A Deductive Refinement Calculus for Differential-Algebraic Equations

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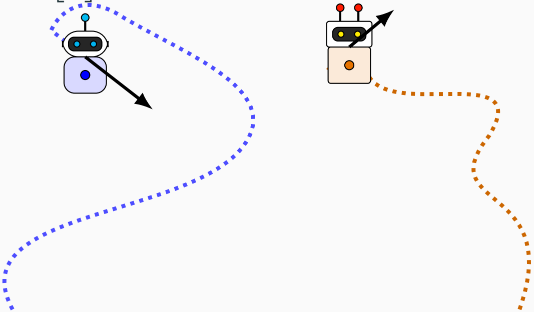
Modeling Physical Systems with DAEs

$$m_1 \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{bmatrix} = \mathbf{F}_1 \qquad m_2 \begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \mathbf{F}_2$$



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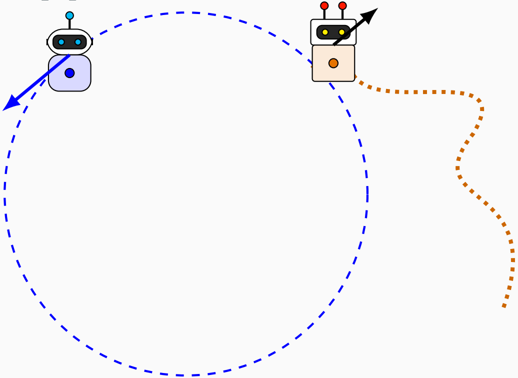

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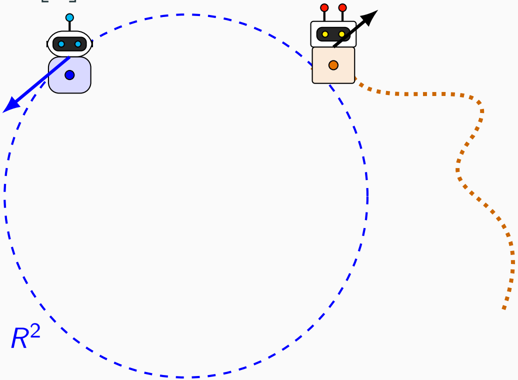
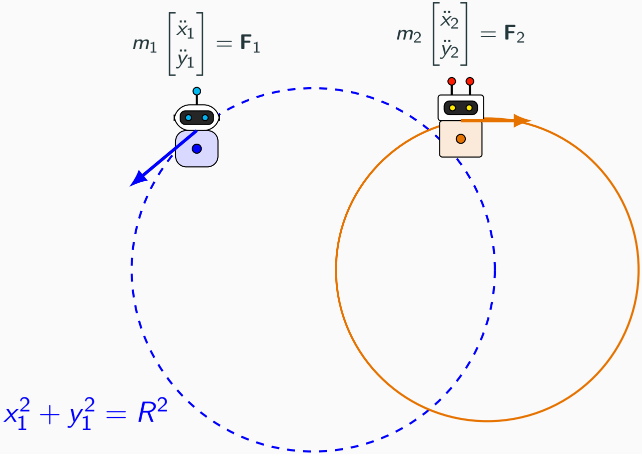


$$x_1^2 + y_1^2 = R^2$$

Modeling Physical Systems with DAEs



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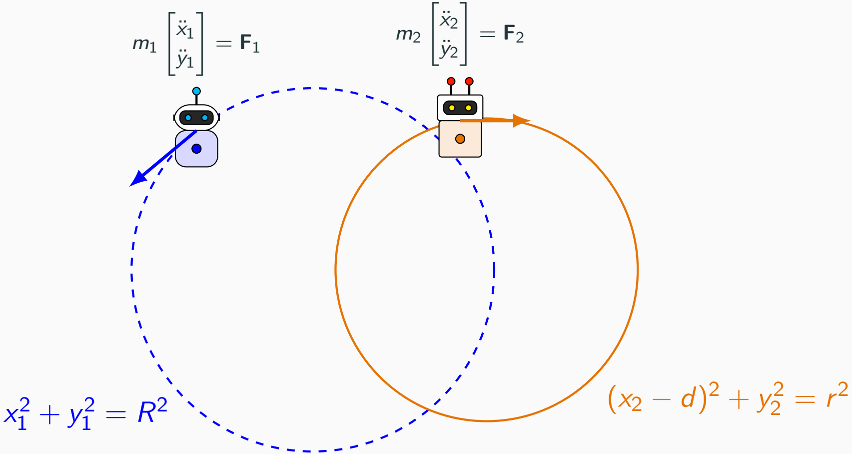

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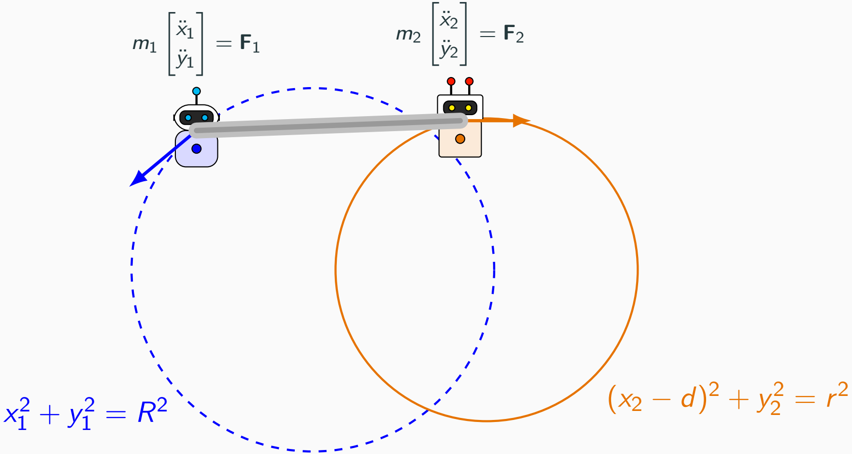

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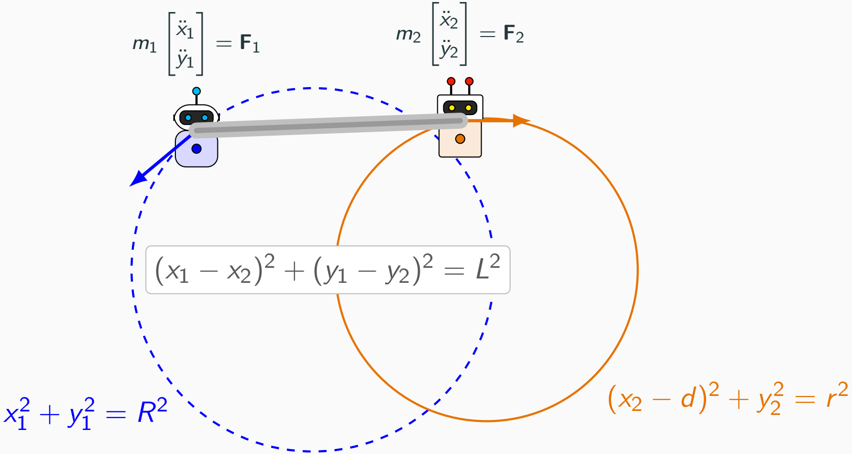

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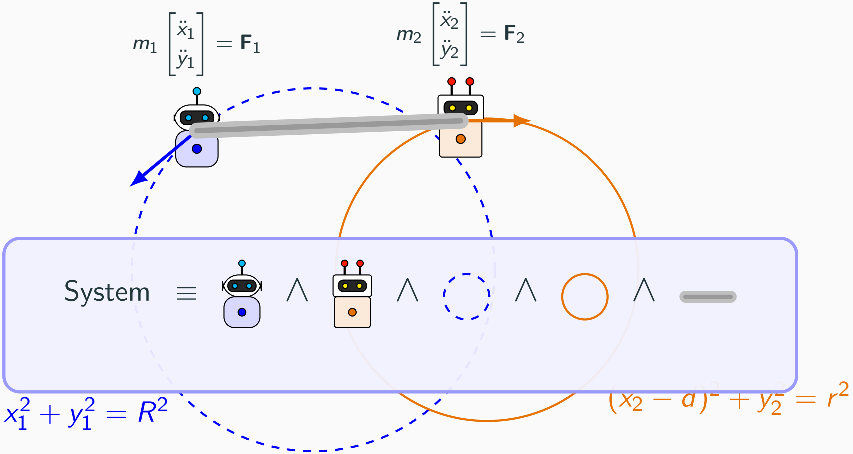

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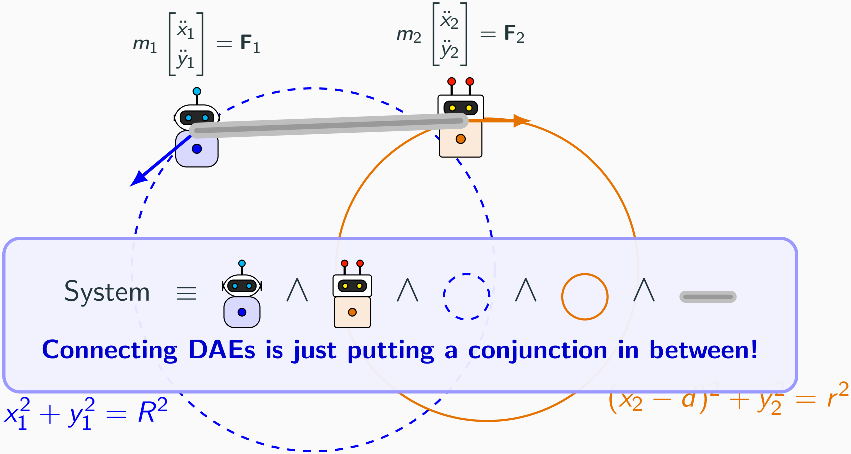
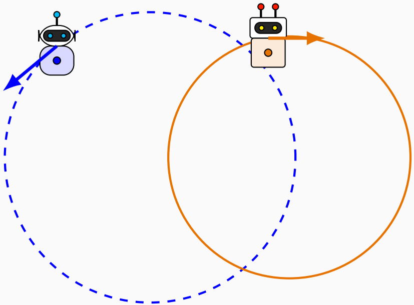


Connecting DAEs is just putting a conjunction in between!

The DAE Verification Problem



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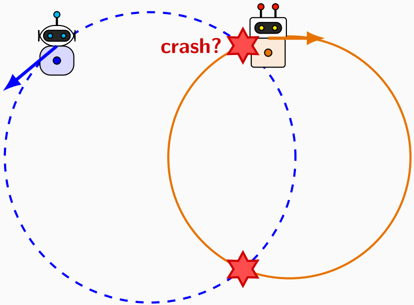

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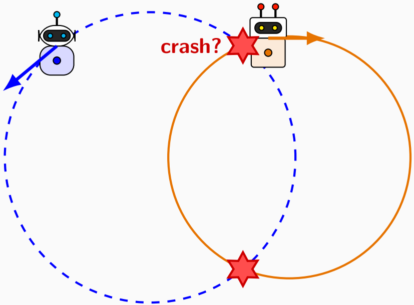

DAEs are a powerful modeling paradigm, but

-  solutions are *not unique* and *non-smooth*,
- $\boxed{x^{100} + xy = 0}$ contain *non-linear algebraic equations*,
- $\boxed{x' = ?}$ leave dynamics *hidden*.

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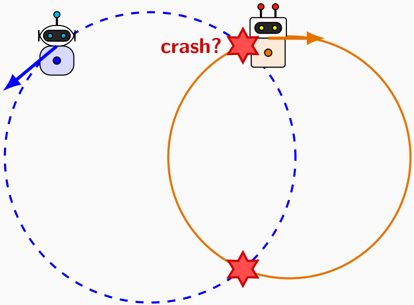

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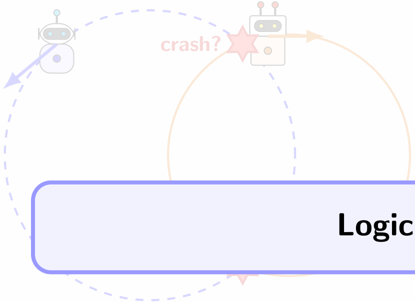
How can we possibly verify such DAE systems?



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Logic to the Rescue!

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Contributions: The dARL Proof Calculus

We introduce **dARL** $= \mathrm{d}\mathcal{L} + \mathrm{DAPs} + \mathrm{Refinement}$:

- C^1 **Trace Semantics** (Generalizes real-analytic DAPs)
  Rigorously defines DAE behavior, handling *non-unique and non-smooth* solutions $(\min, |\cdot|)$.

- $\boxed{x^{100} + xy = 0}$ **Modular Refinement calculus** $(\alpha \leq_{\mathbf{x}} \beta)$ (First for DAEs)
  Avoids solving *non-linear algebraic equations* by reasoning about them deductively.

- $\boxed{x' = ?}$ **Soundly uncovering hidden dynamics**
  Systematically reveals *hidden dynamics* and enables relational reasoning between DAPs.

dARL syntax

$$P, Q ::= \overbrace{e \leq g \mid \neg P \mid P \wedge Q \mid \forall x\, P} \mid \overbrace{[\alpha]P} \mid \overbrace{\alpha \leq_{\mathbf{x}} \beta}$$

$$\alpha, \beta ::= \underbrace{x := e \mid ?Q} \mid \underbrace{\{\mathbf{x} : F\}} \mid \underbrace{\alpha \cup \beta \mid \alpha; \beta \mid (\alpha)^*}$$

Syntax

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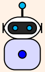
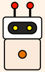
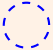
Example: Safety Formula in dARL

$\mathbf{x} \neq \mathbf{y} \rightarrow [\{\mathbf{x}, \mathbf{y} :$  $\wedge$  $\wedge$  $\wedge$  $\wedge$  $\}]\, \mathbf{x} \neq \mathbf{y}$

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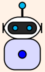
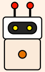
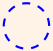
$\underbrace{\mathbf{x} \neq \mathbf{y}}_{\text{Initial Condition}} \rightarrow [\{\mathbf{x}, \mathbf{y} :$  $\wedge$  $\wedge$  $\wedge$  $\wedge$  $\}]\, \underbrace{\mathbf{x} \neq \mathbf{y}}$

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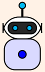
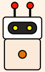
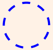
$\underbrace{\mathbf{x} \neq \mathbf{y}}_{\text{Initial Condition}} \rightarrow [\{\mathbf{x}, \mathbf{y} :$  $\wedge$  $\wedge$  $\wedge$  $\wedge$  $\}]\, \underbrace{\mathbf{x} \neq \mathbf{y}}$

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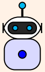
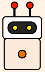
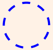
$\underbrace{\mathbf{x} \neq \mathbf{y}}_{\text{Initial Condition}} \rightarrow [\{\mathbf{x}, \mathbf{y} :$  $\wedge$  $\wedge$  $\wedge$  $\wedge$  $\}]\, \underbrace{\mathbf{x} \neq \mathbf{y}}_{\text{Postcondition}}$

DAE

Proof Architecture: Safety via Refinement

Complex DAE Program

α_{DAE}

$$
\begin{tikzcd}
\text{Complex DAE Program } \alpha_{\mathsf{DAE}} \arrow[d, dashed, "\text{Direct proof?}"'] \\
\text{Target DAE Safety } P \to [\alpha_{\mathsf{DAE}}]Q
\end{tikzcd}
$$

Target DAE Safety

$P \to [\alpha_{\mathsf{DAE}}]Q$

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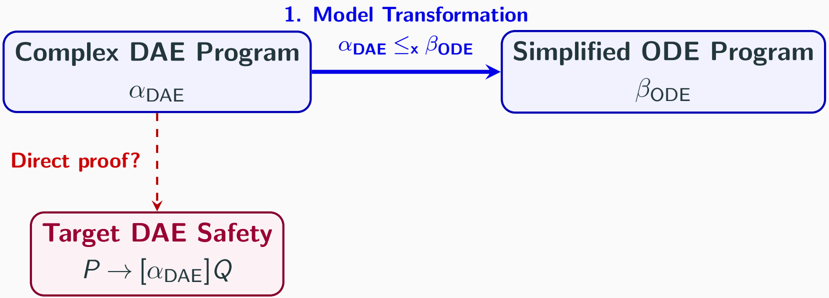

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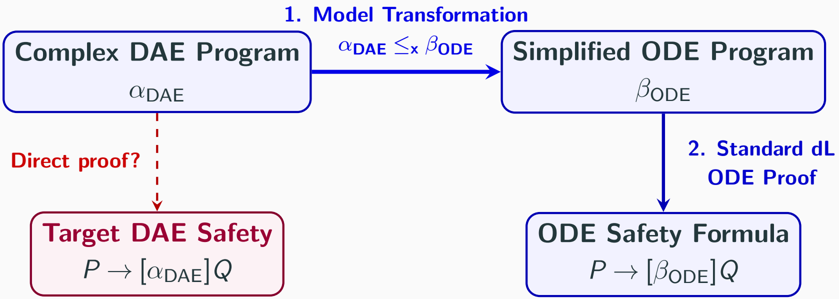

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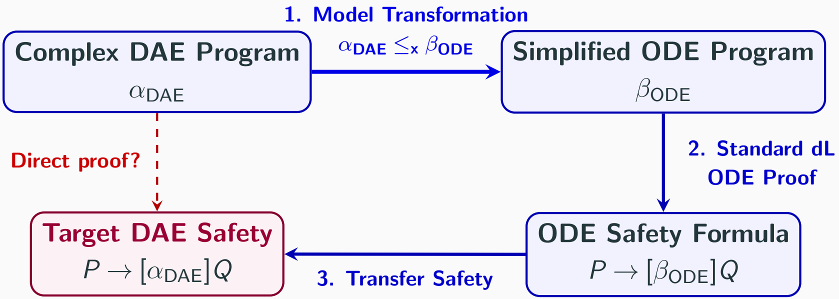

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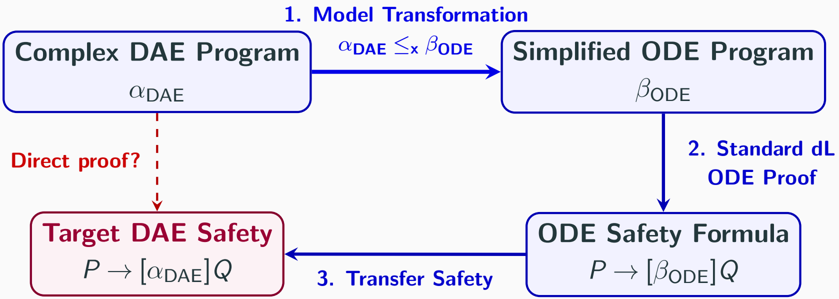

Refinement transfers safety from simplified models to complex DAEs!

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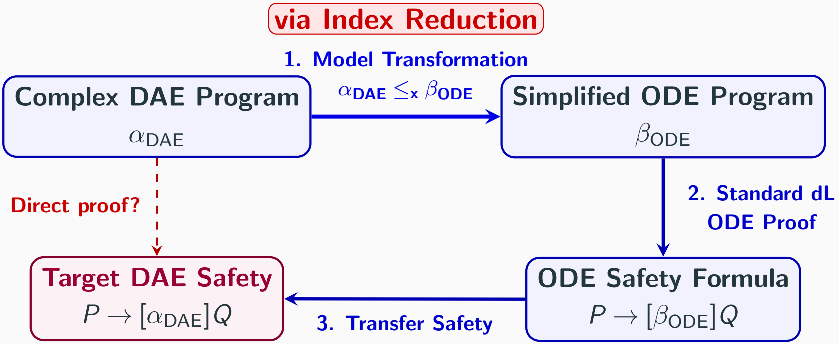

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Index reduction reveals hidden constraints by repeated differentiation.

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1. Pick $e = 0$ → **2. Differentiate** $(e)' = 0$ → **3. Algebraic Reduction** → **4. Augment** DAP

Repeat until explicit ODE

Index Reduction in dARL

Derivable Proof Rule: Index Reduction (IR)

Let $\mathbf{r}_0, \ldots, \mathbf{r}_m$ be a sequence of term vectors. Then IR is derivable in dARL:

$$\mathrm{IR}\ \frac{\Gamma \vdash \bigwedge_{k=0}^{m} (\mathbf{r}_k)' = \mathbf{0} \quad \vdash \bigwedge_{k=0}^{m} (\mathbf{f}_k = \mathbf{0} \land Q \rightarrow \mathbf{r}_k = \mathbf{0})}{\Gamma \vdash \{\mathbf{x} : \mathbf{f}_{m+1} = \mathbf{0} \land Q\} =_{\mathbf{x}^\bullet} \{\mathbf{x} : \mathbf{f}_0 = \mathbf{0} \land Q\}}\ \mathrm{FV}(\mathbf{r}_1, \ldots, \mathbf{r}_m) \subseteq \mathbf{x}$$

Why Informal Index Reduction is Unsound

Naive index reduction produces systems incomparable to the original:

1. **Ignoring initial consistency conditions** (introduces drift / spurious behavior)
2. **Unsound algebraic simplifications**
3. **Violating regularity conditions** (may exclude valid C^1 solutions)

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Why Informal Index Reduction is Unsound

Naive index reduction produces systems incomparable to the original:

1. **Ignoring initial consistency conditions** (introduces drift / spurious behavior)
2. **Unsound algebraic simplifications**
3. **Violating regularity conditions** (may exclude valid C^1 solutions)

Index Reduction in dARL

Derivable Proof Rule: Index Reduction (IR)

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The dARL Proof Calculus (excerpt)

Refinement & Cut Axiom

$[\leq_{\mathsf{x}}]$ $\quad \alpha \leq_{\mathsf{x}} \beta \rightarrow ([\beta]P \rightarrow [\alpha]P)$ [1]

$\leq^t_{\mathsf{x}}$ $\quad \alpha \leq_{\mathsf{x}} \beta \wedge \beta \leq_{\mathsf{x}} \gamma \rightarrow \alpha \leq_{\mathsf{x}} \gamma$

DC $\quad [\{\mathsf{x} : F\}]R \rightarrow \{\mathsf{x} : F\} \leq_{\mathsf{x}^\bullet} \{\mathsf{x} : R\}$

[1] $\mathrm{FV}(P) \cap \mathrm{BV}(\alpha, \beta) \subseteq \mathsf{x}$

Box Axioms

DW $\quad [\{\mathsf{x} : F\}]F$

DHC $\quad (e)' = 0 \rightarrow [\{\mathsf{x} : e = 0\}](e)' = 0 \quad \mathrm{FV}(e) \subseteq \mathsf{x}$

$\mathrm{DI}_{\preceq}$ $\quad e \preceq 0 \rightarrow [\{\mathsf{x} : (e)' \leq 0\}]e \preceq 0 \quad \mathrm{FV}(e) \subseteq \mathsf{x}$

Subsystem Axioms (DAE Monotonicity & Reduction)

DM $\quad \{\mathsf{x} : F\} \leq_{\mathsf{x}^\bullet} \{\mathsf{x} : G\} \rightarrow \{\mathsf{x} : F \wedge R\} \leq_{\mathsf{x}^\bullet} \{\mathsf{x} : G \wedge R\}$

DR $\quad \forall \mathsf{z} \forall \mathsf{z}'(\{\mathsf{x}, \mathsf{z} : R \wedge F\} \leq_{\mathsf{x}^\bullet} \{\mathsf{x} : F\})$

Soundness Theorem: The dARL proof calculus is sound w.r.t. C^1 DAE flow semantics.

Core Engine for Modular Transfer: DC, DHC, DM

DC Differential Cut

$$[\{\mathbf{x} : F\}]R \rightarrow \{\mathbf{x} : F\} \leq_{\mathbf{x}^\bullet} \{\mathbf{x} : R\}$$

Cuts proved continuous invariants R directly into the DAE system.

DHC Differential Hidden Constraint

$$(e)' = 0 \rightarrow [\{\mathbf{x} : e = 0\}](e)' = 0 \qquad \mathrm{FV}(e) \subseteq \mathbf{x}$$

Differentiates algebraic constraints ($e = 0 \implies (e)' = 0$) during index reduction.

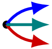
DM Differential Monotonicity

$$\{\mathbf{x} : F\} \leq_{\mathbf{x}^\bullet} \{\mathbf{x} : G\} \rightarrow \{\mathbf{x} : F \wedge R\} \leq_{\mathbf{x}^\bullet} \{\mathbf{x} : G \wedge R\}$$

Preserves refinement $\alpha \leq_{\mathbf{x}} \beta$ under shared context constraints R.

Takeaway: Together, DC + DHC + DM enable modular verification of complex systems.

Summary & Future Work



C^1 **Trace Semantics**

$\boxed{x^{100} + xy = 0}$

Modular Refinement Calculus

$\boxed{x' = ?}$

Sound Index Reduction

Summary & Future Work

C^1 **Trace Semantics**

$\boxed{x^{100} + xy = 0}$

Modular Refinement Calculus

$\boxed{x' = ?}$

Sound Index Reduction

Hidden constraints causing soundness chaos? Logic to the rescue!

Thank You

Questions? Contact: `jonathan.hellwig@kit.edu`

Refinement

Refinement Semantics

$$[\![\alpha \leq_{\mathsf{x}} \beta]\!] = \{\omega \in \mathbb{S} \mid \forall \varphi \in [\![\alpha]\!](\omega)\ \exists \psi \in [\![\beta]\!](\omega) : \psi \sim_{\mathsf{x}} \varphi\}$$

Refinement $(\alpha \leq_{x,y} \beta)$