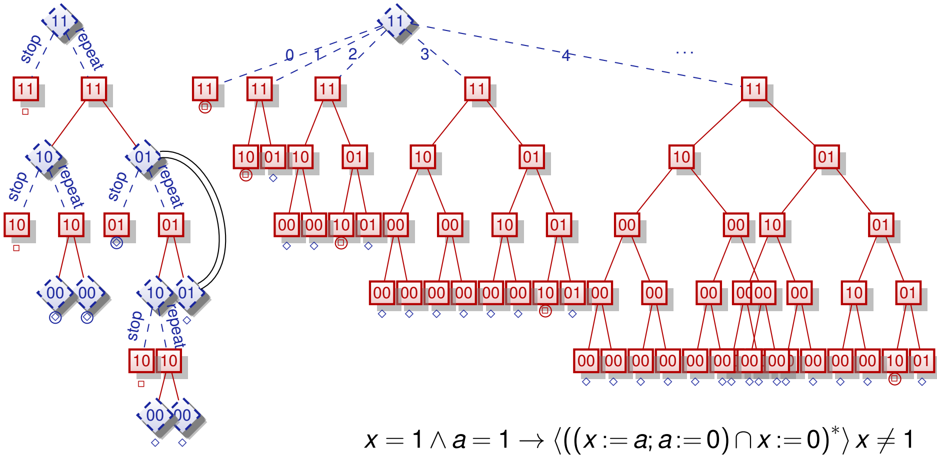
Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ too hard to predict all iterations!

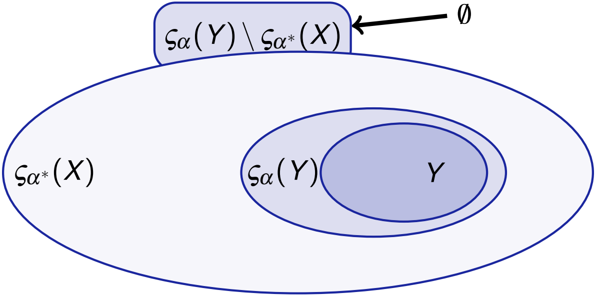


$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle\, x \neq 1$

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one more round away from Y.



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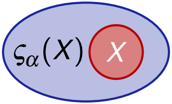

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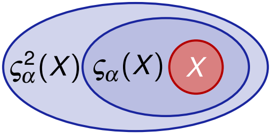

Definition (Hybrid game α)

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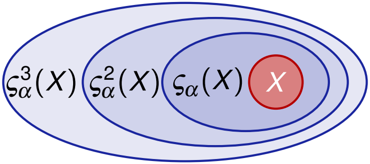

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$ *n* outside the game so Demon won't know

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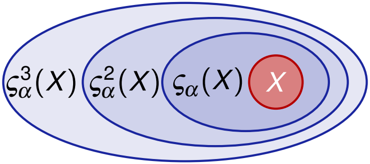

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Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$

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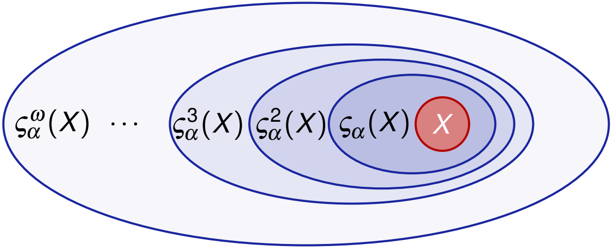

Definition (Hybrid game α)

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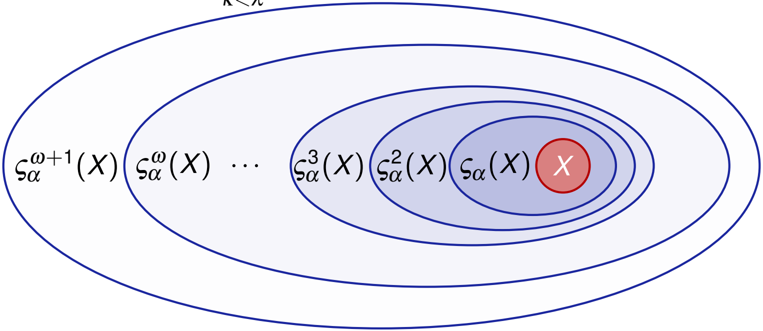

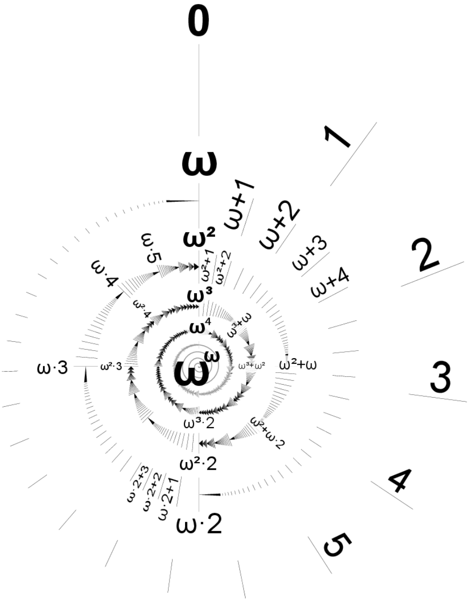

Theorem

Hybrid game closure ordinal $> \omega^\omega$

ω_1^{CK} first nonrecursive ordinal.

Expedition: Ordinal Arithmetic

$$\iota + 0 = \iota$$

$$\iota + (\kappa{+}1) = (\iota + \kappa) + 1 \quad \text{successor } \kappa{+}1$$

$$\iota + \lambda = \bigsqcup_{\kappa<\lambda} \iota + \kappa \quad \text{limit } \lambda$$

$$\iota \cdot 0 = 0$$

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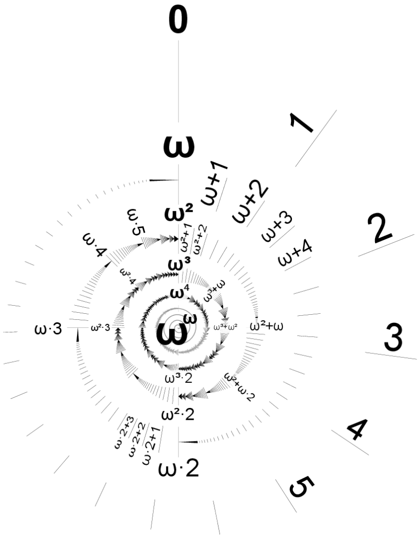
$$2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4$$



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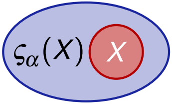

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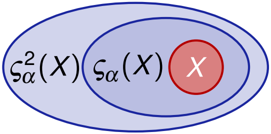

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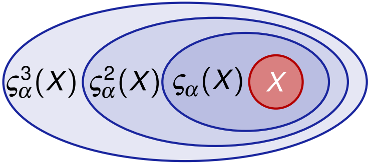

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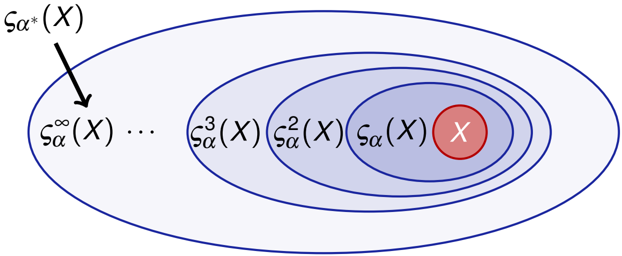
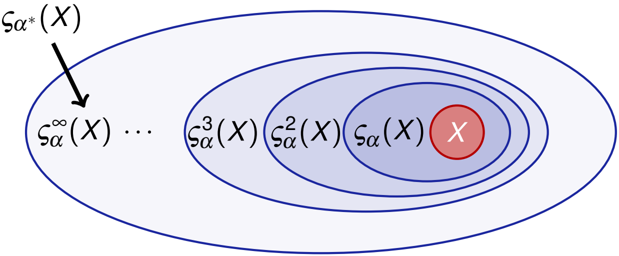


Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ requires transfinite patience



Implicit Definitions

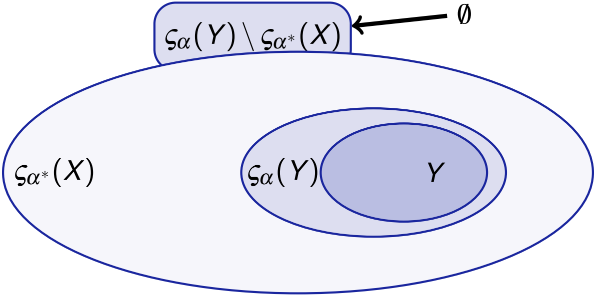
The advantages of implicit definition over construction are roughly those of theft over honest toil.

— Bertrand Russell

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one more round away from Y.



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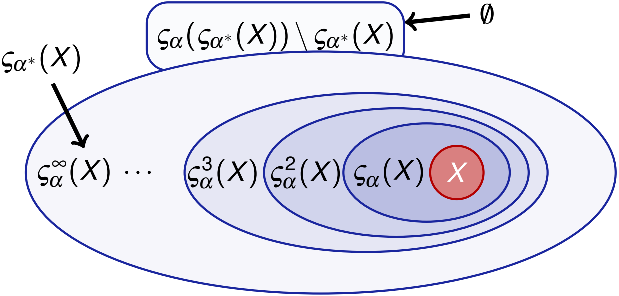
- Still too small: $X \subseteq Z$ since Angel may decide not to repeat

Definition (Pre-fixpoint)

$X \cup \varsigma_\alpha(Z) \subseteq Z$ for the winning region $Z \stackrel{\text{def}}{=} \varsigma_{\alpha^*}(X)$



Fixpoints and Pre-Fixpoints

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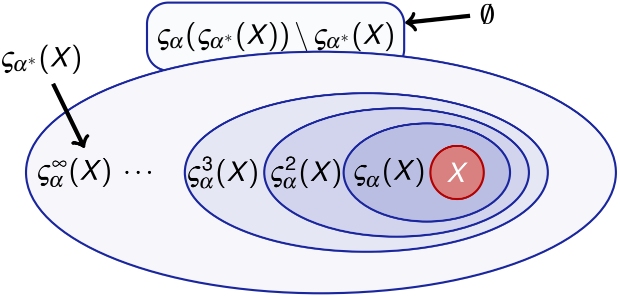

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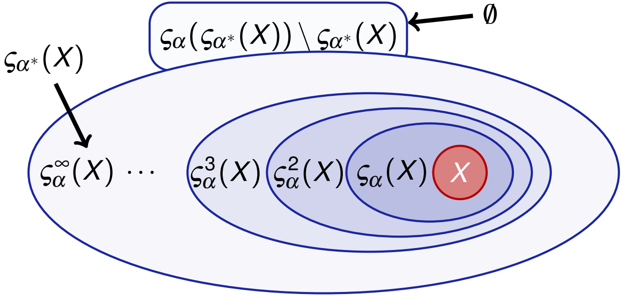

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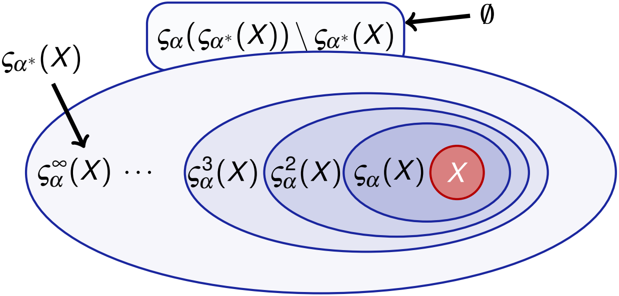

- Which Z is the right one?
- Are there multiple such Z? Does such a Z exist?
- Existence: $Z = \mathscr{S}$ but that's too big and independent of α

Comparing (Pre-)Fixpoints

Lemma ( )

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
$$X \cup \varsigma_\alpha(Z) \subseteq Z$$

are pre-fixpoints, then

Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
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are pre-fixpoints, then $Y \cap Z$ is a smaller pre-fixpoint.

Comparing (Pre-)Fixpoints

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$$X \cup \varsigma_\alpha(Y \cap Z) \overset{\text{mon}}{\subseteq} X \cup (\varsigma_\alpha(Y) \cap \varsigma_\alpha(Z)) \overset{\text{above}}{\subseteq} Y \cap Z \qquad \square$$

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Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!

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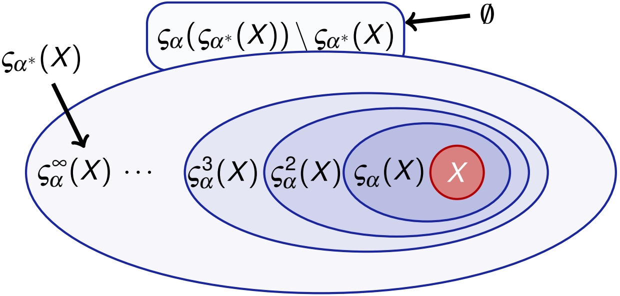
Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!

So: repetition semantics is the smallest pre-fixpoint (well-founded)

Semantics of Repetition

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$$



$X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$

$\varsigma_{\alpha^*}(X)$ intersection of solutions

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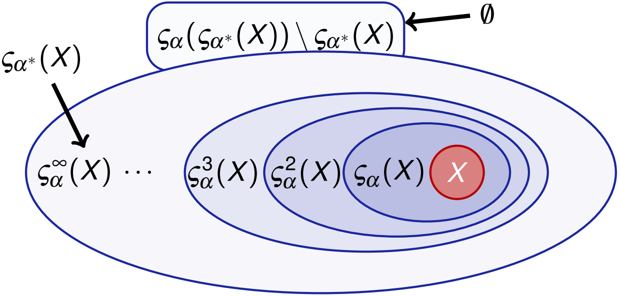

$Z \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$ $\varsigma_{\alpha^*}(X)$ intersection of solutions

$\varsigma_\alpha(Z) \subseteq \varsigma_\alpha(\varsigma_{\alpha^*}(X))$ by mon since $Z \subseteq \varsigma_{\alpha^*}(X)$

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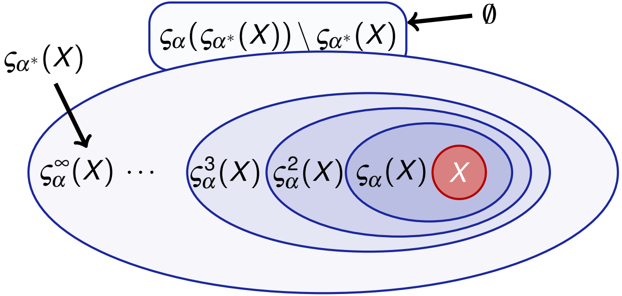

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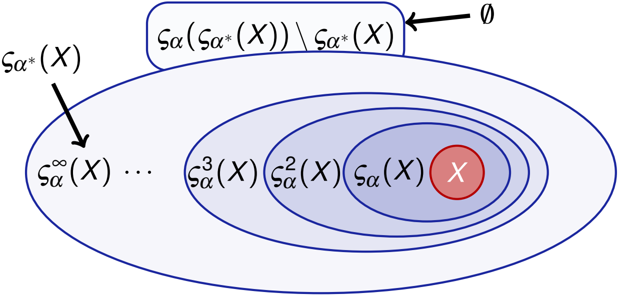

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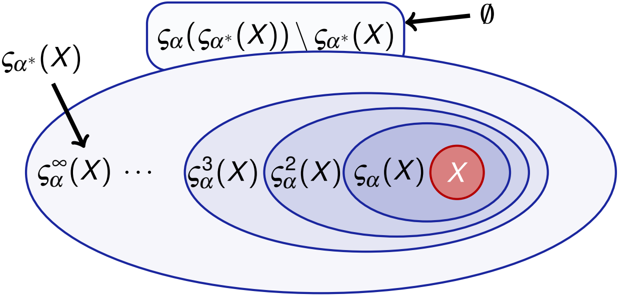

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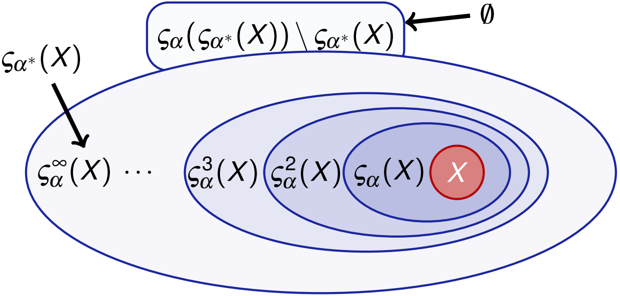

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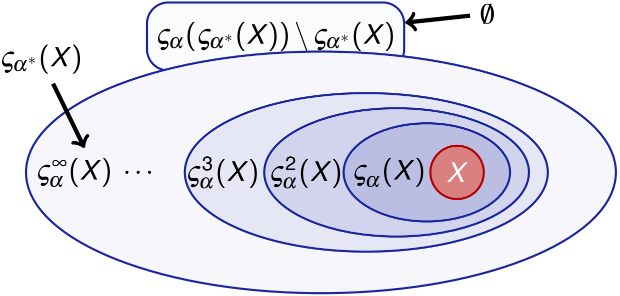

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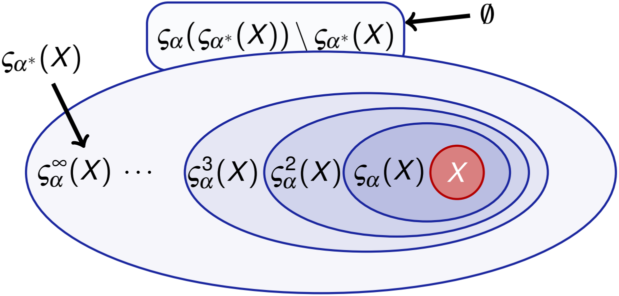
$\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X)) = Z$ — since $\varsigma_{\alpha^*}(X)$ smallest such Z

Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) = Z\} = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ by Knaster-Tarski



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Outline

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathscr{S}) \to \wp(\mathscr{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathscr{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathscr{S} : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X) \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathscr{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
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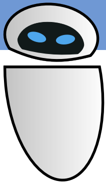
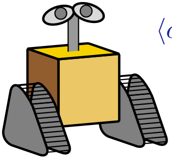
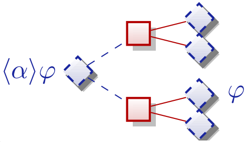
Summary



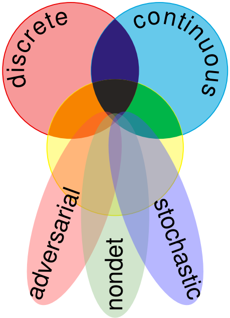
differential game logic

$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG} = \mathsf{dL} + {}^{\mathsf{d}}$





- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint



Next chapter

1. Axiomatics
2. How to win and prove hybrid games

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