21: Virtual Substitution & Real Arithmetic
Logical Foundations of Cyber-Physical Systems

Logical Foundations of Cyber-Physical Systems

André Platzer

🌱 Springer
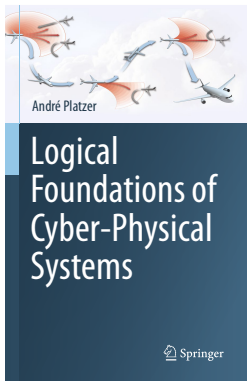
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$\mathcal{R}$ Outline

rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles
infinities & infinitesimals
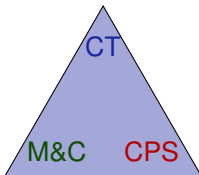
analytic complexity
modeling tradeoffs

verifying CPS at scale

$\mathcal{R}$ Outline

$\mathcal{A}$ Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$

$\Big(\exists x \, (ax^2 + bx + c = 0 \wedge F) \leftrightarrow$

$\quad a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$

$\quad \vee \, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \big) \Big)$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

$\boxed{\text{Extended logic}} \quad F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \quad \boxed{\text{FOL}_{\mathbb{R}}}$

$\omega_x^r \in [\![F]\!]$ iff $\omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!]$ where $r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$\exists x \, (ax^2 + bx + c \leq 0 \wedge F) \leftrightarrow$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$

$\Big(\exists x\, (ax^2 + bx + c \leq 0 \land F) \leftrightarrow$

$\quad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b}$

$\quad \lor\, a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \big) \Big)$

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$\quad \vee \, F_{\bar{x}}^{\text{small}} \Big) \dots$

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$\quad \lor \, F_{\bar{x}}^{-\infty} \Big) \dots$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

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$\quad \vee \, F_{\bar{x}}^{-\infty} \Big) \dots$

$-\infty$ the rubber band number that's smaller on any comparison

> **Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)**
>
> $a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$
>
> $\Big(\exists x \, (ax^2 + bx + c \leq 0 \wedge F) \leftrightarrow$
>
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>
> $\quad \vee \, (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \wedge F_{\bar{x}}^{-\infty} \Big) \ldots$

$-\infty$ needs to satisfy the quadratic inequality (obvious for roots, not $-\infty$)

$\mathcal{R}$ Quadratic Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

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Lemma (Virtual Substitution Lemma for $-\infty$)

$$F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty}$$

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Lemma (Virtual Substitution Lemma for $-\infty$)

Extended logic $FOL_{\mathbb{R} \cup \{-\infty, \infty\}}$ $\quad F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty} \quad$ $FOL_{\mathbb{R}}$

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$\omega_x^r \in \llbracket F \rrbracket$ iff $\omega \in \llbracket F_{\bar{x}}^{-\infty} \rrbracket$ where $r \to -\infty$

Virtual Substitution of $-\infty$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p \leq 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p < 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p \neq 0)_{\bar{x}}^{-\infty} \equiv$$

Virtual Substitution of $-\infty$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{-\infty} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

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$$(p < 0)_{\overline{x}}^{-\infty} \equiv p(\text{-}\infty) < 0$$

$$(p \neq 0)_{\overline{x}}^{-\infty} \equiv$$

Ultimately negative at $-\infty$ $\qquad \lim_{x \to -\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

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Ultimately negative at $-\infty$ $\qquad\qquad \lim_{x \to -\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} & \text{if } \deg(p) \leq 0 \\ & \text{if } \end{cases}$$

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$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ & \text{if } \end{cases}$$

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$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ (-1)^n a_n < 0 & \text{if } \deg(p) > 0 \end{cases}$$

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Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

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Ultimately negative at $-\infty$ $\qquad \lim_{x \to -\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ (-1)^n a_n < 0 \vee \big(a_n = 0 \wedge (\sum_{i=0}^{n-1} a_i x^i)(\text{-}\infty) < 0 \big) & \text{if } \deg(p) > 0 \end{cases}$$

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$$(ax^2 + bx + c < 0)_{\bar{x}}^{-\infty} \equiv (-1)^2 a < 0 \vee a = 0 \wedge ((-1)b < 0 \vee b = 0 \wedge c < 0)$$
$$\equiv a < 0 \vee a = 0 \wedge (b > 0 \vee b = 0 \wedge c < 0)$$

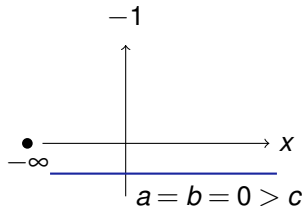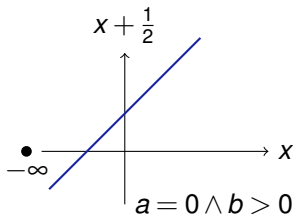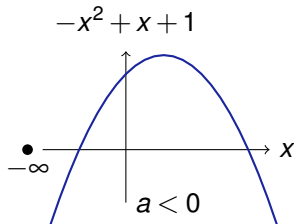Virtual Substitution of $-\infty$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

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$$p(-\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ (-1)^n a_n < 0 \vee \left(a_n = 0 \wedge \left(\sum_{i=0}^{n-1} a_i x^i\right)(-\infty) < 0\right) & \text{if } \deg(p) > 0 \end{cases}$$

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$-x^2 + x + 1$ $a < 0$

$x + \frac{1}{2}$ $a = 0 \wedge b > 0$

-1 $a = b = 0 > c$

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$\omega_x^r \in \llbracket F \rrbracket$ iff $\omega \in \llbracket F_{\bar{x}}^{-\infty} \rrbracket$ where $r \to -\infty$

- Order: $\forall x\, (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$?

$$\infty + x =$$
$$-\infty + x =$$
$$\infty \cdot x =$$
$$\infty \cdot x =$$
$$-\infty \cdot x =$$
$$-\infty \cdot x =$$
$$\infty - \infty =$$
$$0 \cdot \infty =$$
$$\pm\infty / \pm\infty =$$
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$$-\infty \cdot x = -\infty \qquad \text{for all } x > 0$$
$$-\infty \cdot x = \infty \qquad \text{for all } x < 0$$
$$\infty - \infty = \text{undefined} \qquad \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$$
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$$-\infty \cdot x = \infty \qquad \text{for all } x < 0$$
$$\infty - \infty = \text{undefined} \qquad \infty + (-\infty) = \infty + (-\infty+1) = (\infty-\infty)+1$$
$$0 \cdot \infty = \text{undefined}$$
$$\pm\infty / \pm\infty = \text{undefined}$$
$$1/0 =$$

- Order: $\forall x \, (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$? $\infty \leq \infty + 1$ but $\infty + 1 \leq \infty$ by order

$$\infty + x = \infty \qquad \text{for all } x \neq -\infty$$
$$-\infty + x = -\infty \qquad \text{for all } x \neq \infty$$
$$\infty \cdot x = \infty \qquad \text{for all } x > 0$$
$$\infty \cdot x = -\infty \qquad \text{for all } x < 0$$
$$-\infty \cdot x = -\infty \qquad \text{for all } x > 0$$
$$-\infty \cdot x = \infty \qquad \text{for all } x < 0$$
$$\infty - \infty = \text{undefined} \qquad \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$$
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$$\infty + x = \infty \qquad \text{for all } x \neq -\infty$$

$$-\infty + x = -\infty \qquad \text{for all } x \neq \infty$$

$$\infty \cdot x \qquad \qquad \text{for all } x > 0$$

Virtual Substitution

$\infty \cdot x$ Infinities only needed virtually during
$-\infty \cdot x$ virtual substitution, never explicitly.

$$-\infty \cdot x = \infty \qquad \text{for all } x < 0$$

$$\infty - \infty = \text{undefined} \qquad \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$$

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Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$\exists x \, (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$

$\Big(\exists x \, (ax^2 + bx + c < 0 \land F) \leftrightarrow$

$\quad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b}$

$\quad \lor \, a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \big)$

$\quad \lor \, (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty} \Big)$

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$\quad \lor \, (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty} \Big)$

strict inequality never true at the roots but slightly off

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$

$\Big(\exists x \, (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$

$\quad a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b+\varepsilon}$

$\quad \vee \, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \vee F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)+\varepsilon} \big)$

$\quad \vee \, (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \wedge F_{\bar{x}}^{-\infty} \Big)$

ε the rubber band number that's smaller in magnitude on any comparison

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$

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$\quad \vee \, (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \wedge F_{\bar{x}}^{-\infty} \Big)$

Lemma (Virtual Substitution Lemma for ε)

$$F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

$$\Big(\exists x \, (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$$

$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b+\varepsilon}$$

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Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\text{FOL}_{\mathbb{R}[\varepsilon]}$ $\quad F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$ $\quad \text{FOL}_{\mathbb{R}}$

$\mathcal{R}$ Quadratic Strict Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$

$\Big(\exists x \, (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$

$\quad a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b+\varepsilon}$

$\quad \vee \; a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \vee F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)+\varepsilon} \big)$

$\quad \vee \; \big(ax^2 + bx + c \leq 0 \big)_{\bar{x}}^{-\infty} \wedge F_{\bar{x}}^{-\infty} \Big) \ldots$

Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\text{FOL}_{\mathbb{R}[\varepsilon]}$ → $F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$ ← $\text{FOL}_{\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ *iff* $\omega \in [\![F_{\bar{x}}^{e+\varepsilon}]\!]$ *where* $r \searrow e$

ε is "always as small as needed"

- Positive: $\varepsilon > 0$
- Smaller: $\forall x \in \mathbb{R}\,(x > 0 \rightarrow \varepsilon < x)$
- Standard $\mathbb{R}$ are Archimedean: $\cancel{\forall} x \in \mathbb{R} \setminus \{0\}\; \cancel{\exists} n \in \mathbb{N}\; |\underbrace{x + x + \cdots + x}_{n \text{ times}}| > 1$

- $\mathbb{R}[\varepsilon]$ are non-Archimedean: $\underbrace{\varepsilon + \varepsilon + \cdots + \varepsilon}_{\text{any } n \in \mathbb{N} \text{ times}} < 1$

- Infinitesimals as inverses of infinities?

$$\varepsilon \cdot \infty = 1? \quad -\varepsilon \cdot -\infty = 1? \quad (\varepsilon + 1) \cdot (\underbrace{\infty + 2}_{\infty?}) = \ldots$$

- How to order for $x \neq 0$?

$$\varepsilon^2 \qquad \varepsilon \qquad x^2 + \varepsilon \qquad (x + \varepsilon)^2 \qquad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

ε is "always as small as needed"

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- Infinitesimals

  Virtual Substitution

  Infinitesimals only needed virtually during virtual substitution, never explicitly.

  $\varepsilon \cdot \infty = 1? \quad -\varepsilon \cdot -\infty = 1? \quad (\varepsilon + 1) \cdot \underbrace{(\infty + 2)}_{\infty?} = \ldots$

- How to order for $x \neq 0$?

  $$\varepsilon^2 \; < \; \varepsilon \; < \; x^2 + \varepsilon \; < \; (x + \varepsilon)^2 \; < \; x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p \neq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\hspace{2cm} p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

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Virtual Substitution of $e + \varepsilon$ into Comparisons $\hspace{2cm} p = \sum_{i=0}^{n} a_i x^i$

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$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p \neq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \le 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv (p^+ < 0)_{\bar{x}}^{e}$$

$$(p \ne 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

Immediately negative at x $\qquad \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

ordinary virtual $\sqrt{\cdot}$ substitution of e into immediate negativity $p^+ < 0$

Virtual Substitution of $e + \boldsymbol{\varepsilon}$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv (p^+ < 0)_{\bar{x}}^{e}$$

$$(p \neq 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigvee_{i=0}^{n} a_i \neq 0$$

Immediately negative at x $\qquad \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \left\{ \begin{array}{ll} & \text{if} \\ & \text{if} \end{array} \right.$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\quad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \le 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

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Immediately negative at x $\quad\quad \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} & \text{if } \deg(p) \le 0 \\ & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

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Immediately negative at x $\qquad\qquad \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv (p^{+} < 0)_{\bar{x}}^{e}$$

$$(p \neq 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigvee_{i=0}^{n} a_i \neq 0$$

Immediately negative at x $\qquad \lim_{y \searrow x} p(y) < 0$

$$p^{+} < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ p < 0 & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \le 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv (p^+ < 0)_{\bar{x}}^{e}$$

$$(p \ne 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigvee_{i=0}^{n} a_i \ne 0$$

Immediately negative at x $\qquad \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ p < 0 \vee (p = 0 & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\hspace{2cm} p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \le 0)_{\bar{x}}^{e+\varepsilon} \equiv (p < 0)_{\bar{x}}^{e+\varepsilon} \vee (p = 0)_{\bar{x}}^{e+\varepsilon}$$

$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv (p^+ < 0)_{\bar{x}}^{e}$$

$$(p \ne 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigvee_{i=0}^{n} a_i \ne 0$$

Immediately negative at x $\hspace{3cm} \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of $e + \varepsilon$ into Comparisons $\qquad p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

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Immediately negative at x $\qquad \lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \overset{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$

Break ties by successive derivative p' immediately negative at root of p

Immediately negative at e $\qquad\qquad\qquad\qquad\qquad \lim_{x \searrow e} p(x) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if deg}(p) \leq 0 \\ p < 0 \lor (p = 0 \land (p')^+ < 0) & \text{if deg}(p) > 0 \end{cases}$$

$(ax^2 + bx + c)^+ < 0 \equiv ax^2 + bx + c < 0$

$\qquad\qquad \lor \, ax^2 + bx + c = 0 \land (2ax + b < 0 \lor 2ax + b = 0 \land 2a < 0)$

Immediately negative at e	$\lim_{x \searrow e} p(x) < 0$

$$p^+ < 0 \overset{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$

$(ax^2 + bx + c)^+ < 0 \equiv ax^2 + bx + c < 0$

$\qquad \vee\, ax^2 + bx + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0)$

$(ax^2 + bx + c < 0)_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a) + \varepsilon} \equiv ((ax^2 + bx + c)^+ < 0)_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \equiv$

$(ax^2 + bx + c < 0 \vee ax^2 + bx + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0))_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)}$

Immediately negative at e $\qquad\qquad\qquad \lim_{x \searrow e} p(x) < 0$
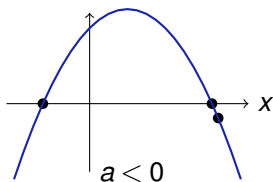
$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$
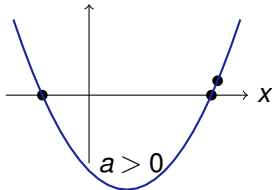
$(ax^2 + bx + c)^+ < 0 \equiv ax^2 + bx + c < 0$
$$\vee\, ax^2 + bx + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0)$$

$(ax^2 + bx + c < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \equiv ((ax^2 + bx + c)^+ < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv$

$(ax^2 + bx + c < 0 \vee ax^2 + bx + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0))_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}$

$\equiv 0 \cdot 1 < 0 \vee 0 = 0 \wedge$

$\qquad \big(\underbrace{(0 < 0 \vee 4a^2 \le 0 \wedge (0 < 0 \vee -4a^2(b^2 - 4ac) < 0))}_{(2ax+b<0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}} \big) \vee \underbrace{0 = 0}_{(2ax+b=0)_{\bar{x}}^{\cdots}} \wedge \underbrace{2a1 < 0}_{(2a<0)_{\bar{x}}^{\cdots}} \big)$

$\equiv 4a^2 \le 0 \wedge -4a^2(b^2 - 4ac) < 0 \vee 2a < 0$

$\equiv \textcolor{red}{a = 0 \wedge 0(b^2 - 0) < 0 \vee 2a < 0 \equiv 2a < 0}$

Immediately negative at e $\qquad\qquad\qquad \lim_{x \searrow e} p(x) < 0$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$

$(ax^2 + bx + c)^+ < 0 \equiv ax^2 + bx + c < 0$
$$\vee \, ax^2 + bx + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0)$$

$(ax^2 + bx + c < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a) + \varepsilon} \equiv ((ax^2 + bx + c)^+ < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv$
$\equiv a = 0 \wedge 0(b^2 - 0) < 0 \vee 2a < 0 \equiv 2a < 0$

$-x^2 + x + 1$

$x^2 - x - 1$

$-\frac{1}{2}x^2 + x - \frac{1}{10}$

$a < 0$ $\qquad\qquad$ $a > 0$ $\qquad\qquad$ $a < 0$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$

$\Big(\exists x \, (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$

$\quad a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b + \varepsilon}$

$\quad \vee \, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \vee F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)+\varepsilon} \big)$

$\quad \vee \, \big(ax^2 + bx + c \leq 0 \big)_{\bar{x}}^{-\infty} \wedge F_{\bar{x}}^{-\infty} \Big) \ldots$

Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\text{FOL}_{\mathbb{R}[\varepsilon]}$ $F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$ $\text{FOL}_{\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ iff $\omega \in [\![F_{\bar{x}}^{e+\varepsilon}]\!]$ where $r \searrow e$

Theorem (Virtual Substitution: Quadratics)   (Weispfenning'97)

*Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$
and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.*

$$\exists x\, F \leftrightarrow$$

$$F_x^{-\infty}$$

$$\vee \bigvee_{(ax^2+bx+c\{\underset{\leq}{=}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_x^{-c/b} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_x^{(-b+\sqrt{d})/(2a)} \vee F_x^{(-b-\sqrt{d})/(2a)})\right)$$

$$\vee \bigvee_{(ax^2+bx+c\{<\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_x^{-c/b+\varepsilon} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_x^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_x^{(-b-\sqrt{d})/(2a)+\varepsilon})\right)$$

Equivalence needs roots and off-roots from **all** atomic formulas in *F*

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$$\exists x \, F \leftrightarrow$$
$$F_x^{-\infty}$$
$$\vee \bigvee_{(ax^2+bx+c\{\stackrel{=}{\leq}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_x^{-c/b} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_x^{(-b+\sqrt{d})/(2a)} \vee F_x^{(-b-\sqrt{d})/(2a)})\right)$$
$$\vee \bigvee_{(ax^2+bx+c\{\stackrel{\neq}{<}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_x^{-c/b+\varepsilon} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_x^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_x^{(-b-\sqrt{d})/(2a)+\varepsilon})\right)$$

Equivalence needs roots and off-roots from **all** atomic formulas in *F*

Theorem (Virtual Substitution: Quadratics)　　　　(Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \overset{\text{def}}{=} b^2 - 4ac$.
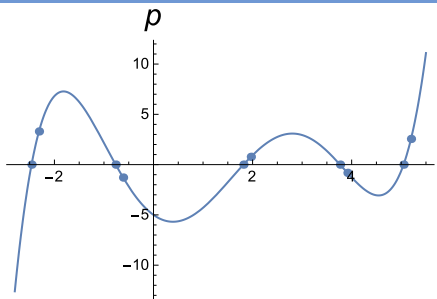
$$\exists x\, F \leftrightarrow$$
$$F_{\bar{x}}^{-\infty}$$
$$\vee \bigvee_{(ax^2+bx+c\{\substack{= \\ \leq}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_{\bar{x}}^{-c/b} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)})\right)$$
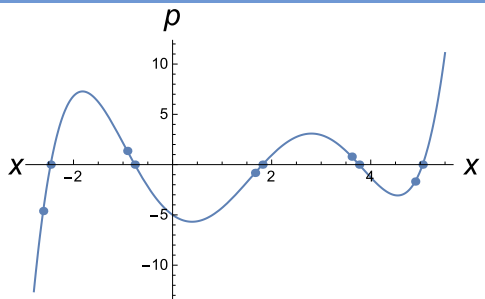$$\vee \bigvee_{(ax^2+bx+c\{\substack{\neq \\ <}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_{\bar{x}}^{-c/b+\varepsilon} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon})\right)$$

Equivalence needs roots and off-roots from **all** atomic formulas in *F*

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Lemma (Virtual Substitution Lemmas)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \qquad F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty} \qquad F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

$-\infty$ and roots e with offsets $e + \varepsilon$ roots e with offsets $e - \varepsilon$ and ∞

No rejection without mention

Other parts of F not satisfied by the points of p have their own polynomial q that contributes different roots $\tilde{e}$ and off-roots $\tilde{e} + \varepsilon$.

Generalizations of quantifier elimination to higher degrees also place a representative point into every region of interest, but derivatives and relationships of derivatives become crucially relevant.

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

$$\exists x\, F \leftrightarrow \quad F_{\overline{x}}^{-\infty}$$

$$\vee \bigvee_{(ax^2+bx+c\{\stackrel{=}{\leq}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_{\overline{x}}^{-c/b} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_{\overline{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\overline{x}}^{(-b-\sqrt{d})/(2a)}) \right)$$

$$\vee \bigvee_{(ax^2+bx+c\{\stackrel{\neq}{<}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_{\overline{x}}^{-c/b+\varepsilon} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_{\overline{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_{\overline{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon}) \right)$$

"Proof" Sketch.

"←" simple from virtual substitution lemma with (extended) term witness

"→" Valid iff true in every state, so all variables have real numeric value

- o-minimal: solutions form finite union of disjoint intervals (univariate)
- WLOG endpoints are the roots since all polynomials quadratic
- All side conditions have to be met otherwise can't be solution
- Non-point intervals contain ε offset since smaller than endpoint □

$$a \neq 0 \to \big(\exists x\,(ax^2 + bx + c = 0 \land x \geq 0)$$
$$\leftrightarrow b^2 - 4ac \geq 0 \land (2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$
$$\lor 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0)\big)$$

$$-(-b + \sqrt{b^2 - 4ac})/(2a) = ((-1 + 0\sqrt{b^2 - 4ac})/1) \cdot ((-b + \sqrt{b^2 - 4ac})/(2a)$$
$$= (b - \sqrt{b^2 - 4ac})/(2a)$$

$(-x \leq 0)_{\overline{x}}^{(b - \sqrt{b^2 - 4ac})/(2a)}$

$\equiv b2a \leq 0 \land \cancel{b^2} - (-1)^2(\cancel{b^2} - 4ac) \geq 0 \lor -1 \cdot 2a \leq 0 \land \cancel{b^2} - (-1)^2(\cancel{b^2} - 4ac) \leq 0$

$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$

$(-x \leq 0)_{\overline{x}}^{(b + \sqrt{b^2 - 4ac})/(2a)}$

$\equiv b2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \geq 0 \lor 1 \cdot 2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \leq 0$

$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0$

$$\exists x\,(x^2 - x + c = 0 \land x \geq 0)$$
$$\leftrightarrow (-1)^2 - 4c \geq 0 \land (c \geq 0 \lor c \leq 0$$
$$\underline{\lor c \geq 0} \lor \cancel{\text{false} \land c \leq 0})$$

$$\exists x \, (x^2 - x + c = 0 \land x \geq 0)$$
$$\leftrightarrow 1 - 4c \geq 0$$

$$\exists x\,(x^2 - x + c = 0 \wedge x \geq 0)$$
$$\leftrightarrow 1 - 4c \geq 0$$
$$\exists x\,(x^2 - x + c \leq 0 \wedge x \geq 0)$$
$$\leftrightarrow \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\bar{x}}^{-\infty}}_{\text{false}} \vee 1 - 4c \geq 0 \vee \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\bar{x}}^{0}}_{c \leq 0 \wedge 0 \geq 0 \text{ (subsumed)}}$$

$$\exists x\,(x^2 - x + c = 0 \wedge x \geq 0)$$

$$\leftrightarrow 1 - 4c \geq 0$$

$$\exists x\,(x^2 - x + c \leq 0 \wedge x \geq 0)$$

$$\leftrightarrow \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\bar{x}}^{-\infty}}_{\textit{false}} \vee 1 - 4c \geq 0 \vee \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\bar{x}}^{0}}_{c \leq 0 \wedge 0 \geq 0 \ \text{(subsumed)}}$$

$$\exists x\,(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0) \qquad \textcolor{red}{\text{more roots!}}$$

$$\leftrightarrow \underbrace{(x^2 - x + c \leq 0 \wedge \ldots)_{\bar{x}}^{-\infty}}_{\textit{false}}$$

$$\vee\, 1 - 4c \geq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)_{\bar{x}}^{(-1 \pm \sqrt{1-4c})/2}}_{8 - 4c = 0}$$

$$\vee\, -1 \neq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)_{x}^{0}}_{c \leq 0 \wedge 0 \geq 0 \wedge 2 = 0}$$

$$\vee\, -1 \neq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)_{x}^{2}}_{2 + c \leq 0} \qquad \equiv 2 + c \leq 0$$

$\mathcal{R}$ Outline

$\mathcal{A}$ Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$

$\Big(\exists x \, (ax^2 + bx + c = 0 \wedge F) \leftrightarrow$

$\quad a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$

$\quad \vee \, a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \big) \Big)$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

$\boxed{\text{Extended logic}} \longrightarrow F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \longleftarrow \boxed{\text{FOL}_\mathbb{R}}$

$\omega_x^r \in [\![F]\!]$ iff $\omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!]$ where $r = (\omega[\![a]\!] + \omega[\![b]\!] \sqrt{\omega[\![c]\!]}) / (\omega[\![d]\!]) \in \mathbb{R}$

Theorem (Virtual Substitution: Quadratics)   (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$
and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

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$$\vee \bigvee_{(ax^2+bx+c\{\stackrel{=}{\leq}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_{\bar{x}}^{-c/b} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)})\right)$$

$$\vee \bigvee_{(ax^2+bx+c\{\stackrel{\neq}{<}\}0)\in F} \left(a{=}0 \wedge b{\neq}0 \wedge F_{\bar{x}}^{-c/b+\varepsilon} \vee a{\neq}0 \wedge d{\geq}0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon})\right)$$

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$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \qquad F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty} \qquad F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

$\mathcal{A}$ Summary

- Miracle: $FOL_\mathbb{R}$ is decidable: Tarski'31
  Algorithm decides whether (closed) formula valid or not

- Quantifier elimination computes quantifier-free equivalent

- Successive quantifier elimination decides $FOL_\mathbb{R}$ (after universal closure)

- QE accepts free variables, giving equivalent that identifies the requirements for truth (synthesis)

- Virtual substitution does QE for degree ≤ 3 by equivalent syntactic rephrasing of semantics   Weispfenning'97

- QE proceeds inside out, so degree ≤ 3 needed on *each* iteration

- Important fragments permit many optimizations   your research?

- Universally quantified weak inequalities / existentially quantified strict inequalities are easier since infinitesimals/infinities don't satisfy =

- Cylindrical algebraic decomposition (CAD) any degree   Collins'75

- Simplify arithmetic to relevant parts, transform to fit together

André Platzer.
Logical Foundations of Cyber-Physical Systems.
Springer, Cham, 2018.
doi:10.1007/978-3-319-63588-0.

Volker Weispfenning.
Quantifier elimination for real algebra — the quadratic case and beyond.
Appl. Algebra Eng. Commun. Comput., 8(2):85–101, 1997.
doi:10.1007/s002000050055.

André Platzer.
Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.
Springer, Heidelberg, 2010.
doi:10.1007/978-3-642-14509-4.

Jacek Bochnak, Michel Coste, and Marie-Francoise Roy.
Real Algebraic Geometry, volume 36 of *Ergeb. Math. Grenzgeb.*
Springer, Berlin, 1998.
doi:10.1007/978-3-662-03718-8.

📄 Saugata Basu, Richard Pollack, and Marie-Françoise Roy.
Algorithms in Real Algebraic Geometry.
Springer, Berlin, 2nd edition, 2006.
`doi:10.1007/3-540-33099-2.`

📄 Alfred Tarski.
A Decision Method for Elementary Algebra and Geometry.
University of California Press, Berkeley, 2nd edition, 1951.
`doi:10.1007/978-3-7091-9459-1_3.`

📄 George E. Collins.
Quantifier elimination for real closed fields by cylindrical algebraic decomposition.
In H. Barkhage, editor, *Automata Theory and Formal Languages*, volume 33 of *LNCS*, pages 134–183, Berlin, 1975. Springer.
`doi:10.1007/3-540-07407-4_17.`

📄 George E. Collins and Hoon Hong.
Partial cylindrical algebraic decomposition for quantifier elimination.
J. Symb. Comput., 12(3):299–328, 1991.
`doi:10.1016/S0747-7171(08)80152-6.`