The Complete Proof Theory of Hybrid Systems

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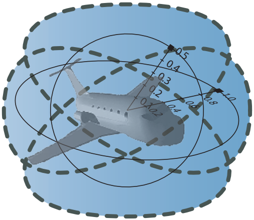
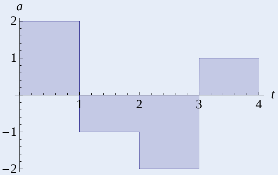


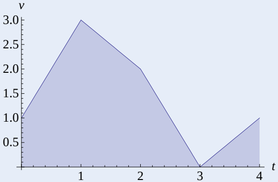
Hybrid Systems: e.g., Car Control

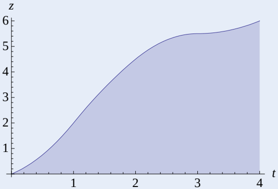
Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)







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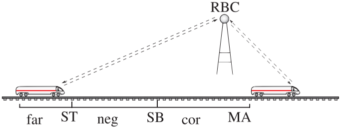
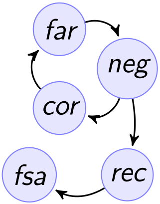
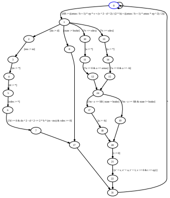
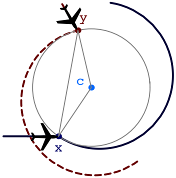
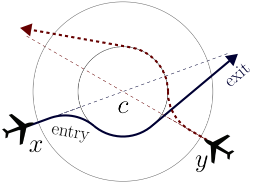
Successful Hybrid Systems Proofs

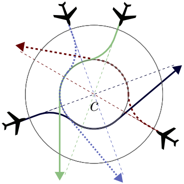


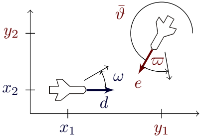


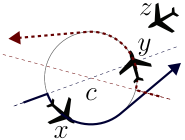


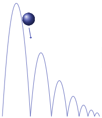


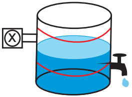




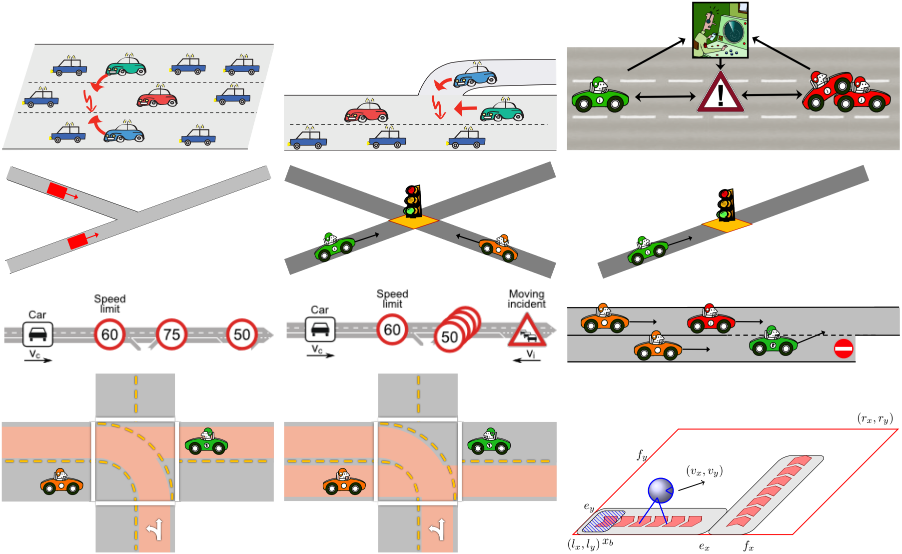








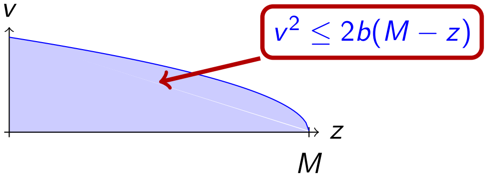
Successful Hybrid Systems Proofs



Proof theory: hybrid = continuous = discrete

differential dynamic logic

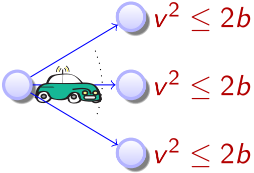
$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$



differential dynamic logic

$d\mathcal{L} = FOL_{\mathbb{R}} + DL + HP$

$$\mathcal{C} \rightarrow [\underbrace{\texttt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \leq 2b$$



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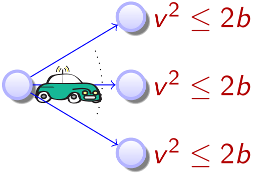

Initial condition

System dynamics

Post condition

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

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All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic d$\mathcal{L}$: Semantics

Definition (Hybrid program α)

$$
\begin{aligned}
\rho(x := \theta) &= \{(v,w) \ : \ w = v \text{ except } [\![x]\!]_w = [\![\theta]\!]_v\} \\
\rho(?H) &= \{(v,v) \ : \ v \models H\} \\
\rho(x' = f(x)) &= \{(\varphi(0),\varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha;\beta) &= \rho(\beta) \circ \rho(\alpha) \\
\rho(\alpha^*) &= \bigcup_{n \in \mathbb{N}} \rho(\alpha^n)
\end{aligned}
$$

Definition (d$\mathcal{L}$ Formula ϕ)

$v \models \theta_1 \geq \theta_2$	iff	$[\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v$
$v \models [\alpha]\phi$	iff	$w \models \phi$ for all w with $(v,w) \in \rho(\alpha)$
$v \models \langle\alpha\rangle\phi$	iff	$w \models \phi$ for some w with $(v,w) \in \rho(\alpha)$
$v \models \forall x\, \phi$	iff	$w \models \phi$ for all w that agree with v except for x
$v \models \exists x\, \phi$	iff	$w \models \phi$ for some w that agrees with v except for x
$v \models \phi \wedge \psi$	iff	$v \models \phi$ and $v \models \psi$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

[:=]	$[x := \theta]\phi(x) \leftrightarrow \phi(\theta)$
[?]	$[?H]\phi \leftrightarrow (H \rightarrow \phi)$
[']	$[x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\, [x := y(t)]\phi \qquad (y'(t) = f(y))$
[∪]	$[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$
[;]	$[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$
[*]	$[\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi$
K	$[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$
I	$[\alpha^*](\phi \rightarrow [\alpha]\phi) \rightarrow (\phi \rightarrow [\alpha^*]\phi)$
C	$[\alpha^*]\forall v{>}0\, (\varphi(v) \rightarrow \langle\alpha\rangle\varphi(v-1)) \rightarrow \forall v\, (\varphi(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\, \varphi(v))$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

G $\dfrac{\phi}{[\alpha]\phi}$

MP $\dfrac{\phi \rightarrow \psi \quad \phi}{\psi}$

$\forall$ $\dfrac{\phi}{\forall x\, \phi}$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

G $\quad \dfrac{\phi}{[\alpha]\phi}$

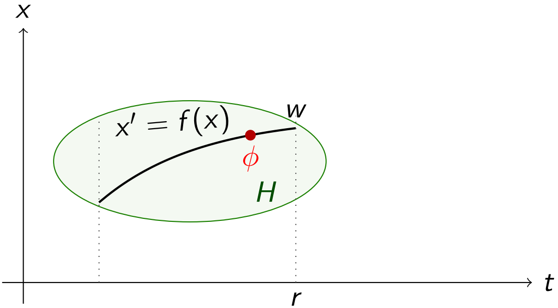
MP $\quad \dfrac{\phi \to \psi \quad \phi}{\psi}$

$\forall \quad \dfrac{\phi}{\forall x\, \phi}$

B $\quad \forall x\, [\alpha]\phi \to [\alpha]\forall x\, \phi \qquad (x \notin \alpha)$

V $\quad \phi \to [\alpha]\phi \qquad (FV(\phi) \cap BV(\alpha) = \emptyset)$

[&] $[x' = f(x)\,\&\,H]\phi \leftrightarrow [x' = f(x)](\phi)$



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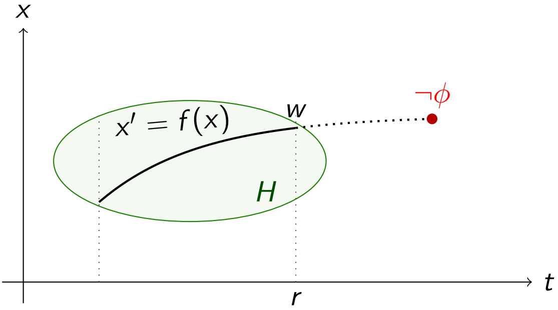
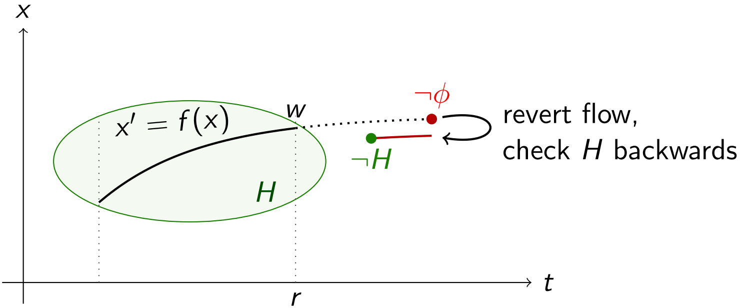


$$[\&] \quad [x' = f(x) \,\&\, H]\phi \leftrightarrow [x' = f(x)]\big([x' = -f(x)](H) \rightarrow \phi\big)$$



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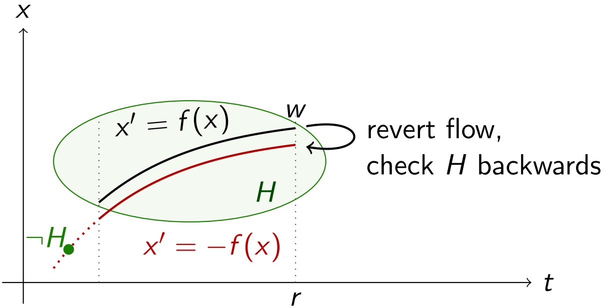
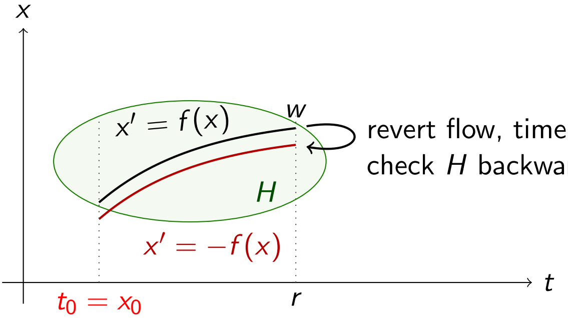


[&] $[x' = f(x) \,\&\, H]\phi$
$\leftrightarrow \forall t_0{=}x_0[x' = f(x)]\big([x' = -f(x)](x_0 \geq t_0 \rightarrow H) \rightarrow \phi\big)$



revert flow, time x_0;
check H backwards

"There and Back Again" Axiom of d$\mathcal{L}$

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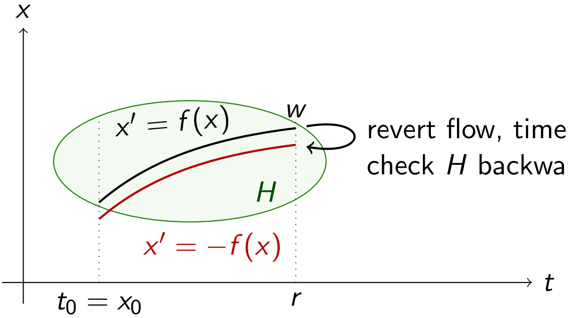

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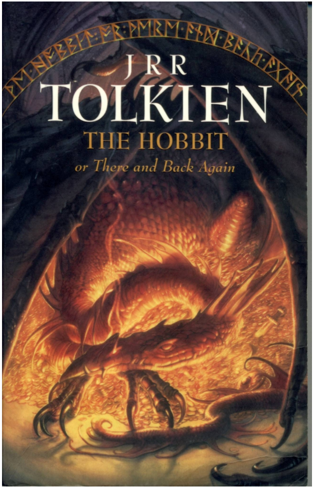

Lemma

Evolution domain axiomatizable

Soundness

Theorem (Soundness)

d$\mathcal{L}$ *calculus is sound, i.e., all provable* d$\mathcal{L}$ *formulas are valid:*

$$\vdash \phi \textit{ implies } \vDash \phi$$

What about the converse?

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$$(s := s + 2n + 1; n := n + 1)^* \quad \rightsquigarrow \quad s = n^2$$

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What about the converse?

$$\begin{array}{lcl} (s := s + 2n + 1; n := n + 1)^* & \leadsto & s = n^2 \\ x' = 5 & \leadsto & x(t) = 5t + x_0 \end{array}$$

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$$x' = 5 \quad \rightsquigarrow \quad x(t) = 5t + x_0$$
$$x' = x \quad \rightsquigarrow \quad x(t) = x_0 e^t$$

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(s := s + 2n + 1; n := n + 1)^* & \leadsto & s = n^2 \\
x' = 5 & \leadsto & x(t) = 5t + x_0 \\
x' = x & \leadsto & x(t) = x_0 e^t \\
x'' = -x & \leadsto & x(t) = x_0 \cos t + x_0' \sin t
\end{array}$$

Continuous Completeness

Theorem (Relative Completeness / Continuous)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

▸ Proof Outline

$$\vDash \phi \text{ iff } Taut_{FOD} \vdash \phi$$

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Proof Outline

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$$\text{FOD} \quad = \quad \text{FOL} + [x' = f(x)]F$$

Continuous Completeness

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$$\vDash \phi \text{ iff } Taut_{FOD} \vdash \phi$$

Corollary (Proof-theoretical Alignment)

proving hybrid systems = proving continuous dynamical systems!

Corollary (Compositionality)

hybrid systems can be verified by recursive decomposition

Discrete Completeness

Theorem (Relative Completeness / Continuous)

dℒ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

Proof Outline

$$\vDash \phi \text{ iff } Taut_{FOD} \vdash \phi$$

Theorem (Relative Completeness / Discrete)

dℒ *calculus is a sound & complete axiomatization of hybrid systems relative to discrete dynamics.*

Proof Outline

$$\vDash \phi \text{ iff } Taut_{DL} \vdash \phi$$

Discrete Completeness

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▸ Proof Outline

$$\vDash \phi \text{ iff } Taut_{DL} \vdash \phi$$

Corollary (Complete Proof-theoretical Alignment)

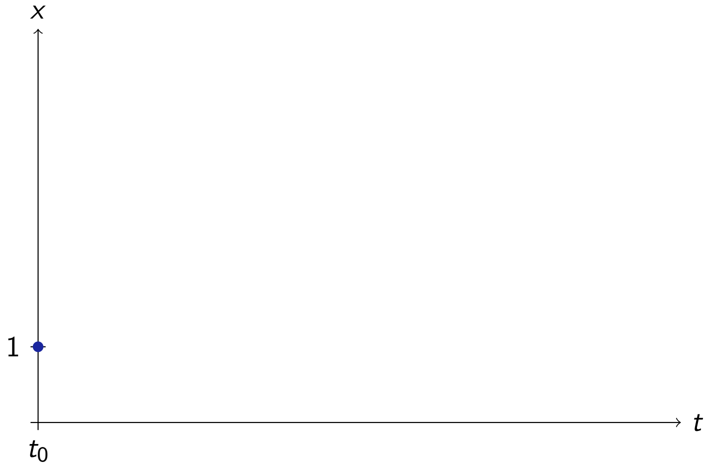
hybrid = continuous = discrete

Corollary (Interdisciplinary Integrability)

"Discrete computer science + continuous control are integrable"

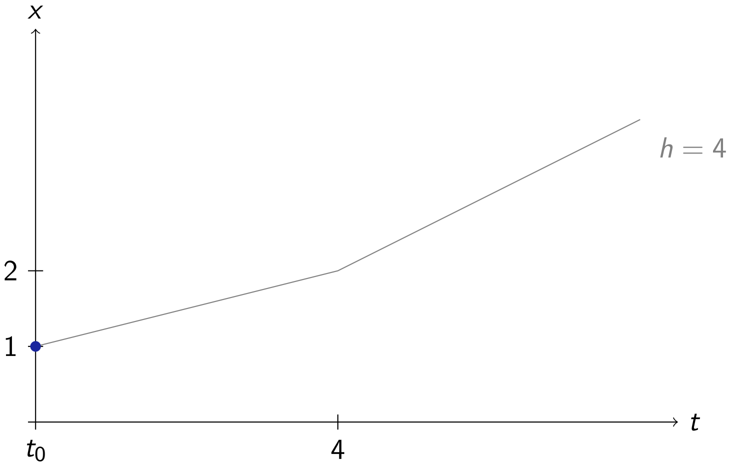
Proof of "hybrid = continuous = discrete"

$$[x' = \frac{x}{4}]F$$



Continuous Dynamics $\neq$ Discrete Dynamics

$$[x' = \frac{x}{4}]F \qquad [(x := x + h\frac{x}{4})^*]F$$



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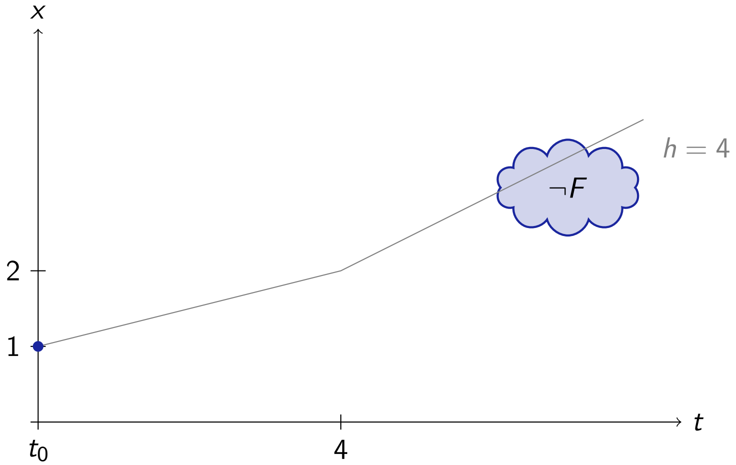

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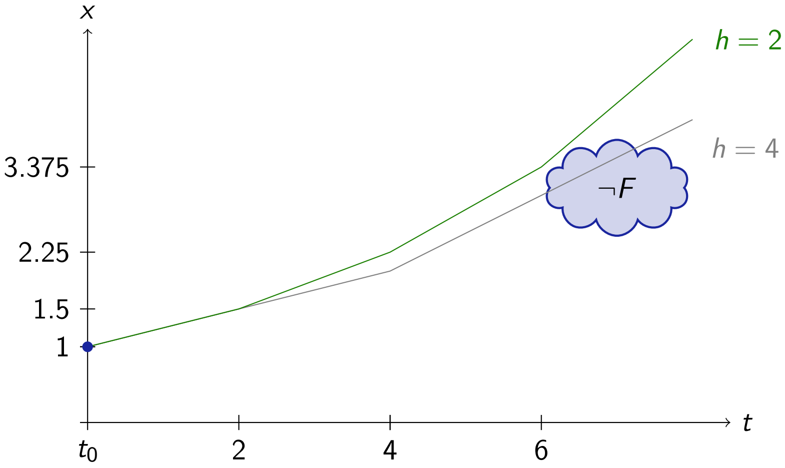

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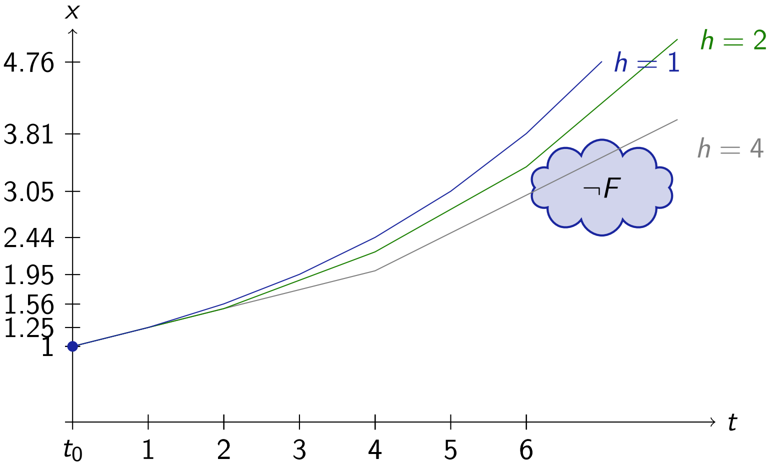

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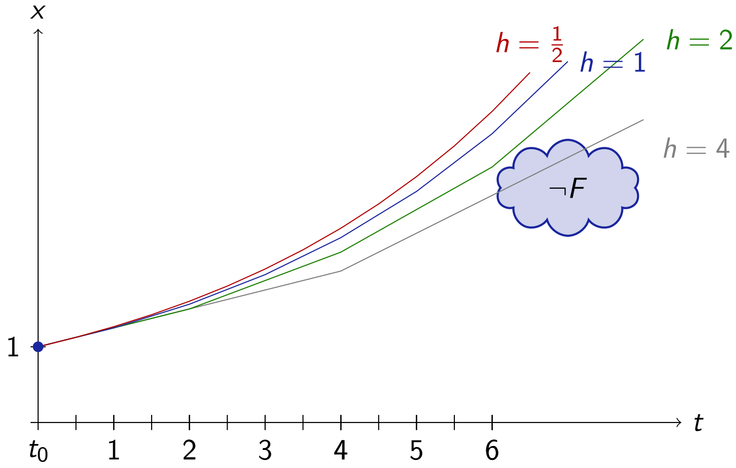
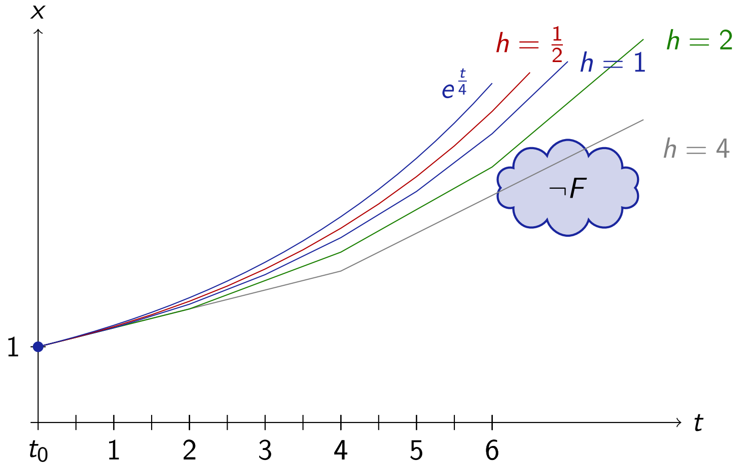


$$[x' = \frac{x}{4}]F \quad \text{vs.} \quad [(x := x + h\frac{x}{4})^*]F$$



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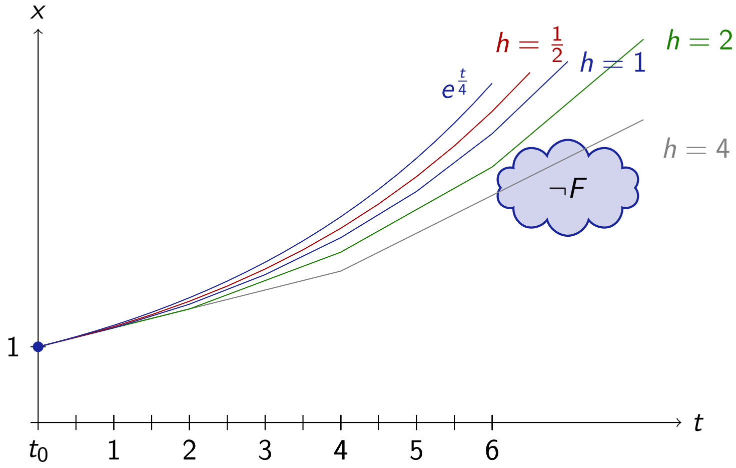

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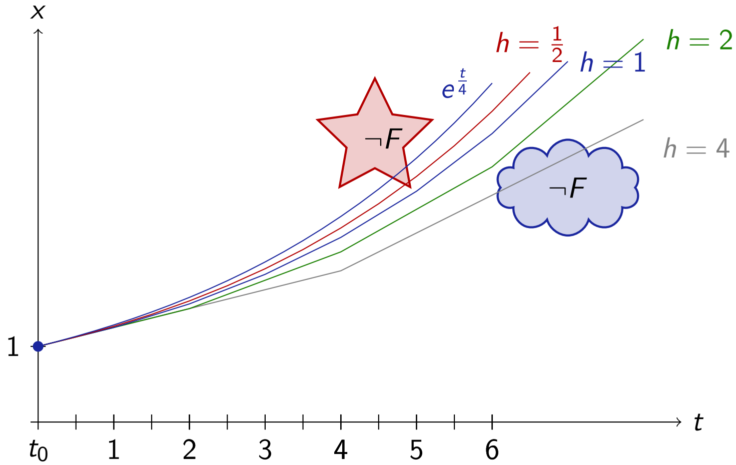

Discrete Euler Approximation Axiom $\overleftarrow{\Delta}$

$$\overleftarrow{\Delta} \quad [x' = f(x)]F \leftarrow \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*]F$$

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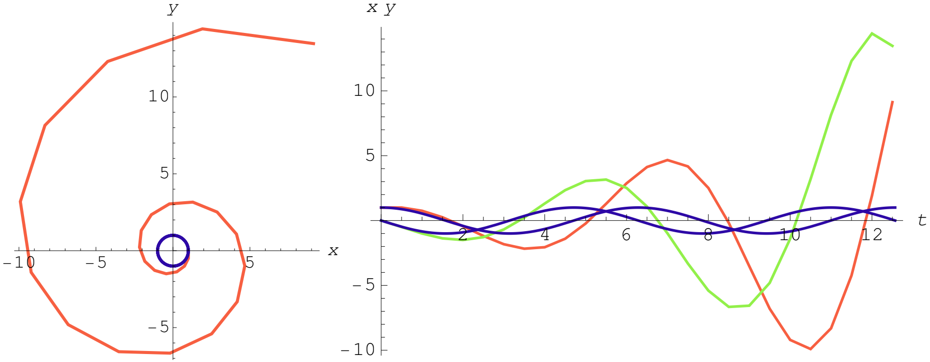
Example (Insufficient, not global)

$$\vDash x^2 + y^2 \leq 1.1 \rightarrow [x' = y, y' = -x]x^2 + y^2 \leq 1.1$$



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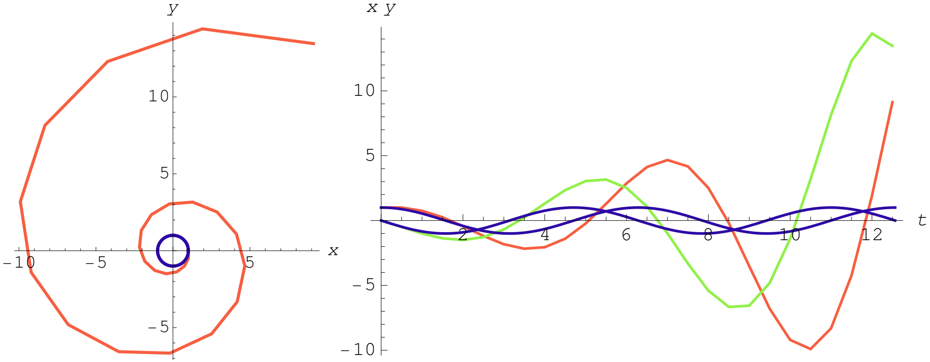
Example (Unsound for open F, only in closure)

$$\nvDash x = 1 \land y = 0 \to [x' = y, y' = -x](x \le 0 \to x^2 + y^2 > 1)$$



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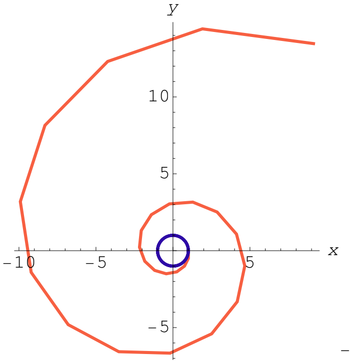

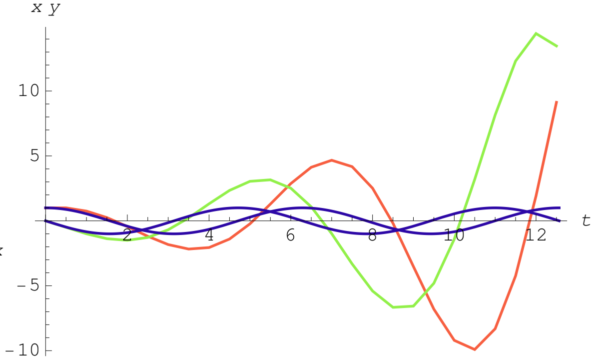

$\overrightarrow{\Delta}$ $[x' = f(x)]F$
$\rightarrow \forall t{\geq}0\, \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*](t \geq 0 \rightarrow F)$

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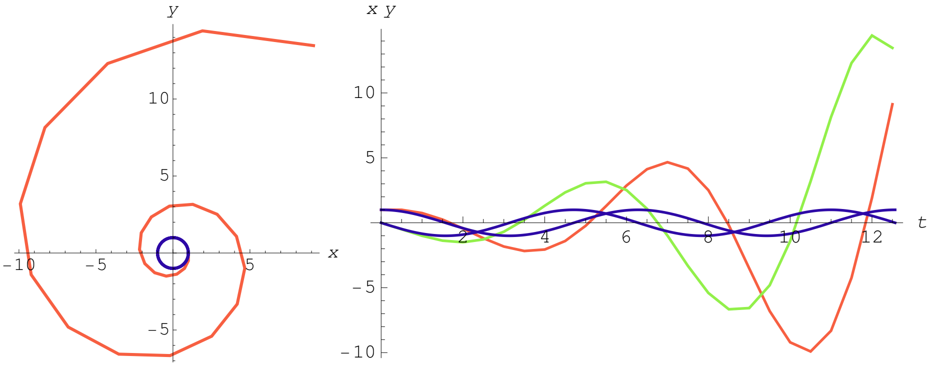
Example (Converse unsound for open F — $\overleftarrow{\Delta}$ for closed F)

$$\nvDash x = 1 \land y = 0 \rightarrow [x' = y, y' = -x](x \le 0 \rightarrow x^2 + y^2 > 1)$$



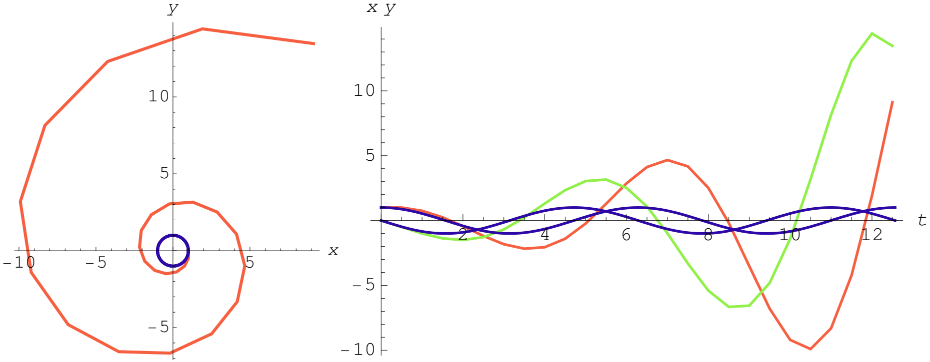
$\mathcal{R}$ Discrete Euler Approximation Axiom $\overrightarrow{\Delta}$ $t' = -1$

$$\overrightarrow{\Delta} \quad [x' = f(x)]F \rightarrow \forall t{\geq}0\, \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*](t \geq 0 \rightarrow F)$$

(open)

Example (Unsound for closed F, only holds in the limit)

$$\vDash x^2 + y^2 = 1 \rightarrow [x' = y, y' = -x]x^2 + y^2 = 1$$



$$\overleftrightarrow{\Delta} \quad [x' = f(x)]F$$
$$\leftrightarrow \forall t{\geq}0 \exists \varepsilon{>}0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x + hf(x))^*] \big(t{\geq}0 \rightarrow \neg \mathcal{U}_\varepsilon(\neg F)\big)$$

$\overleftrightarrow{\Delta}$ $[x' = f(x)]F$
$\leftrightarrow \forall t{\geq}0 \exists \varepsilon{>}0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*] (t{\geq}0 \rightarrow \neg \mathcal{U}_\varepsilon(\neg F))$

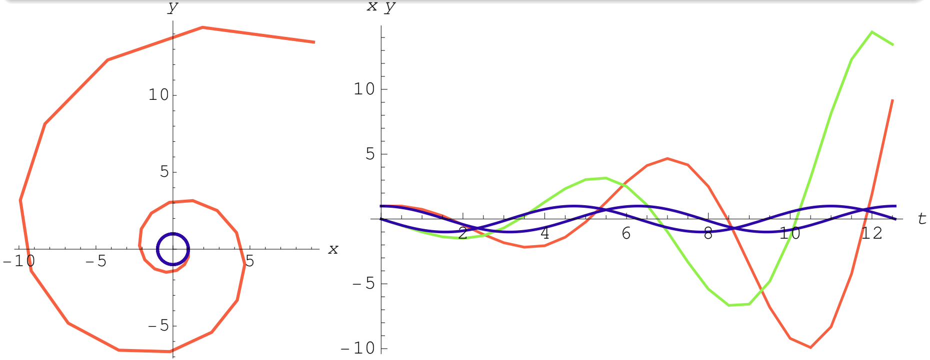
Example ()

$$\vDash x^2 + y^2 < 1.1 \rightarrow [x' = y, y' = -x] x^2 + y^2 < 1.1$$

$\overleftrightarrow{\Delta}$ $[x' = f(x)]F$
$\leftrightarrow \forall t{\geq}0 \exists \varepsilon{>}0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x + hf(x))^*]\big(t{\geq}0 \to \neg\mathcal{U}_\varepsilon(\neg F)\big)$

Example (Insufficient for closed F)

$$\vDash x^2 + y^2 \leq 1 \to [x' = y, y' = -x] x^2 + y^2 \leq 1$$



$\overleftrightarrow{\Delta}$ $[x' = f(x)]F$ *(open)*

$\leftrightarrow \forall t{\geq}0 \exists \varepsilon{>}0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*] \big(t{\geq}0 \rightarrow \neg\mathcal{U}_\varepsilon(\neg F)\big)$

Example (Insufficient for closed F)

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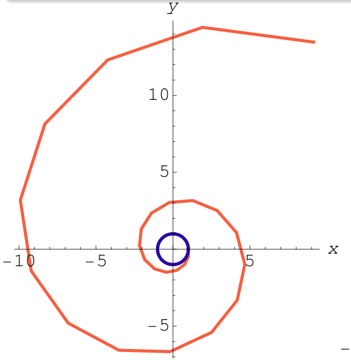
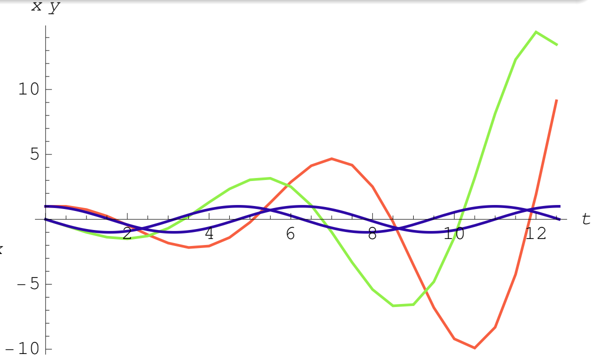
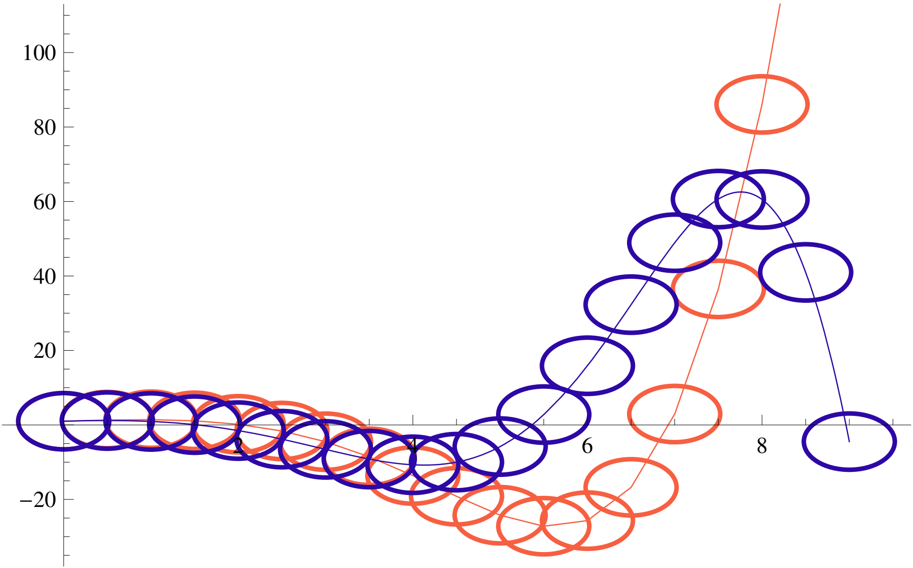




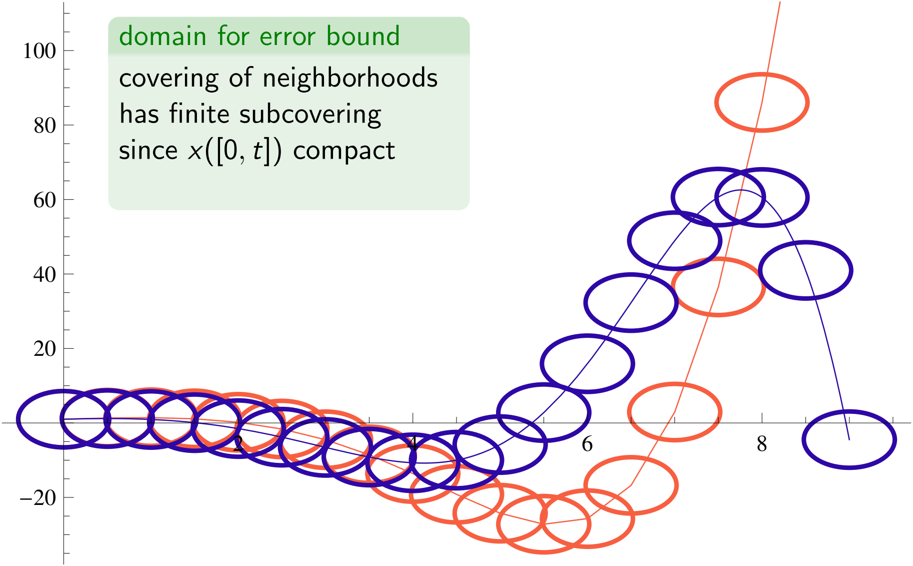
$\mathcal{A}$ $\overleftrightarrow{\Delta}$ Proof: Partial Covering for Solution, Approximation



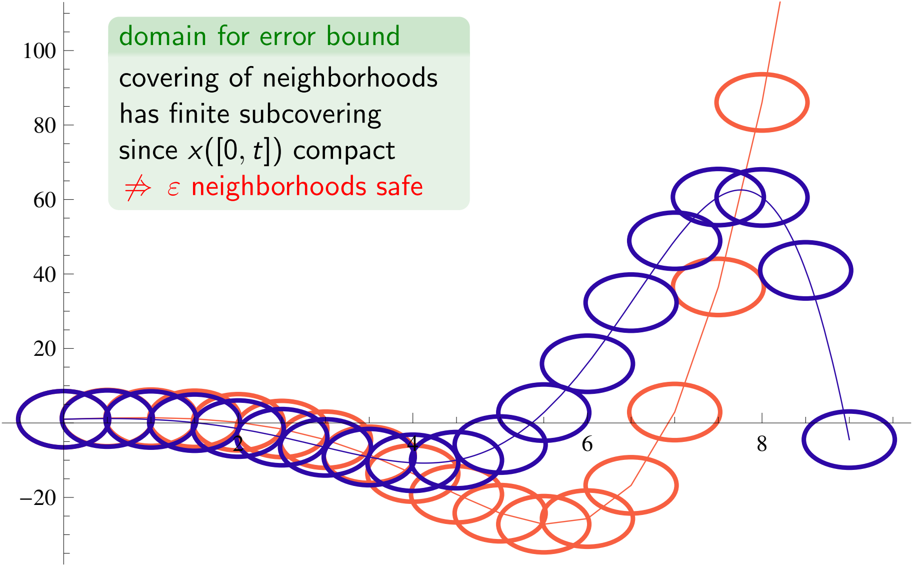
Proof: Partial Covering for Solution, Approximation

domain for error bound

covering of neighborhoods has finite subcovering since $x([0, t])$ compact



$\overleftrightarrow{\Delta}$ Proof: Partial Covering for Solution, Approximation



domain for error bound

covering of neighborhoods
has finite subcovering
since $x([0, t])$ compact
$\not\Rightarrow$ ε neighborhoods safe

$\overleftrightarrow{\Delta}$ axiom for open F, but F may be closed

$\overleftrightarrow{\Delta}$ $[x' = f(x)]F$ (*open*)

$\leftrightarrow \forall t{\geq}0 \exists \varepsilon{>}0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*]\big(t{\geq}0 \to \neg\mathcal{U}_\varepsilon(\neg F)\big)$

Example (Insufficient for closed F)

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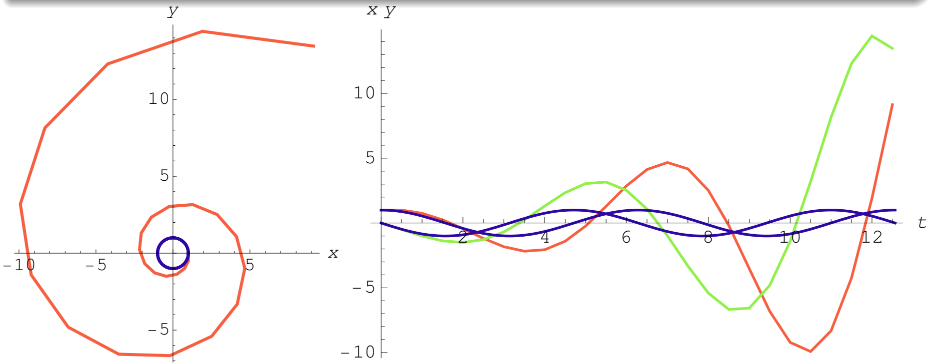


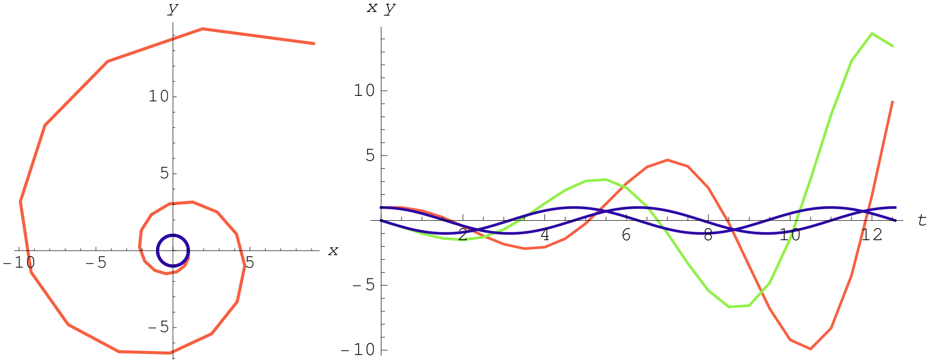
Closed Discrete Completeness (derivable)

$\mathring{U}$ $[x' = f(x)]F \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)]\mathcal{U}_{\check{\varepsilon}}(F)$ $(\Leftarrow$ B,V,G,K$)$

$\mathring{U}$ $[x' = f(x)]F \leftrightarrow \forall \check{\varepsilon}{>}0\,[x' = f(x)]\mathcal{U}_{\check{\varepsilon}}(F)$ $\quad$ ($\Leftarrow$ B,V,G,K)

Example (Closed $\leadsto$ Quantified Open)

$$\vDash x^2 + y^2 \leq 1 \rightarrow [x' = y, y' = -x]x^2 + y^2 \leq 1$$



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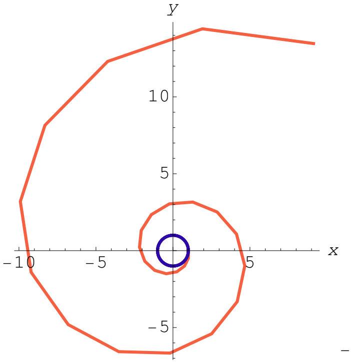
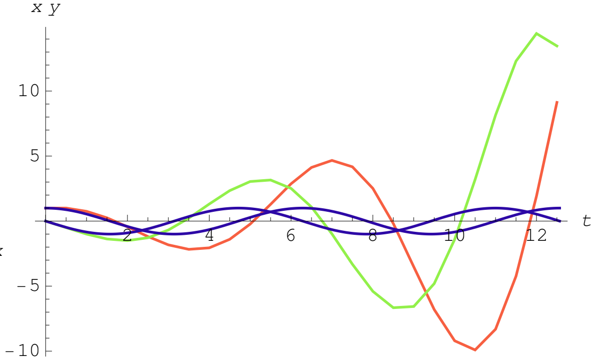
Example (Closed $\rightsquigarrow$ Quantified Open)

$$\vDash x^2 + y^2 \le 1 \to [x' = y, y' = -x]\forall \check{\varepsilon}{>}0\, x^2 + y^2 < 1 + \check{\varepsilon}$$





$\mathring{U}$ $[x' = f(x)]F \leftrightarrow \forall \check{\varepsilon}{>}0\,[x' = f(x)]\mathcal{U}_{\check{\varepsilon}}(F)$ $(\Leftarrow \text{B,V,G,K})$

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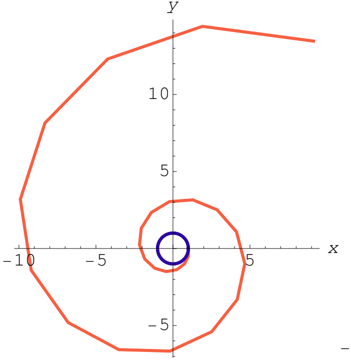

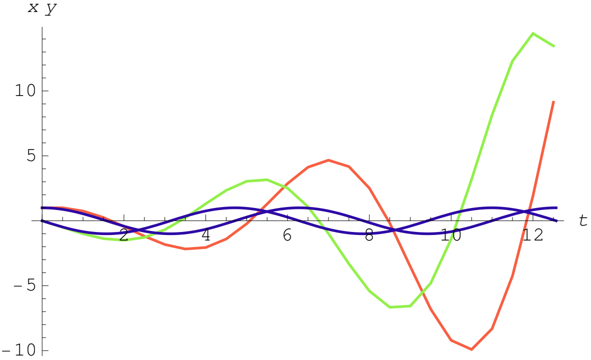

$\overleftrightarrow{\Delta}$ axiom for open/closed F, but otherwise?

Example (Locally Closed $\rightsquigarrow$ Open, Closed)

$$\vDash O \land C \rightarrow [x' = y, y' = -x](O \land C)$$

Locally Closed Discrete Completeness (derivable)

$[]\wedge \quad [\alpha](O \wedge C) \leftrightarrow [\alpha]O \wedge [\alpha]C \qquad (\Leftarrow \text{K})$

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Example (Locally Closed $\rightsquigarrow$ Open, Closed)

$$\vDash O \wedge C \rightarrow [x' = y, y' = -x]O \wedge [x' = y, y' = -x]C$$

Semialgebraic Discrete Completeness (derivable)

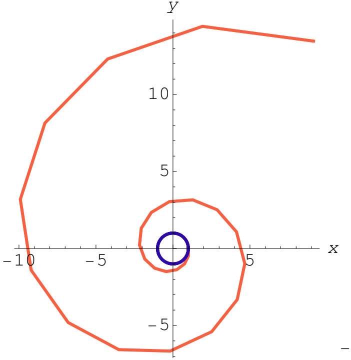
$\check{U}$ $[x' = f(x)](O \lor C) \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)](O \lor \mathcal{U}_{\check{\varepsilon}}(C))$ $(\Leftarrow$ B,V,G,K)

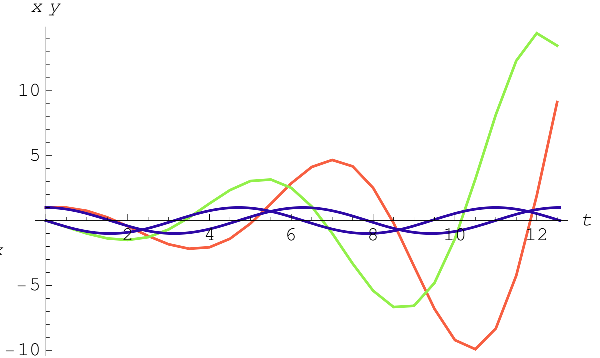
Semialgebraic Discrete Completeness (derivable)

$\check{U} \quad [x' = f(x)](O \vee C) \leftrightarrow \forall \check{\varepsilon}{>}0\,[x' = f(x)](O \vee \mathcal{U}_{\check{\varepsilon}}(C)) \qquad (\Leftarrow \text{ B,V,G,K})$

Example ((Open $\vee$ Closed) $\leadsto$ Quantified Open)

$$\vDash O \vee C \rightarrow [x' = y, y' = -x](O \vee C)$$

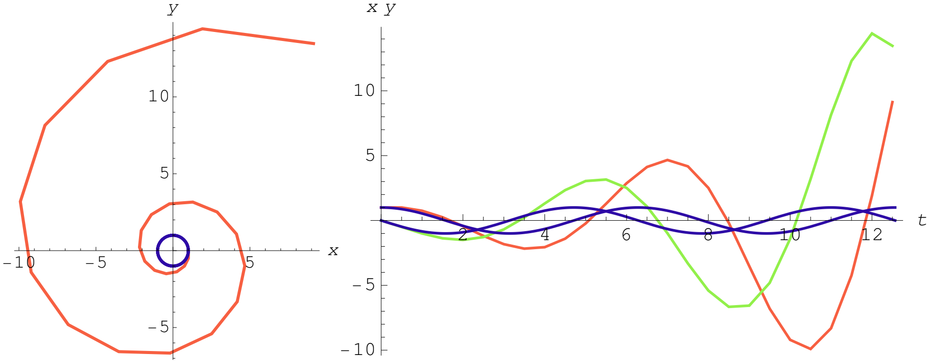




$\check{U}$ $[x' = f(x)](O \vee C) \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)](O \vee \mathcal{U}_{\check{\varepsilon}}(C))$ $(\Leftarrow$ B,V,G,K$)$

Example ((Open ∨ Closed) ⇝ Quantified Open)

$$\vDash O \vee C \to [x' = y, y' = -x](O \vee \forall \check{\varepsilon}{>}0\, \mathcal{U}_{\check{\varepsilon}}(C))$$



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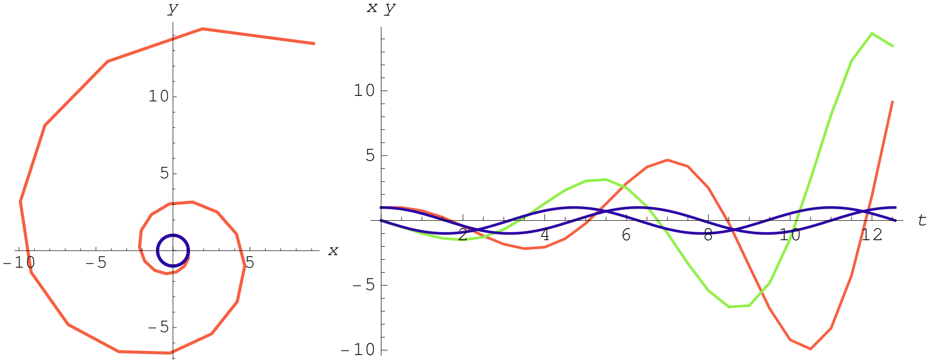

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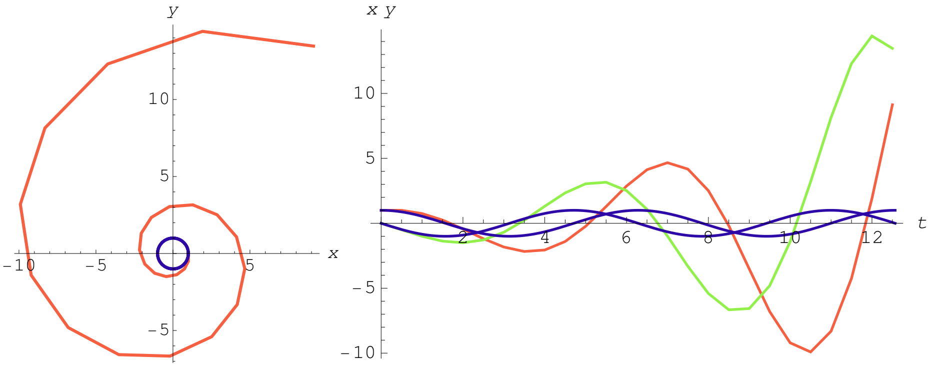


$\overleftrightarrow{\Delta}$ axiom for semialgebraic F, but otherwise?

Discrete Completeness

Theorem (Relative Completeness / Continuous)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

▸ Proof Outline 6p

$$\vDash \phi \text{ implies } Taut_{FOD} \vdash \phi$$

Theorem (Relative Completeness / Discrete)

d$\mathcal{L}$ *calculus* $+\overleftrightarrow{\Delta}$ *is a sound & complete axiomatization of hybrid systems relative to discrete dynamics.*

▸ Proof Outline +5p

$$\vDash \phi \text{ implies } Taut_{DL} \vdash \phi$$

Proof Sketch.

Talked about 0-order semialgebraic

Paper proves $\forall, \exists$ ...

Paper proves $[\alpha], \langle\alpha\rangle$ with hybrid system α ...

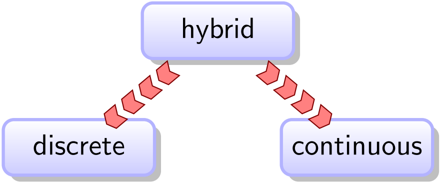
Paper proves nesting ... □

Equi-expressibility

Theorem (Equi-expressibility)

d$\mathcal{L}$ (constructively) expressible in FOD and in DL:

$$\forall\phi \;\; \exists\phi^\flat \in FOD \;\; \vDash \phi \leftrightarrow \phi^\flat$$
$$\forall\phi \;\; \exists\phi^\# \in DL \;\; \vDash \phi \leftrightarrow \phi^\#$$



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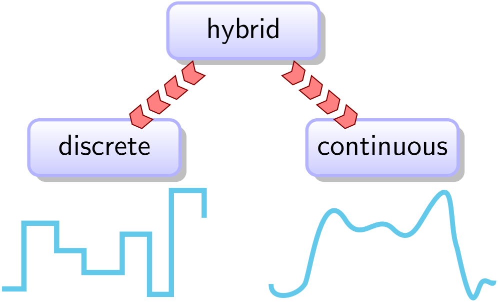

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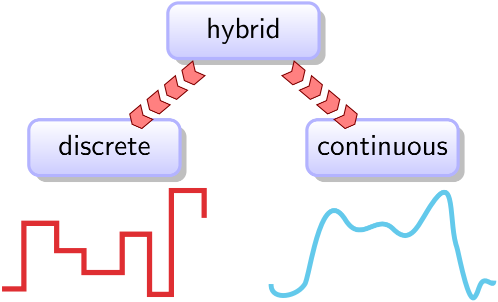

Equi-expressibility

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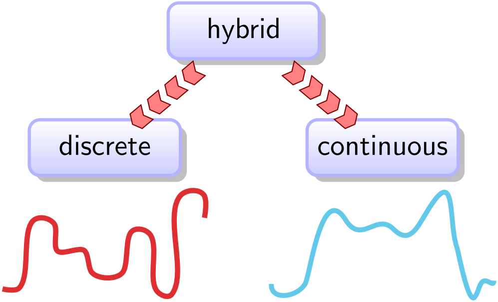
dℒ *(constructively) expressible in FOD and in DL:*

$$\begin{array}{lll} \forall\phi\ \exists\phi^\flat \in FOD & \vDash \phi \leftrightarrow \phi^\flat \\ \forall\phi\ \exists\phi^\# \in DL & \vDash \phi \leftrightarrow \phi^\# \end{array}$$



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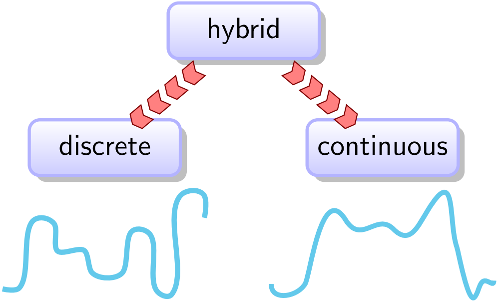

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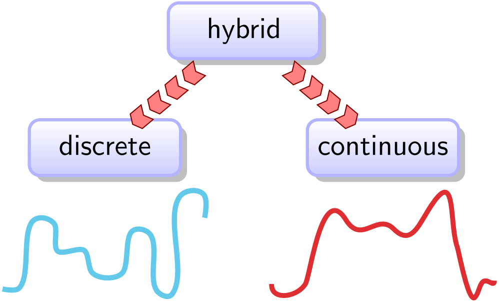

Equi-expressibility

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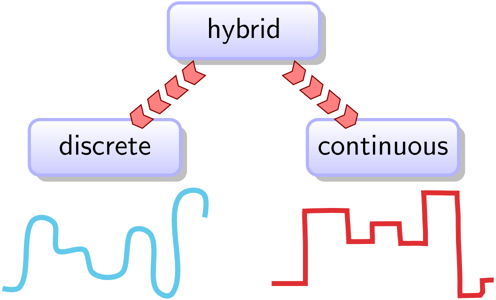

Equi-expressibility

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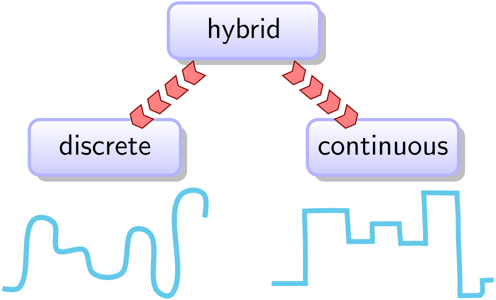


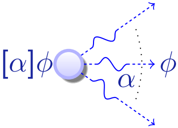
Relative Decidability

Theorem (Relative Decidability)

Validity of $\mathrm{d}\mathcal{L}$ *sentences is decidable relative to FOD or DL.*

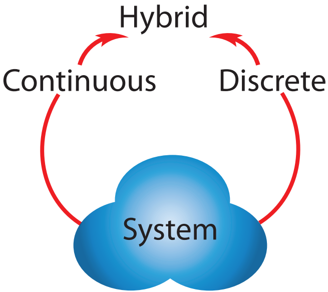
differential dynamic logic

$d\mathcal{L} = DL + HP$



proof-theoretical alignment

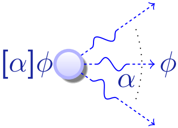
hybrid = continuous = discrete



The Complete Proof Theory of Hybrid Systems

differential dynamic logic

$d\mathcal{L} = \text{DL} + \text{HP}$



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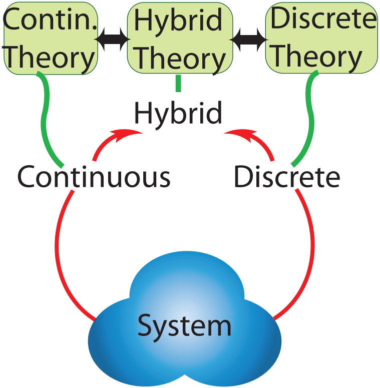

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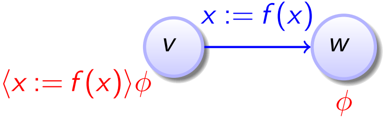
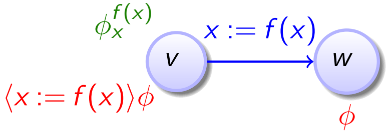
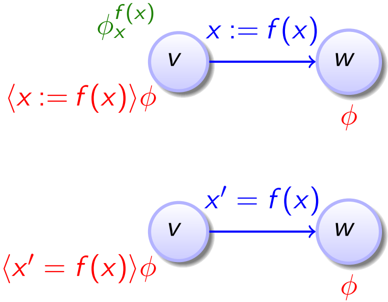
Proof by Symbolic Decomposition



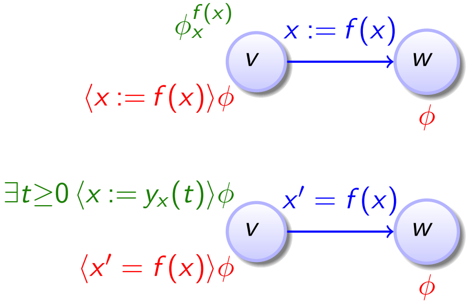
Proof by Symbolic Decomposition



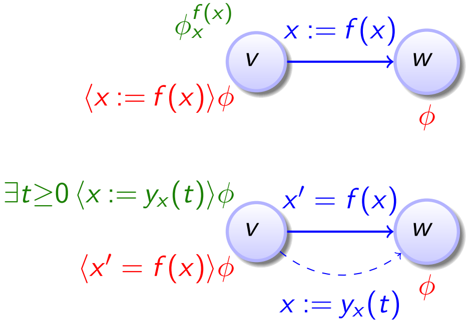
Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



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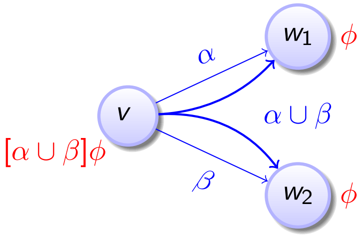
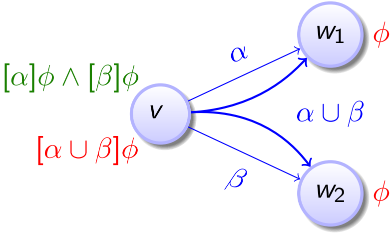


Proof by Symbolic Decomposition



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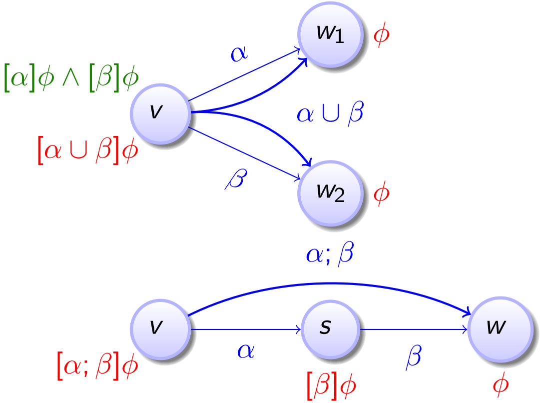
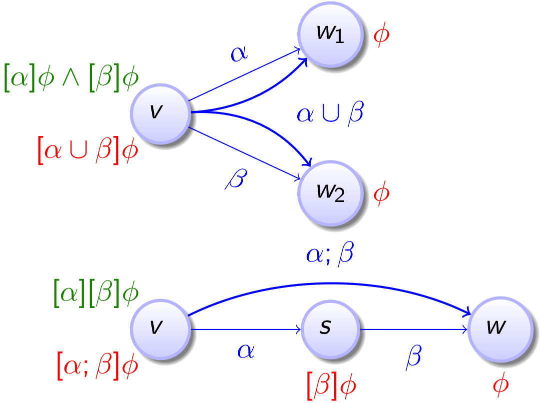
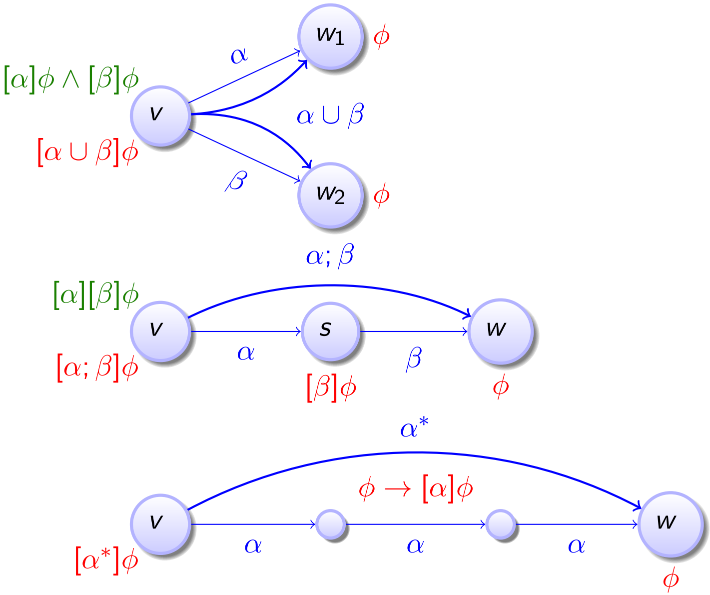


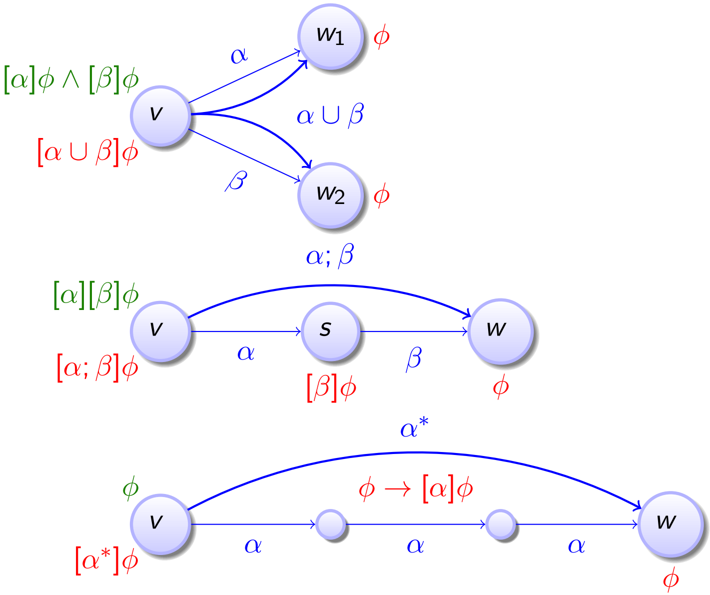
Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



Proof by Symbolic Decomposition