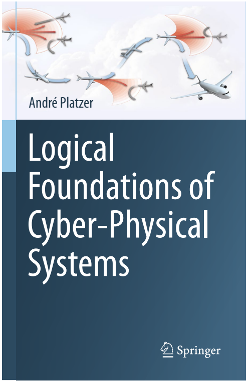
21: Virtual Substitution & Real Arithmetic

Logical Foundations of Cyber-Physical Systems



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Outline

1. Learning Objectives

2. Real Arithmetic
    - Recap: Quadratic Equations
    - Quadratic Weak Inequalities
    - Infinity ∞ Virtual Substitution
    - Expedition: Infinities
    - Quadratic Strict Inequalities
    - Infinitesimal ε Virtual Substitution

3. Quantifier Elimination by Virtual Substitution of Quadratics

4. Summary

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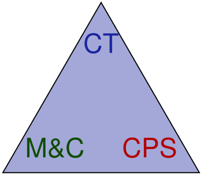
Learning Objectives

Virtual Substitution & Real Equations

rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles
infinities & infinitesimals



analytic complexity
modeling tradeoffs

verifying CPS at scale

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4. Summary

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$\begin{aligned}
&a \neq 0 \lor b \neq 0 \lor c \neq 0 \to \\
&\quad \Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow \\
&\qquad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b} \\
&\qquad \lor a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)
\end{aligned}$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$ $\text{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \textit{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \textit{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$$

Quadratic Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x\,(ax^2 + bx + c \leq 0 \land F) \leftrightarrow$$

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$$\lor\, a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\right)\Big)$$

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$$\lor\; F_{\bar{x}}^{\text{small}}\Big)\ldots$$

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$$\lor\ F_{\bar{x}}^{-\infty}\Big) \ldots$$

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& \quad \lor \; F_{\bar{x}}^{-\infty} \Big) \ldots
\end{aligned}
$$

$-\infty$ the rubber band number that's smaller on any comparison

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& \qquad \lor \ (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty} \Big) \ldots
\end{aligned}
$$

$-\infty$ needs to satisfy the quadratic inequality (obvious for roots, not $-\infty$)

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Lemma (Virtual Substitution Lemma for $-\infty$)

$$F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty}$$

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Extended logic $\mathrm{FOL}_{\mathbb{R} \cup \{-\infty, \infty\}}$

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$\mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{-\infty}]\!] \text{ where } r \rightarrow -\infty$$

Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p \leq 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p < 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p \neq 0)_{\bar{x}}^{-\infty} \equiv$$

Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

$$(p = 0)_{\bar{x}}^{-\infty} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

$$(p \leq 0)_{\bar{x}}^{-\infty} \equiv$$

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$$(p < 0)_{\bar{x}}^{-\infty} \equiv p(\text{-}\infty) < 0$$

$$(p \neq 0)_{\bar{x}}^{-\infty} \equiv$$

Ultimately negative at $-\infty$ $\lim_{x \to -\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \overset{\text{def}}{\equiv} \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons

$p = \sum_{i=0}^{n} a_i x^i$

$$
\begin{aligned}
(p = 0)_{\bar{x}}^{-\infty} &\equiv \bigwedge_{i=0}^{n} a_i = 0 \\
(p \leq 0)_{\bar{x}}^{-\infty} &\equiv (p < 0)_{\bar{x}}^{-\infty} \lor (p = 0)_{\bar{x}}^{-\infty} \\
(p < 0)_{\bar{x}}^{-\infty} &\equiv p(\text{-}\infty) < 0 \\
(p \neq 0)_{\bar{x}}^{-\infty} &\equiv \bigvee_{i=0}^{n} a_i \neq 0
\end{aligned}
$$

Ultimately negative at $-\infty$

$\lim_{x \to -\infty} p(x) < 0$

$$
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Ultimately negative at $-\infty$ $\lim_{x\to-\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \overset{\text{def}}{\equiv} \begin{cases} & \text{if } \deg(p) \leq 0 \\ & \text{if } \end{cases}$$

Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

$$(p=0)_{\bar{x}}^{-\infty} \equiv \bigwedge_{i=0}^{n} a_i = 0$$

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$$(p<0)_{\bar{x}}^{-\infty} \equiv p(\text{-}\infty) < 0$$

$$(p \ne 0)_{\bar{x}}^{-\infty} \equiv \bigvee_{i=0}^{n} a_i \ne 0$$

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$$p(\text{-}\infty) < 0 \overset{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ & \text{if} \end{cases}$$

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$$p(\text{-}\infty) < 0 \overset{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ (-1)^n a_n < 0 & \text{if } \deg(p) > 0 \end{cases}$$

Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

$$
\begin{aligned}
(p = 0)_{\bar{x}}^{-\infty} &\equiv \bigwedge_{i=0}^{n} a_i = 0 \\
(p \leq 0)_{\bar{x}}^{-\infty} &\equiv (p < 0)_{\bar{x}}^{-\infty} \lor (p = 0)_{\bar{x}}^{-\infty} \\
(p < 0)_{\bar{x}}^{-\infty} &\equiv p(\text{-}\infty) < 0 \\
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Example: Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

$$(p < 0)_{\bar{x}}^{-\infty} \equiv (p < 0)_{\bar{x}}^{-\infty}$$

Ultimately negative at $-\infty$ $\lim_{x \to -\infty} p(x) < 0$

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$$\begin{aligned} (ax^2 + bx + c < 0)_{\bar{x}}^{-\infty} &\equiv (-1)^2 a < 0 \lor a = 0 \land ((-1)b < 0 \lor b = 0 \land c < 0) \\ &\equiv a < 0 \lor a = 0 \land (b > 0 \lor b = 0 \land c < 0) \end{aligned}$$

Example: Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons $p = \sum_{i=0}^{n} a_i x^i$

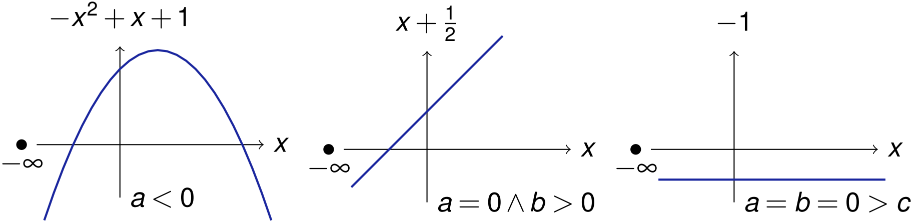
$$(p<0)_{\bar{x}}^{-\infty} \equiv (p<0)_{\bar{x}}^{-\infty}$$

Ultimately negative at $-\infty$ $\lim_{x\to-\infty} p(x) < 0$

$$p(\text{-}\infty) < 0 \overset{\text{def}}{\equiv} \begin{cases} p<0 & \text{if } \deg(p)\le 0 \\ (-1)^n a_n<0 \lor \big(a_n=0 \land (\sum_{i=0}^{n-1} a_i x^i)(\text{-}\infty) < 0\big) & \text{if } \deg(p)>0 \end{cases}$$

$$(ax^2+bx+c<0)_{\bar{x}}^{-\infty} \equiv (-1)^2 a<0 \lor a=0 \land ((-1)b<0 \lor b=0 \land c<0)$$

$$\equiv a<0 \lor a=0 \land (b>0 \lor b=0 \land c<0)$$



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Expedition: Infinite Challenges with Infinities $\mathbb{R}\cup\{-\infty,\infty\}$

- Order: $\forall x\,(-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$?

$$\begin{aligned}
\infty + x &= \\
-\infty + x &= \\
\infty \cdot x &= \\
\infty \cdot x &= \\
-\infty \cdot x &= \\
-\infty \cdot x &= \\
\infty - \infty &= \\
0 \cdot \infty &= \\
\pm\infty / \pm\infty &= \\
1/0 &=
\end{aligned}$$

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$$\begin{aligned}
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\infty \cdot x &= \\
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-\infty \cdot x &= \infty && \text{for all } x < 0 \\
\infty - \infty &= \text{undefined} && \infty + (-\infty) = \infty + (-\infty+1) = (\infty - \infty) + 1 \\
0 \cdot \infty &= \\
\pm\infty / \pm\infty &= \\
1/0 &=
\end{aligned}$$

Expedition: Infinite Challenges with Infinities $\mathbb{R}\cup\{-\infty,\infty\}$

- Order: $\forall x\,(-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty+1$? $\infty \leq \infty+1$ but $\infty+1 \leq \infty$ by order

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$\infty + x = \infty$ for all $x \neq -\infty$

$-\infty + x = -\infty$ for all $x \neq \infty$

$\infty \cdot x$

$\infty \cdot x$

$-\infty \cdot x$

$-\infty \cdot x = \infty$ for all $x < 0$

$\infty - \infty = \text{undefined}$ $\infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$

$0 \cdot \infty = \text{undefined}$

$\pm\infty / \pm\infty = \text{undefined}$

$1/0 = \text{undefined}$

Virtual Substitution

Infinities only needed virtually during virtual substitution, never explicitly.

Quadratic Strict Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x\,(ax^2 + bx + c < 0 \land F) \leftrightarrow$$

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\begin{aligned}
& a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow \\
& \quad \Big(\exists x \, (ax^2 + bx + c < 0 \land F) \leftrightarrow \\
& \qquad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b} \\
& \quad \lor a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)} \big) \\
& \quad \lor \; (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty} \Big)
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strict inequality never true at the roots but slightly off

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ε the rubber band number that's smaller in magnitude on any comparison

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Lemma (Virtual Substitution Lemma for ε)

$$F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

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Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\mathrm{FOL}_{\mathbb{R}[\varepsilon]}$

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$\mathrm{FOL}_{\mathbb{R}}$

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$\mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{e+\varepsilon}]\!] \text{ where } r \searrow e$$

Expedition: Infinitesimal Nonstandard Field Ext. $\mathbb{R}[\varepsilon]$

ε is "always as small as needed"

- Positive: $\varepsilon > 0$
- Smaller: $\forall x \in \mathbb{R}\,(x > 0 \to \varepsilon < x)$
- Standard $\mathbb{R}$ are Archimedean: $\forall x \in \mathbb{R} \setminus \{0\}\ \exists n \in \mathbb{N}\ |\underbrace{x + x + \cdots + x}_{n \text{ times}}| > 1$
- $\mathbb{R}[\varepsilon]$ are non-Archimedean: $\underbrace{\varepsilon + \varepsilon + \cdots + \varepsilon}_{\text{any } n \in \mathbb{N} \text{ times}} < 1$
- Infinitesimals as inverses of infinities?

$$\varepsilon \cdot \infty = 1? \quad -\varepsilon \cdot -\infty = 1? \quad (\varepsilon + 1) \cdot (\underbrace{\infty + 2}_{\infty?}) = \ldots$$

- How to order for $x \neq 0$?

$$\varepsilon^2 \qquad \varepsilon \qquad x^2 + \varepsilon \qquad (x + \varepsilon)^2 \qquad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

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- Infinitesimals

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Virtual Substitution of Infinitesimals

Virtual Substitution of $e+\varepsilon$ into Comparisons $p=\sum_{i=0}^{n} a_i x^i$

$$(p=0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p\leq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

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Immediately negative at x $\lim_{y \searrow x} p(y) < 0$

$$p^+ < 0 \overset{\text{def}}{\equiv} \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

ordinary virtual $\sqrt{\cdot}$ substitution of e into immediate negativity $p^+ < 0$

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Break ties by successive derivative p' immediately negative at root of p

Example: Virtual Substitution of Infinitesimals

Immediately negative at e $\lim_{x \searrow e} p(x) < 0$

$$p^+ < 0 \overset{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \leq 0 \\ p < 0 \lor (p = 0 \land (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$

$$\begin{aligned}(ax^2+bx+c)^+ < 0 \equiv{} & ax^2+bx+c < 0 \\ & \lor ax^2+bx+c = 0 \land (2ax+b < 0 \lor 2ax+b = 0 \land 2a<0)\end{aligned}$$

Example: Virtual Substitution of Infinitesimals

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$$(ax^2+bx+c<0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \equiv ((ax^2+bx+c)^+ < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv$$
$$(ax^2+bx+c<0 \lor ax^2+bx+c=0 \land (2ax+b<0 \lor 2ax+b=0 \land 2a<0))_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}$$

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$$\equiv 0 \cdot 1 < 0 \lor 0=0 \land$$
$$\Big(\underbrace{(0<0 \lor 4a^2 \leq 0 \land (0<0 \lor -4a^2(b^2-4ac)<0))}_{(2ax+b<0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}} \lor \underbrace{0=0}_{(2ax+b=0)_{\bar{x}}^{\cdots}} \land \underbrace{2a1 < 0}_{(2a<0)_{\bar{x}}^{\cdots}}\Big)$$
$$\equiv 4a^2 \leq 0 \land -4a^2(b^2-4ac)<0 \lor 2a<0$$
$$\equiv a = 0 \land 0(b^2 - 0) < 0 \lor 2a < 0 \equiv 2a < 0$$

Example: Virtual Substitution of Infinitesimals

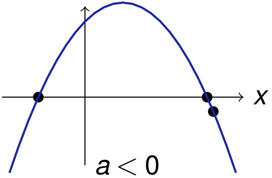
Immediately negative at e $\lim_{x \searrow e} p(x) < 0$

$$p^+ < 0 \overset{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) \le 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \deg(p) > 0 \end{cases}$$

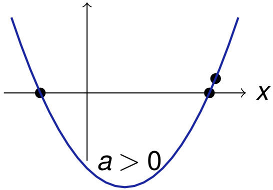
$$(ax^2+bx+c)^+ < 0 \equiv ax^2+bx+c < 0$$
$$\vee\, ax^2+bx+c = 0 \wedge (2ax+b < 0 \vee 2ax+b = 0 \wedge 2a<0)$$

$$(ax^2+bx+c<0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \equiv ((ax^2+bx+c)^+ < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv$$
$$\equiv a = 0 \wedge 0(b^2 - 0) < 0 \vee 2a < 0 \equiv 2a < 0$$

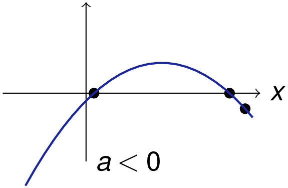
$-x^2+x+1$



x^2-x-1



$-\frac{1}{2}x^2+x-\frac{1}{10}$



Quadratic Strict Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$
\begin{aligned}
& a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow \\
& \Big(\exists x \, (ax^2 + bx + c < 0 \land F) \leftrightarrow \\
& \quad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b+\varepsilon} \\
& \quad \lor a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)+\varepsilon} \big) \\
& \quad \lor \ (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} \land F_{\bar{x}}^{-\infty} \Big) \ldots
\end{aligned}
$$

Lemma (Virtual Substitution Lemma for ε)

Nonstandard analysis $\mathrm{FOL}_{\mathbb{R}[\varepsilon]}$

$$F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

$\mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{e+\varepsilon}]\!] \text{ where } r \searrow e$$

Outline

1. Learning Objectives

2. Real Arithmetic
   - Recap: Quadratic Equations
   - Quadratic Weak Inequalities
   - Infinity ∞ Virtual Substitution
   - Expedition: Infinities
   - Quadratic Strict Inequalities
   - Infinitesimal ε Virtual Substitution

3. Quantifier Elimination by Virtual Substitution of Quadratics

4. Summary

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \overset{\text{def}}{=} b^2 - 4ac$.

$$
\begin{aligned}
&\exists x\, F \leftrightarrow \\
&F_x^{-\infty} \\
\lor &\bigvee_{(ax^2+bx+c\{_{\leq}\}0)\in F} \left(a{=}0 \land b{\neq}0 \land F_x^{-c/b} \lor a{\neq}0 \land d{\geq}0 \land \left(F_x^{(-b+\sqrt{d})/(2a)} \lor F_x^{(-b-\sqrt{d})/(2a)}\right)\right) \\
\lor &\bigvee_{(ax^2+bx+c\{_{<}\}0)\in F} \left(a{=}0 \land b{\neq}0 \land F_x^{-c/b+\varepsilon} \lor a{\neq}0 \land d{\geq}0 \land \left(F_x^{(-b+\sqrt{d})/(2a)+\varepsilon} \lor F_x^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)
\end{aligned}
$$

Equivalence needs roots and off-roots from **all** atomic formulas in F

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ *with* $x \notin a, b, c$ *and* $\sim \in \{=, \leq, <, \neq\}$ *and its discriminant* $d \overset{\text{def}}{=} b^2 - 4ac$.

$$\exists x\, F \leftrightarrow$$

$$F_x^{-\infty}$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{=\\ \leq}\}0)\in F} \left(a{=}0\land b{\neq}0 \land F_x^{-c/b} \lor a{\neq}0\land d{\geq}0 \land (F_x^{(-b+\sqrt{d})/(2a)} \lor F_x^{(-b-\sqrt{d})/(2a)})\right)$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{\neq\\ <}\}0)\in F} \left(a{=}0\land b{\neq}0\land F_x^{-c/b+\varepsilon} \lor a{\neq}0\land d{\geq}0\land(F_x^{(-b+\sqrt{d})/(2a)+\varepsilon}\lor F_x^{(-b-\sqrt{d})/(2a)+\varepsilon})\right)$$

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$$F_{\bar{x}}^{-\infty}$$

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$$\lor \bigvee_{(ax^2+bx+c\{\substack{\neq\\ <}\}0)\in F} \big(a{=}0 \land b{\neq}0 \land F_{\bar{x}}^{-c/b+\varepsilon} \lor a{\neq}0 \land d{\geq}0 \land (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon})\big)$$

Equivalence needs roots and off-roots from **all** atomic formulas in *F*

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2+bx+c\sim 0$ with $x\notin a,b,c$ and $\sim\,\in\{=,\leq,<,\neq\}$ and its discriminant $d\overset{\text{def}}{=}b^2-4ac$.

$$\exists x\, F \leftrightarrow$$

$$F_{\bar{x}}^{-\infty}$$

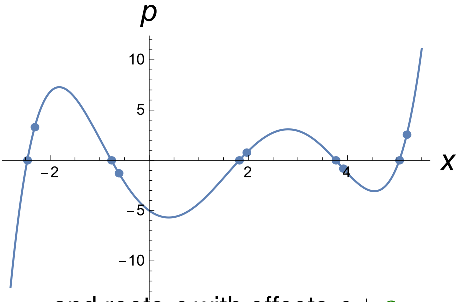
$$\lor \bigvee_{(ax^2+bx+c\{\substack{=\\ \leq}\}0)\in F} \left(a{=}0\land b{\neq}0\land F_{\bar{x}}^{-c/b}\lor a{\neq}0\land d{\geq}0\land\left(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)}\lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}\right)\right)$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{\neq\\ <}\}0)\in F} \left(a{=}0\land b{\neq}0\land F_{\bar{x}}^{-c/b+\varepsilon}\lor a{\neq}0\land d{\geq}0\land\left(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon}\lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon}\right)\right)$$

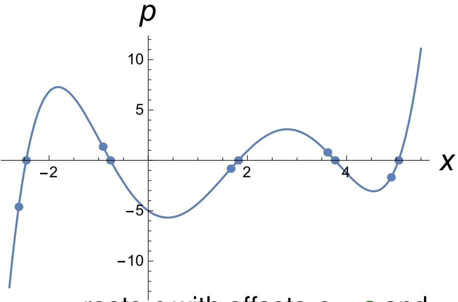
Lemma (Virtual Substitution Lemmas)

$$F_x^{(a+b\sqrt{c})/d}\equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}\qquad F_x^{-\infty}\equiv F_{\bar{x}}^{-\infty}\qquad F_x^{e+\varepsilon}\equiv F_{\bar{x}}^{e+\varepsilon}$$

Alternative Formulations



$-\infty$ and roots e with offsets $e+\varepsilon$



roots e with offsets $e-\varepsilon$ and ∞

No rejection without mention

Other parts of F not satisfied by the points of p have their own polynomial q that contributes different roots $\tilde{e}$ and off-roots $\tilde{e}+\varepsilon$.

Generalizations of quantifier elimination to higher degrees also place a representative point into every region of interest, but derivatives and relationships of derivatives become crucially relevant.

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

$$\exists x\, F \leftrightarrow \quad F_{\bar{x}}^{-\infty}$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{=\\ \leq}\}0)\in F} \big(a=0\land b\neq 0 \land F_{\bar{x}}^{-c/b} \lor a\neq 0\land d\geq 0 \land (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)})\big)$$

$$\lor \bigvee_{(ax^2+bx+c\{\substack{\neq\\ <}\}0)\in F} \big(a=0\land b\neq 0\land F_{\bar{x}}^{-c/b+\varepsilon} \lor a\neq 0\land d\geq 0\land(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon}\lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon})\big)$$

"Proof" Sketch.

"$\leftarrow$" simple from virtual substitution lemma with (extended) term witness

"$\rightarrow$" Valid iff true in every state, so all variables have real numeric value

- o-minimal: solutions form finite union of disjoint intervals (univariate)
- WLOG endpoints are the roots since all polynomials quadratic
- All side conditions have to be met otherwise can't be solution
- Non-point intervals contain ε offset since smaller than endpoint □

Example: Nonnegative Roots of Quadratic Polynomials

$$a \neq 0 \to \big(\exists x\,(ax^2+bx+c=0 \land x \geq 0)$$
$$\leftrightarrow b^2-4ac \geq 0 \land \big(2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$
$$\lor 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0)\big)$$

$$-(-b+\sqrt{b^2-4ac})/(2a) = ((-1+0\sqrt{b^2-4ac})/1)\cdot((-b+\sqrt{b^2-4ac})/(2a)$$
$$= (b-\sqrt{b^2-4ac})/(2a)$$

$$(-x \leq 0)_{\bar{x}}^{(b-\sqrt{b^2-4ac})/(2a)}$$
$$\equiv b2a\leq 0 \land \cancel{b^2}-(-1)^2(\cancel{b^2}-4ac)\geq 0 \lor -1\cdot 2a\leq 0 \land \cancel{b^2}-(-1)^2(\cancel{b^2}-4ac)\leq 0$$
$$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$

$$(-x \leq 0)_{\bar{x}}^{(b+\sqrt{b^2-4ac})/(2a)}$$
$$\equiv b2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \geq 0 \lor 1\cdot 2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \leq 0$$
$$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0$$

Example: Nonnegative Roots of Quadratic Polynomials

$$\begin{aligned}
&\exists x\,(x^2 - x + c = 0 \land x \ge 0)\\
&\leftrightarrow (-1)^2 - 4c \ge 0 \land (c \ge 0 \lor c \le 0\\
&\qquad\qquad\qquad \cancel{\lor c \ge 0 \lor \mathit{false} \land c \le 0})
\end{aligned}$$

Example: Nonnegative Roots of Quadratic Polynomials

$$\begin{aligned}&\exists x\,(x^2-x+c=0 \land x \geq 0)\\ \leftrightarrow\, &1-4c \geq 0\end{aligned}$$

Example: Nonnegative Roots

$$
\begin{aligned}
&\exists x\,(x^2-x+c=0\land x\geq 0)\\
\leftrightarrow\; &1-4c\geq 0\\
&\exists x\,(x^2-x+c\leq 0\land x\geq 0)\\
\leftrightarrow\; &\underbrace{(x^2-x+c\leq 0\land x\geq 0)_{\bar{x}}^{-\infty}}_{\textit{false}}\lor 1-4c\geq 0\lor\underbrace{(x^2-x+c\leq 0\land x\geq 0)_{\bar{x}}^{0}}_{c\leq 0\land 0\geq 0\ \text{(subsumed)}}
\end{aligned}
$$

Example: Nonnegative Roots

$$\exists x\,(x^2-x+c=0\land x\ge 0)$$

$$\leftrightarrow 1-4c\ge 0$$

$$\exists x\,(x^2-x+c\le 0\land x\ge 0)$$

$$\leftrightarrow \underbrace{(x^2-x+c\le 0\land x\ge 0)_{\bar{x}}^{-\infty}}_{\textit{false}}\lor 1-4c\ge 0\lor \underbrace{(x^2-x+c\le 0\land x\ge 0)_{\bar{x}}^{0}}_{c\le 0\land 0\ge 0\ \text{(subsumed)}}$$

$$\exists x\,(x^2-x+c\le 0\land x\ge 0\land -x+2=0)$$

more roots!

$$\leftrightarrow \underbrace{(x^2-x+c\le 0\land\ldots)_{\bar{x}}^{-\infty}}_{\textit{false}}$$

$$\lor 1-4c\ge 0\land \underbrace{(x^2-x+c\le 0\land x\ge 0\land -x+2=0)_{\bar{x}}^{(-1\pm\sqrt{1-4c})/2}}_{8-4c=0}$$

$$\lor -1\ne 0\land \underbrace{(x^2-x+c\le 0\land x\ge 0\land -x+2=0)_{x}^{0}}_{c\le 0\land 0\ge 0\land 2=0}$$

$$\lor -1\ne 0\land \underbrace{(x^2-x+c\le 0\land x\ge 0\land -x+2=0)_{x}^{2}}_{2+c\le 0}\qquad\equiv 2+c\le 0$$

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4. Summary

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$
\begin{aligned}
& a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow \\
& \quad \Big(\exists x \, (ax^2 + bx + c = 0 \land F) \leftrightarrow \\
& \qquad a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b} \\
& \quad \lor a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)
\end{aligned}
$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $\rightarrow$ $F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$ $\leftarrow$ FOL$_\mathbb{R}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \text{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$$

Quantifier Elimination by Virtual Substitution

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$$
\begin{aligned}
&\exists x\, F \leftrightarrow \\
&F_{\bar{x}}^{-\infty} \\
\lor &\bigvee_{(ax^2+bx+c\left\{\substack{=\\ \leq}\right\}0)\in F} \left(a{=}0\land b{\neq}0 \land F_{\bar{x}}^{-c/b} \lor a{\neq}0\land d{\geq}0 \land (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)})\right) \\
\lor &\bigvee_{(ax^2+bx+c\left\{\substack{\neq\\ <}\right\}0)\in F} \left(a{=}0\land b{\neq}0\land F_{\bar{x}}^{-c/b+\varepsilon} \lor a{\neq}0\land d{\geq}0\land(F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon}\lor F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon})\right)
\end{aligned}
$$

Lemma (Virtual Substitution Lemmas)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \qquad F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty} \qquad F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

Summary

- Miracle: $\mathrm{FOL}_{\mathbb{R}}$ is decidable: Tarski'31
  Algorithm decides whether (closed) formula valid or not
- Quantifier elimination computes quantifier-free equivalent
- Successive quantifier elimination decides $\mathrm{FOL}_{\mathbb{R}}$ (after universal closure)
- QE accepts free variables, giving equivalent that identifies the requirements for truth (synthesis)
- Virtual substitution does QE for degree ≤ 3 by equivalent syntactic rephrasing of semantics Weispfenning'97
- QE proceeds inside out, so degree ≤ 3 needed on *each* iteration
- Important fragments permit many optimizations your research?
- Universally quantified weak inequalities / existentially quantified strict inequalities are easier since infinitesimals/infinities don't satisfy =
- Cylindrical algebraic decomposition (CAD) any degree Collins'75
- Simplify arithmetic to relevant parts, transform to fit together

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