# Semi-Competitive Differential Hybrid Games

Julia Butte, André Platzer

# Introduction

- Cyber-physical systems are not alone in their environment:

# Introduction

- Cyber-physical systems are not alone in their environment:

# Introduction

- Cyber-physical systems are not alone in their environment:

Traffic

Air Traffic

# Introduction

- Cyber-physical systems are not alone in their environment:



# Introduction

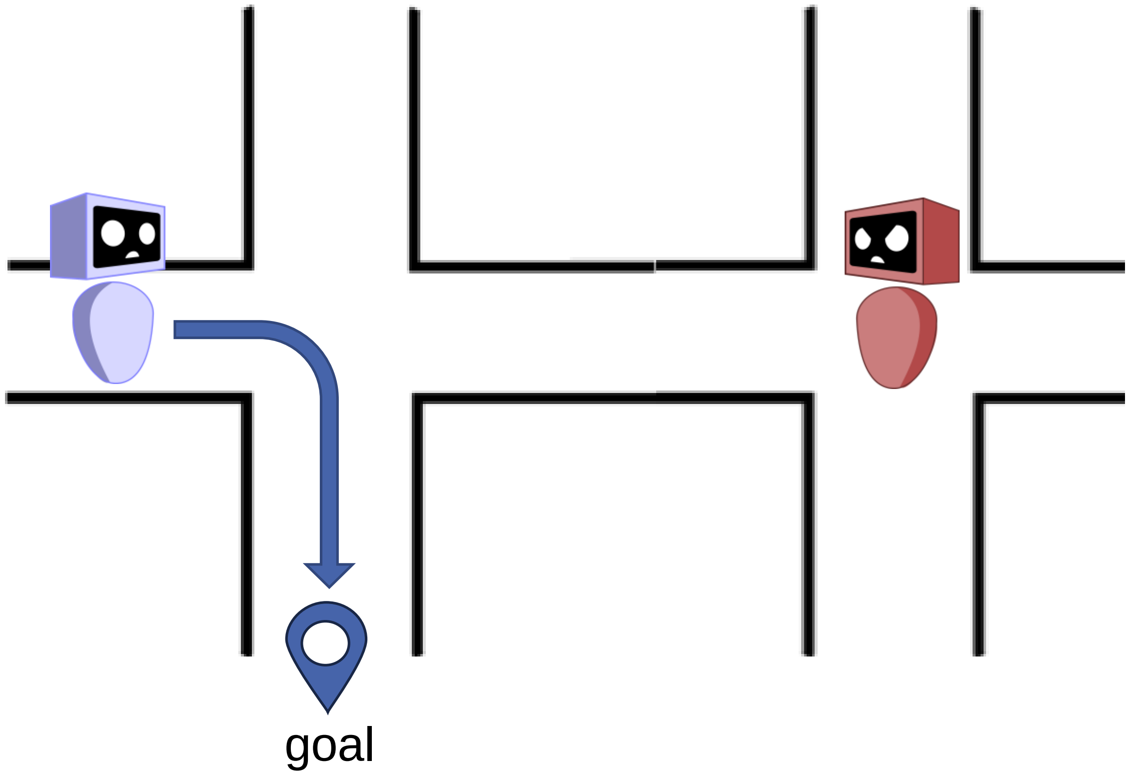
- Cyber-physical systems are not alone in their environment:




➡ The safety of their interactions needs to be verified!

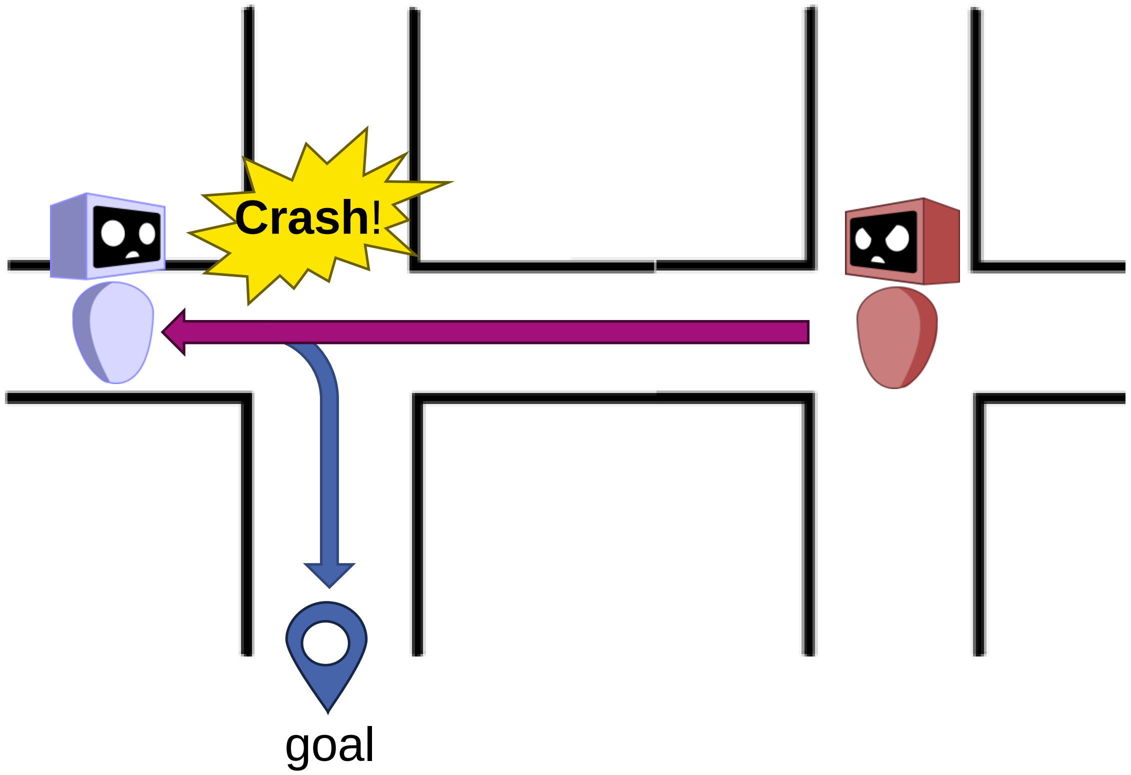
# Motivation

Adversarial players



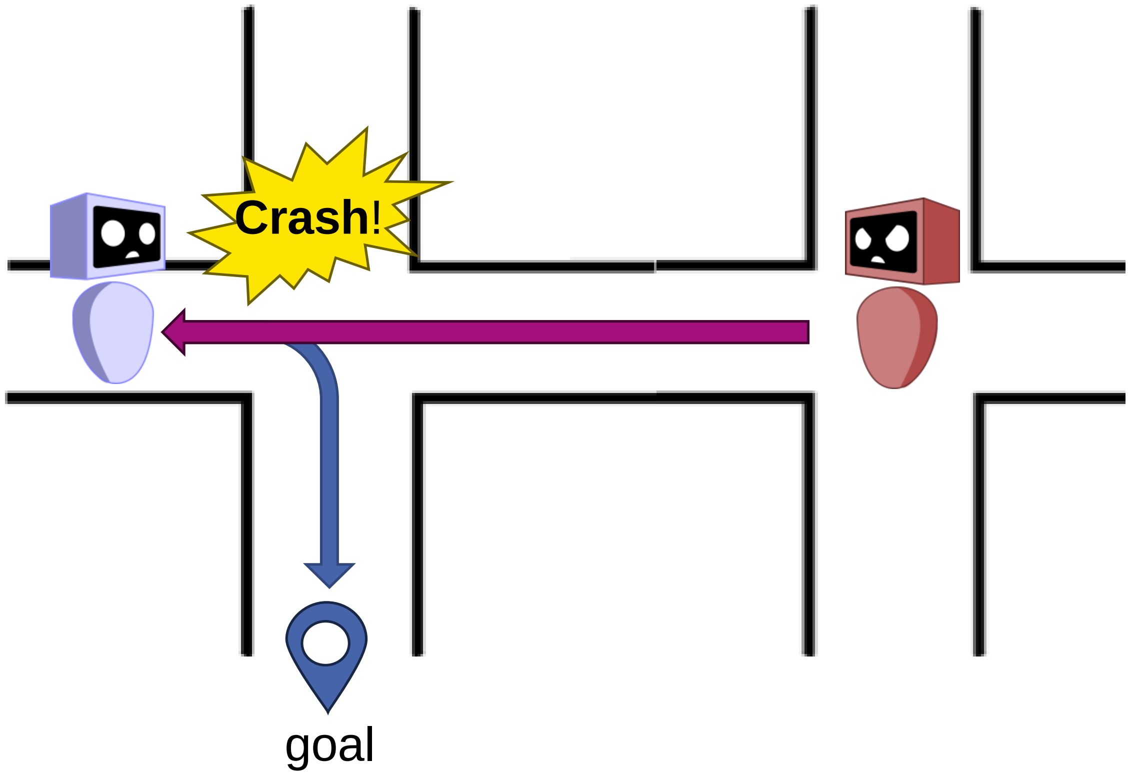
# Motivation

Adversarial players

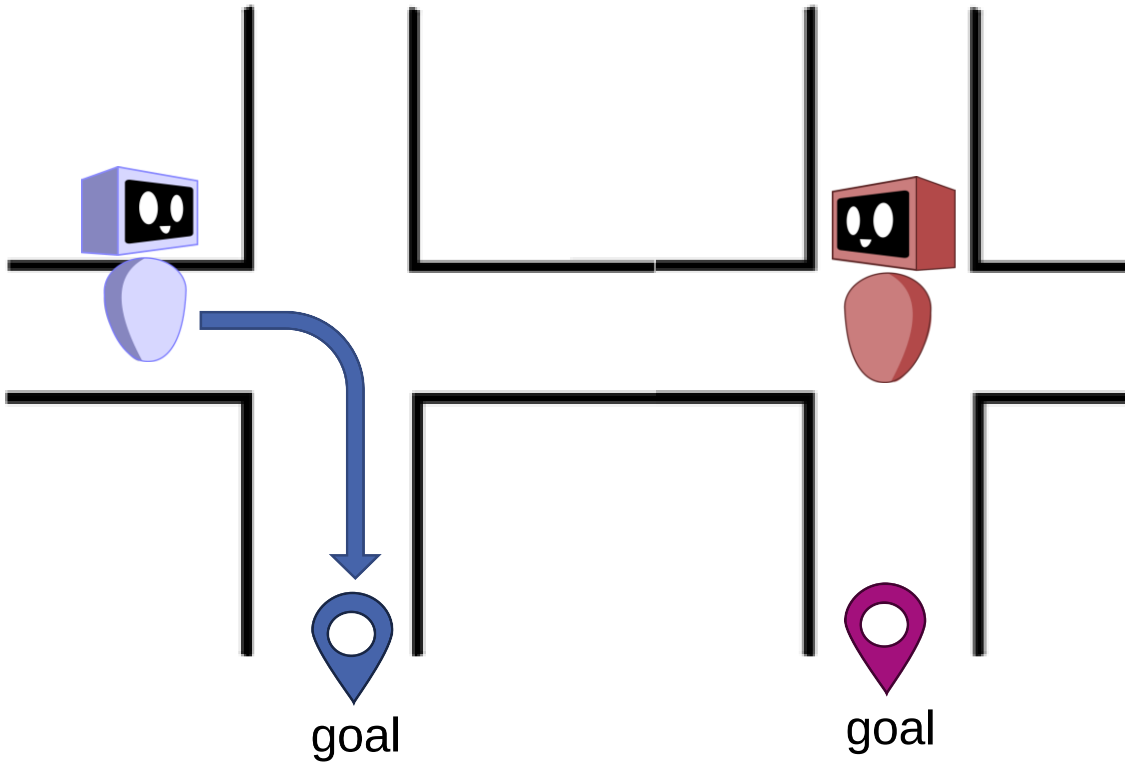


# Motivation

Adversarial players

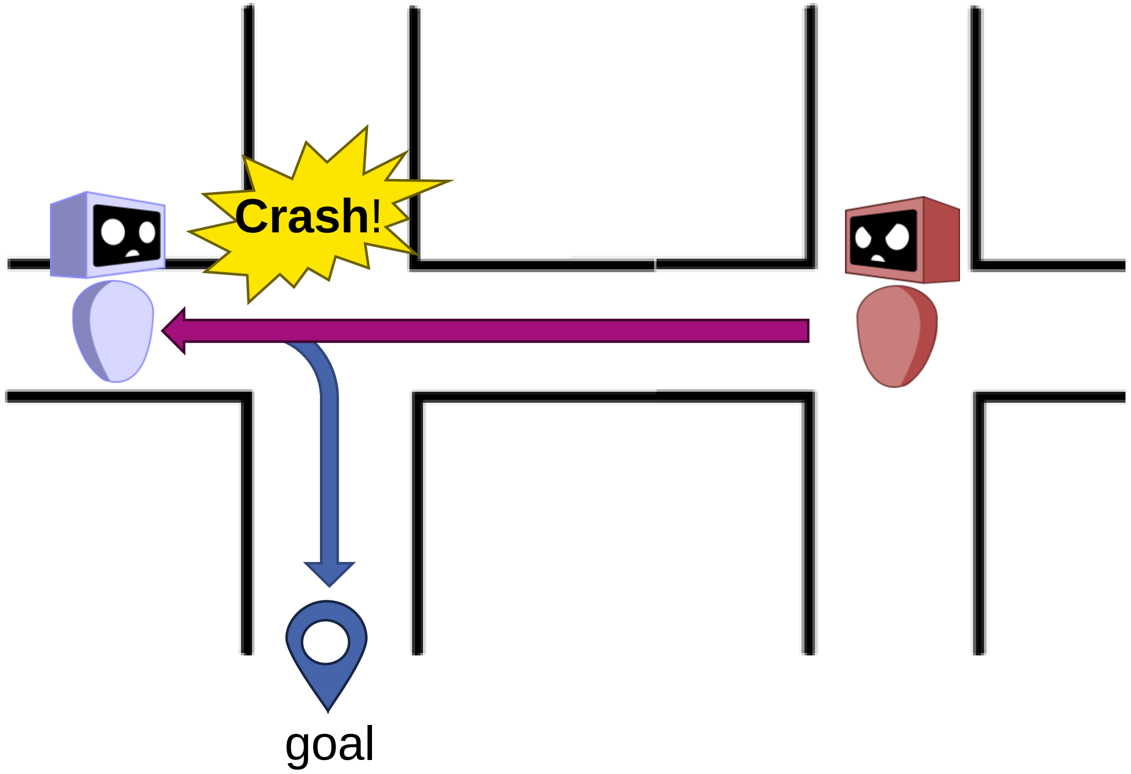



With goals

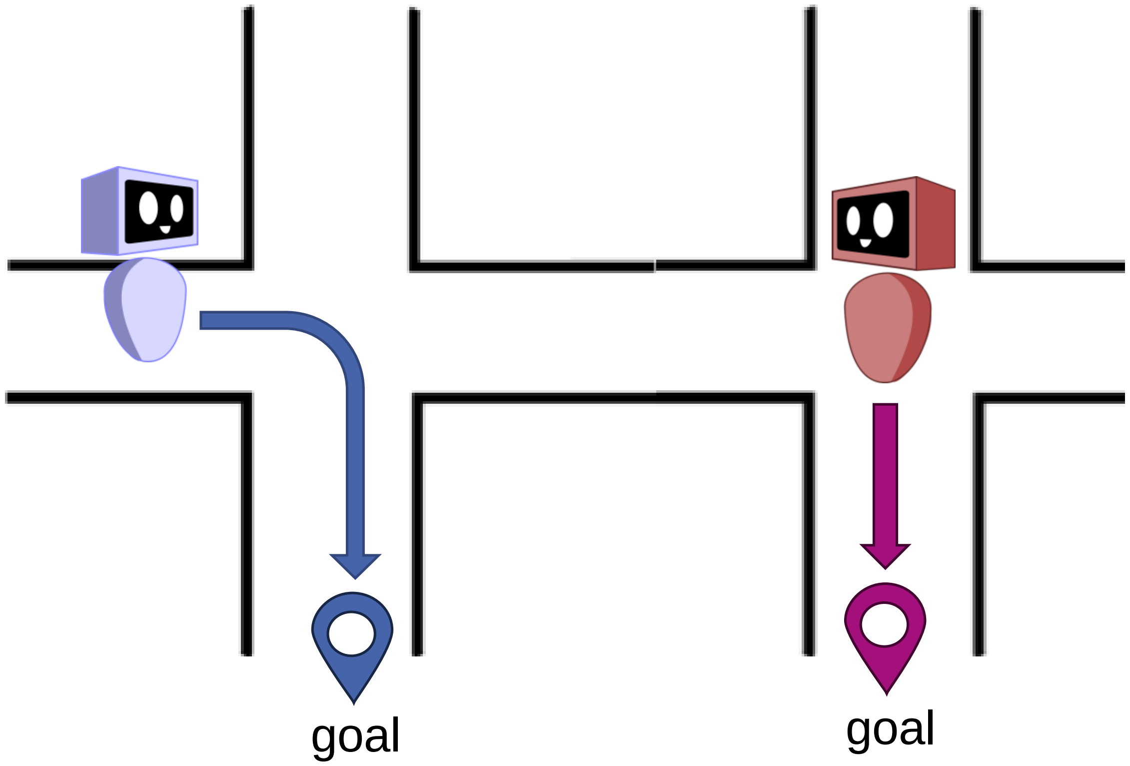


# Motivation

Adversarial players




With goals

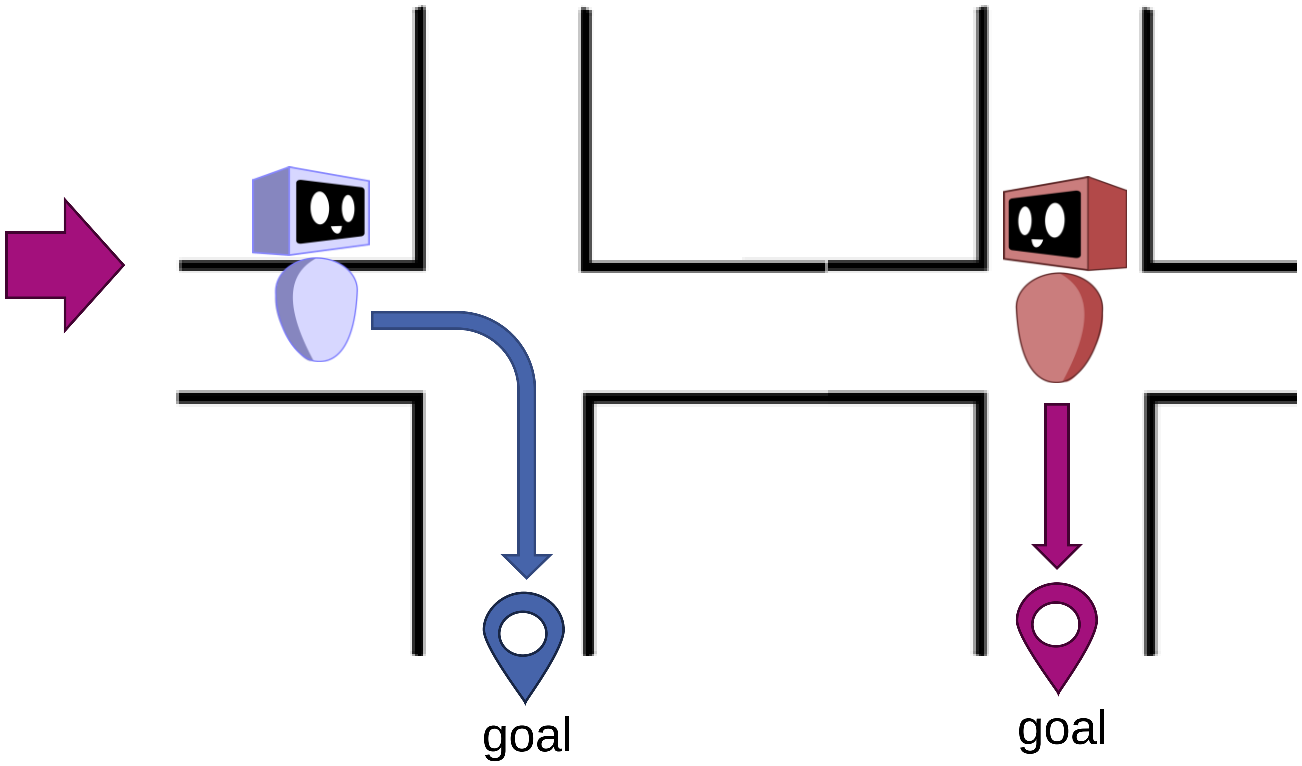


# Motivation

To handle these situations:

- Semi-Competitive Differential Game Logic ($dGL_{sc}$)

With goals



# Structure

- Semi-Competitiveness
- Syntax
- Semantics
- Proof Calculus
- Summary

# Semi-Competitiveness

If a player can win, they will help the other player win too

Players cooperate where it is helpful

Players compete where it is necessary

Players will not lose on purpose

Greatest-most cooperation amongst players

Ensures monotonicity

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

$$\alpha, \beta \;::= x \coloneqq e \mid x' = f(x)\,\&Q \mid ?\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

$$\alpha, \beta \ ::= \boxed{x := e} \mid x' = f(x)\ \&Q \mid ?\,Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Assignment
game

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

Continuous game

$$\alpha, \beta \ ::= \boxed{x := e} \mid \boxed{x' = f(x)\ \&Q} \mid ?Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Assignment game

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

Continuous game

$$\alpha, \beta \ ::= \boxed{x := e} \mid \boxed{x' = f(x)\ \& Q} \mid \boxed{?\,Q} \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^{*} \mid \alpha^{d}$$

Assignment game

Test game

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

Continuous game

Choice game

$$\alpha, \beta \ ::= \boxed{x := e} \mid \boxed{x' = f(x)\ \&Q} \mid \boxed{?\,Q} \mid \boxed{\alpha \cup \beta} \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Assignment game

Test game

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

Sequential game

Continuous game

Choice game

$$\alpha, \beta \ ::= \boxed{x \coloneqq e} \mid \boxed{x' = f(x)\ \&Q} \mid \boxed{?\,Q} \mid \boxed{\alpha \cup \beta} \mid \boxed{\alpha; \beta} \mid \alpha^* \mid \alpha^d$$

Assignment game

Test game

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

Sequential game

Continuous game

Choice game

Repetition game

$$\alpha, \beta \ ::= \boxed{x := e} \mid \boxed{x' = f(x)\ \&Q} \mid \boxed{?\,Q} \mid \boxed{\alpha \cup \beta} \mid \boxed{\alpha; \beta} \mid \boxed{\alpha^*} \mid \alpha^d$$

Assignment game

Test game

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Hybrid games**

$$\alpha,\beta ::= \underbrace{\boxed{x := e}}_{\text{Assignment game}} \mid \overbrace{\boxed{x' = f(x)\ \&Q}}^{\text{Continuous game}} \mid \underbrace{\boxed{?\,Q}}_{\text{Test game}} \mid \overbrace{\boxed{\alpha \cup \beta}}^{\text{Choice game}} \mid \overbrace{\boxed{\alpha;\beta}}^{\text{Sequential game}} \mid \overbrace{\boxed{\alpha^*}}^{\text{Repetition game}} \mid \overbrace{\boxed{\alpha^d}}^{\text{Dual game}}$$

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

$$P, Q \ ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle (P\,, Q) \mid [\alpha](P\,, Q)$$

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

$$P, Q \ ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \boxed{\langle\alpha\rangle(P\,, Q)} \mid [\alpha](P\,, Q)$$

Angel can win

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

$$P, Q \ ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \underbrace{\langle \alpha \rangle (\underbrace{P}_{\text{Angel's goal}}, Q)}_{\text{Angel can win}} \mid [\alpha](P, Q)$$

Angel can win

Angel‘s goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

Demon‘s goal

$$P, Q \; ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \boxed{\langle \alpha \rangle (\boxed{P}, \boxed{Q})} \mid [\alpha](P, Q)$$

Angel can win

Angel‘s goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

Demon‘s goal

$$P, Q \; ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x\, P \mid \exists x\, P \mid \boxed{\langle \alpha \rangle (\boxed{P}, \boxed{Q})} \mid \boxed{[\alpha](P, Q)}$$

Angel can win

Demon can win

Angel‘s goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

Demon's goal

$$P, Q \ ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \boxed{\langle \alpha \rangle (\boxed{P}, \boxed{Q})} \mid \boxed{[\alpha](\boxed{P}, Q)}$$

Angel can win

Demon can win

Angel's goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Syntax**
  - **Formulas**

Demon‘s goal

$$P, Q \ ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle (P, Q) \mid [\alpha](P, Q)$$

Angel can win

Demon can win

Angel‘s goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Example: Drinking Game**

$$o = 0 \land t = 0 \to \langle \{t' = 1\}; \{o' = 1\}^d \rangle (o = 3, t = 5)$$

## Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Example: Drinking Game**

$$o = 0 \land t = 0 \rightarrow \langle \{t' = 1\}; \{o' = 1\}^d \rangle (o = 3, t = 5)$$

Angel evolves

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

## Example: Drinking Game

$$o = 0 \land t = 0 \rightarrow \langle \{t' = 1\}; \{o' = 1\}^d \rangle (o = 3, t = 5)$$

Angel evolves

Demon evolves

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Example: Drinking Game**

Angel's goal

$$o = 0 \land t = 0 \to \langle \{t' = 1\}; \{o' = 1\}^d \rangle (o = 3, t = 5)$$

Angel evolves

Demon evolves

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- **Example: Drinking Game**

Angel‘s goal

$$o = 0 \land t = 0 \rightarrow \langle \{t' = 1\}; \{o' = 1\}^d \rangle (o = 3, t = 5)$$

Angel evolves

Demon evolves

Demon‘s goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

## Example: Drinking Game

Can Angel win?

Angel‘s goal

$$o = 0 \land t = 0 \rightarrow \langle\{t' = 1\};\{o' = 1\}^d\rangle(o = 3, t = 5)$$

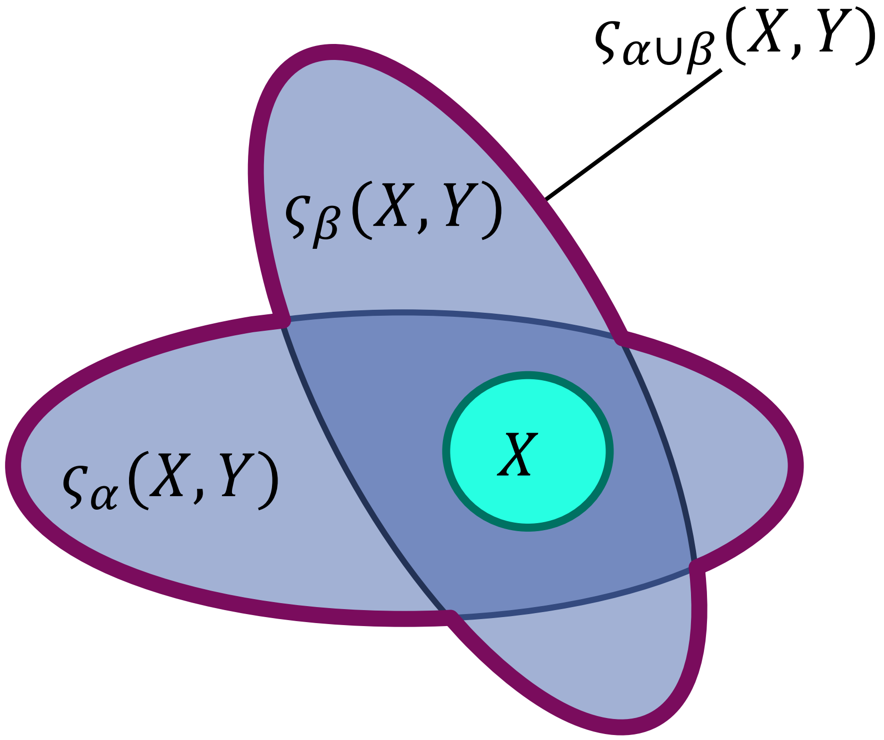
Angel evolves

Demon evolves

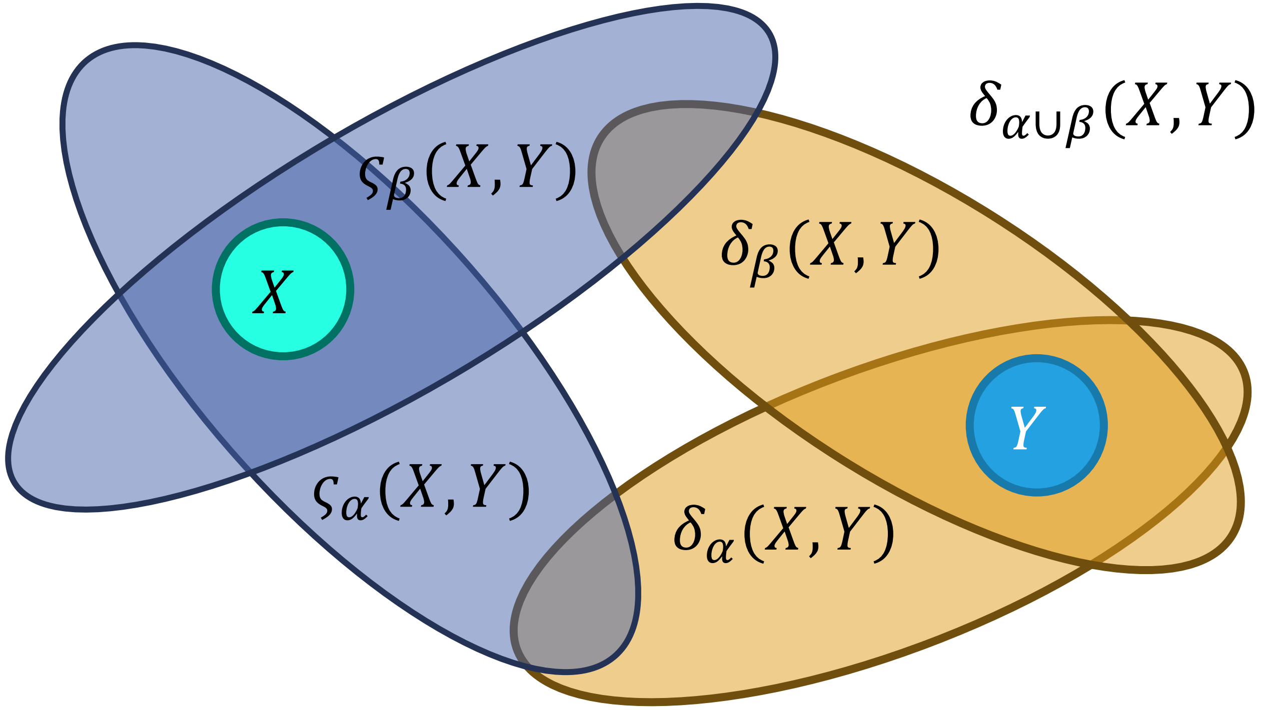
Demon‘s goal

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha\cup\beta}(X,Y) = \varsigma_\alpha(X,Y) \cup \varsigma_\beta(X,Y)$

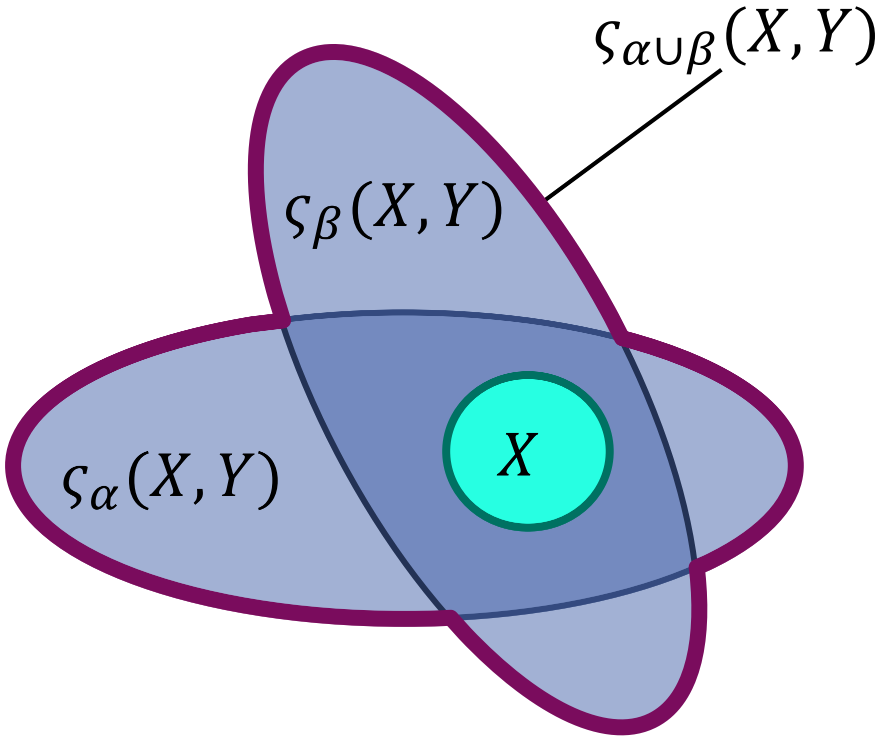



- $$\delta_{\alpha\cup\beta}(X,Y) = \big(\delta_\alpha(X,Y) \cap \delta_\beta(X,Y)\big) \cup \big(\delta_\alpha(X,Y) \cap \varsigma_\alpha(X,Y)\big) \cup \big(\delta_\beta(X,Y) \cap \varsigma_\beta(X,Y)\big)$$

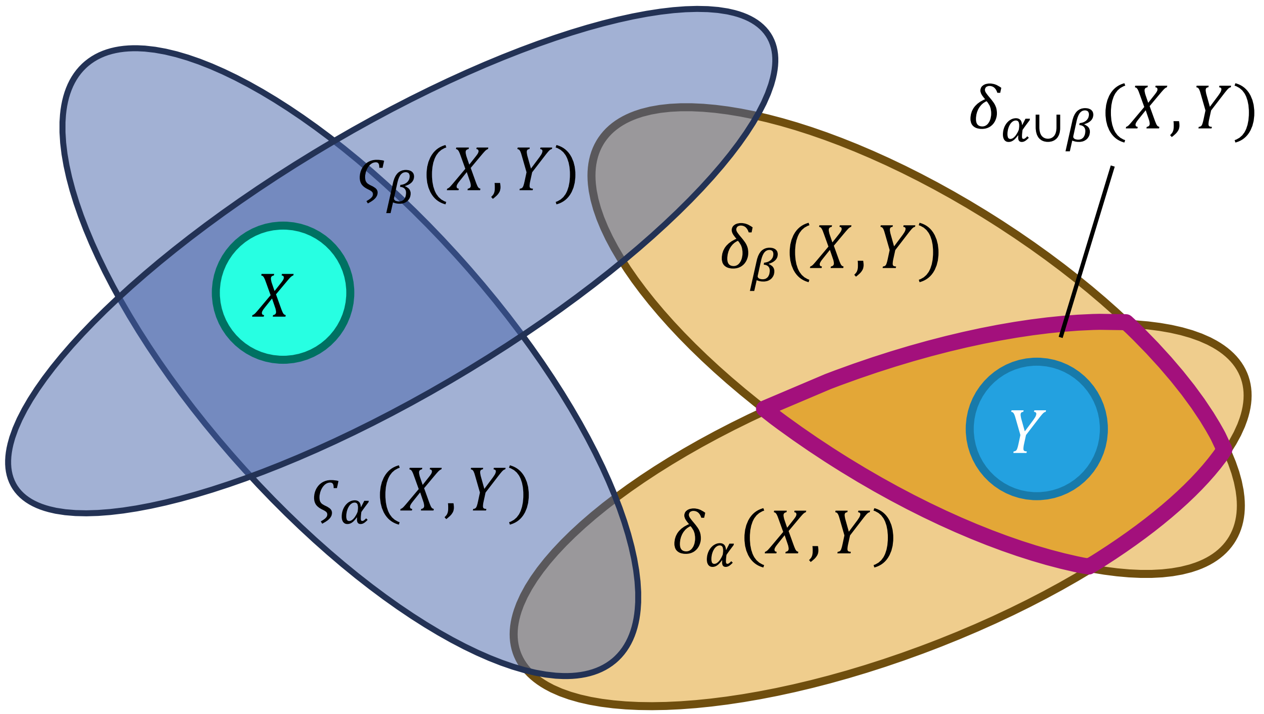


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha\cup\beta}(X,Y) = \varsigma_\alpha(X,Y) \cup \varsigma_\beta(X,Y)$

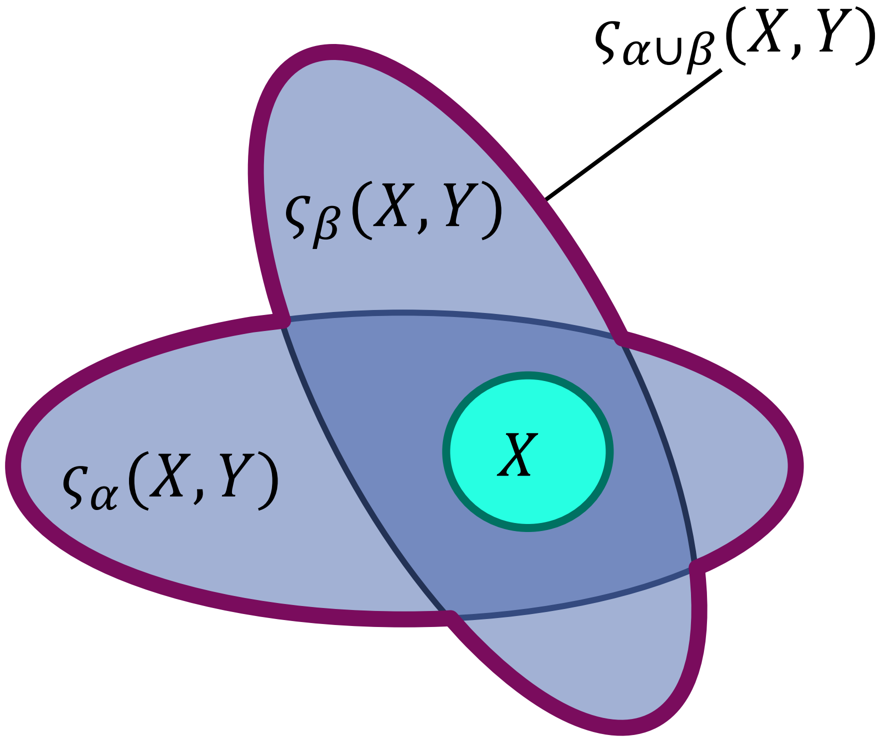



- $\delta_{\alpha\cup\beta}(X,Y) = \boxed{\big(\delta_\alpha(X,Y) \cap \delta_\beta(X,Y)\big)}$
$\quad \cup \big(\delta_\alpha(X,Y) \cap \varsigma_\alpha(X,Y)\big)$
$\quad \cup \big(\delta_\beta(X,Y) \cap \varsigma_\beta(X,Y)\big)$

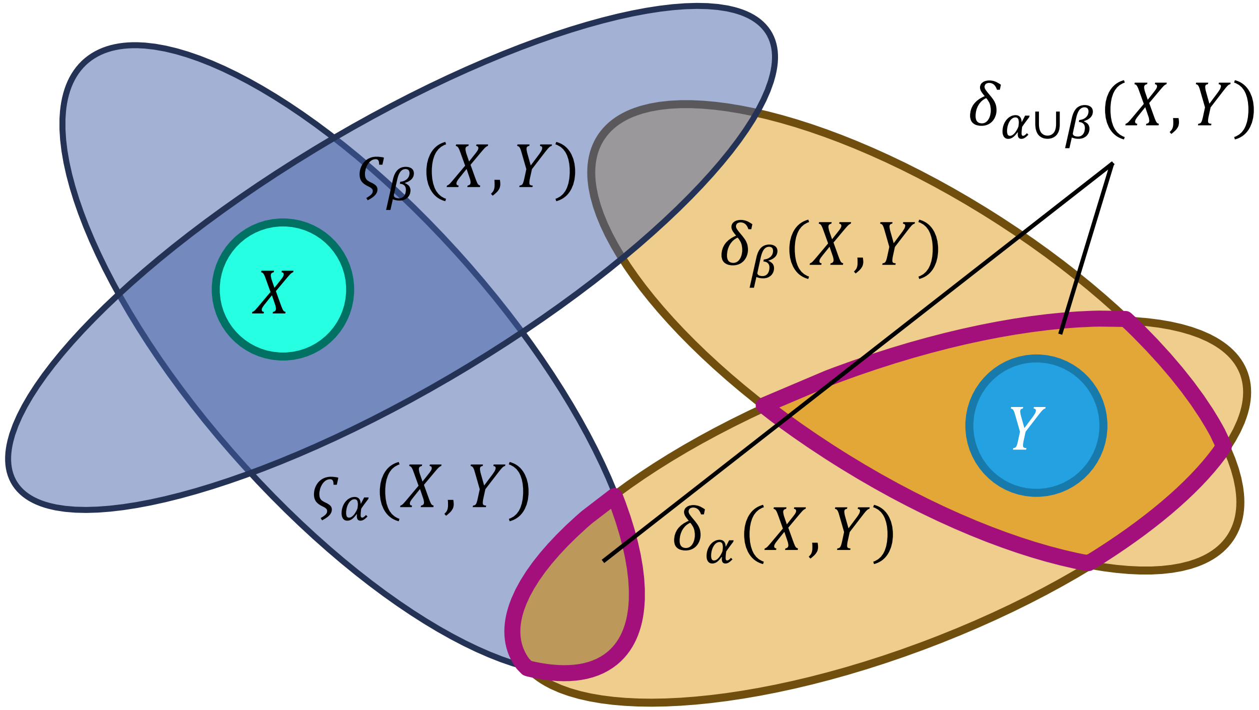


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha\cup\beta}(X,Y) = \varsigma_\alpha(X,Y) \cup \varsigma_\beta(X,Y)$

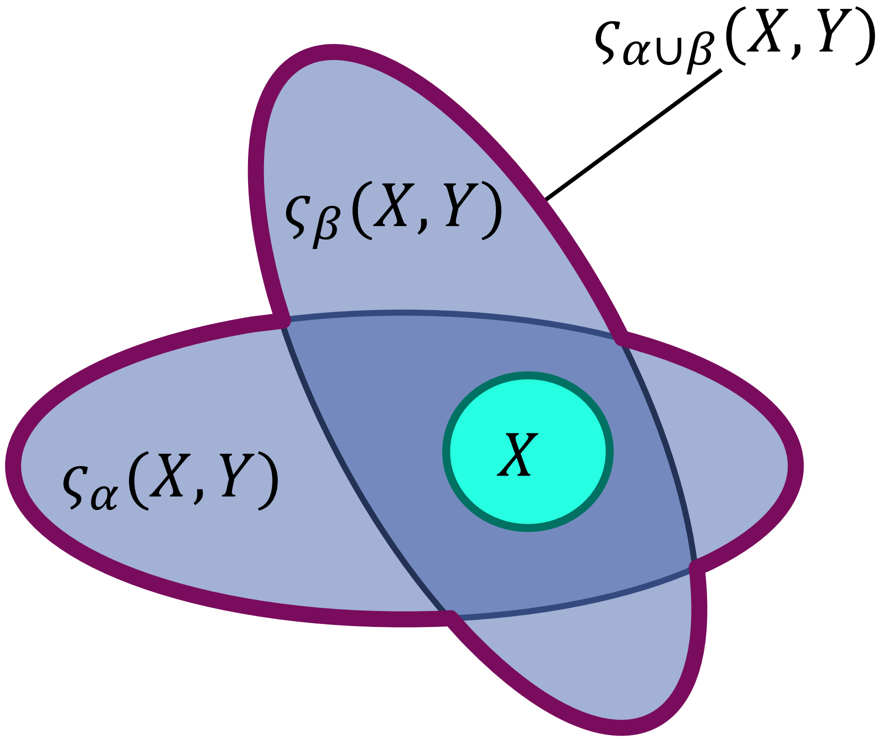



- $\delta_{\alpha\cup\beta}(X,Y) = \big(\delta_\alpha(X,Y) \cap \delta_\beta(X,Y)\big)$
  $\cup \boxed{\big(\delta_\alpha(X,Y) \cap \varsigma_\alpha(X,Y)\big)}$
  $\cup \big(\delta_\beta(X,Y) \cap \varsigma_\beta(X,Y)\big)$

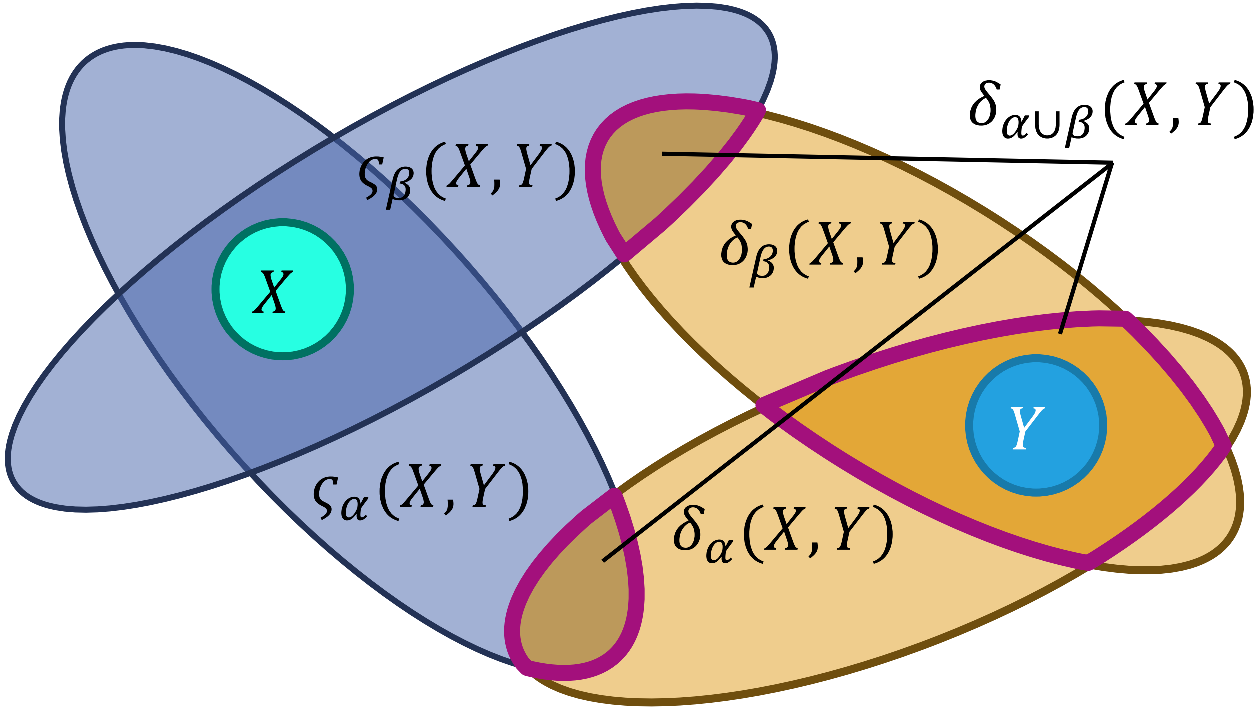


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha\cup\beta}(X,Y) = \varsigma_\alpha(X,Y) \cup \varsigma_\beta(X,Y)$

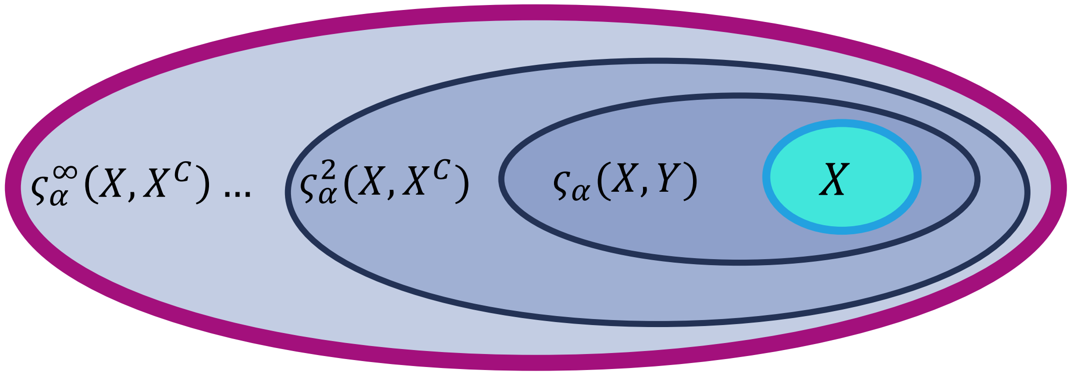



- $\delta_{\alpha\cup\beta}(X,Y) = \big(\delta_\alpha(X,Y) \cap \delta_\beta(X,Y)\big)$
  $\cup \big(\delta_\alpha(X,Y) \cap \varsigma_\alpha(X,Y)\big)$
  $\cup \boxed{\big(\delta_\beta(X,Y) \cap \varsigma_\beta(X,Y)\big)}$

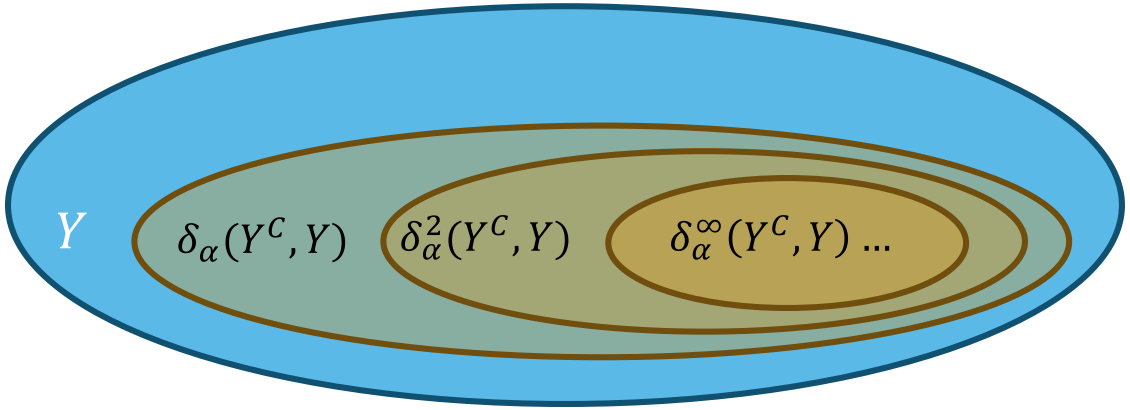


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha^*}(X,Y) = \boxed{\bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z,Z^C) \subseteq Z\}} \cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \left(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\right) \subseteq Z\}$

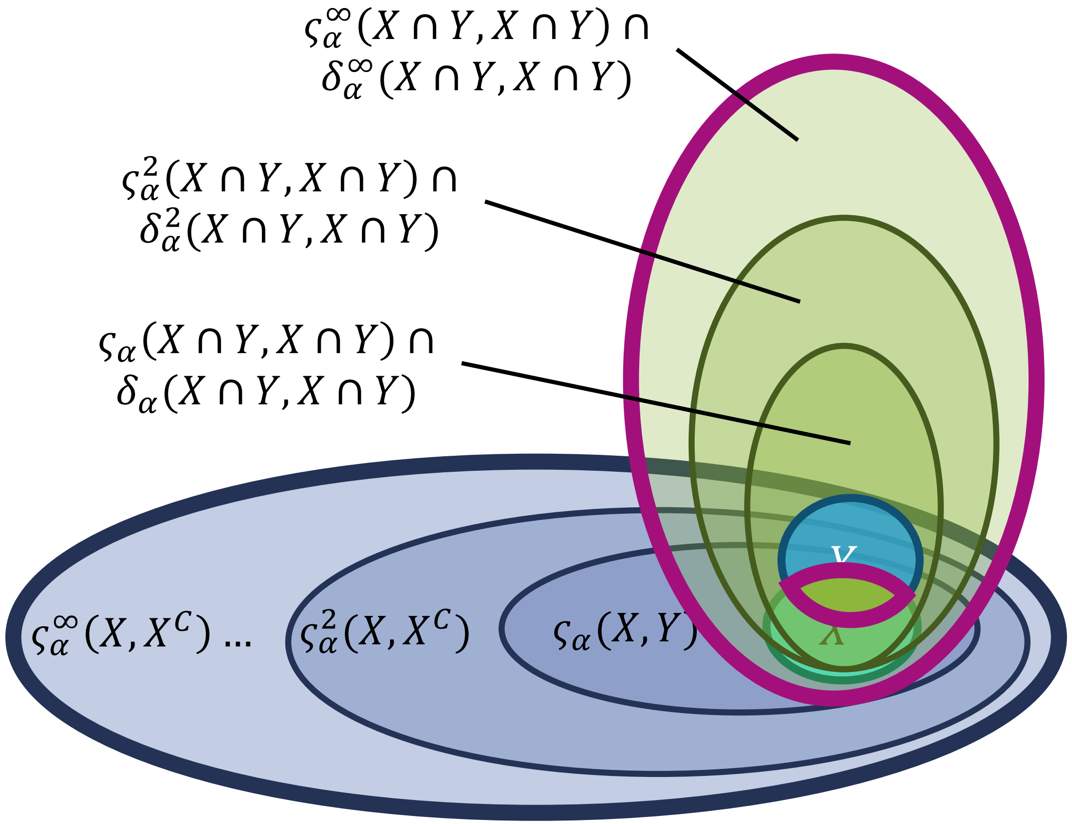



- $\delta_{\alpha^*}(X,Y) = \bigcup\{Z \subseteq \mathcal{S} \mid Z \subseteq Y \cap \delta_\alpha(Z^C,Z)\} \cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \left(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\right) \subseteq Z\}$

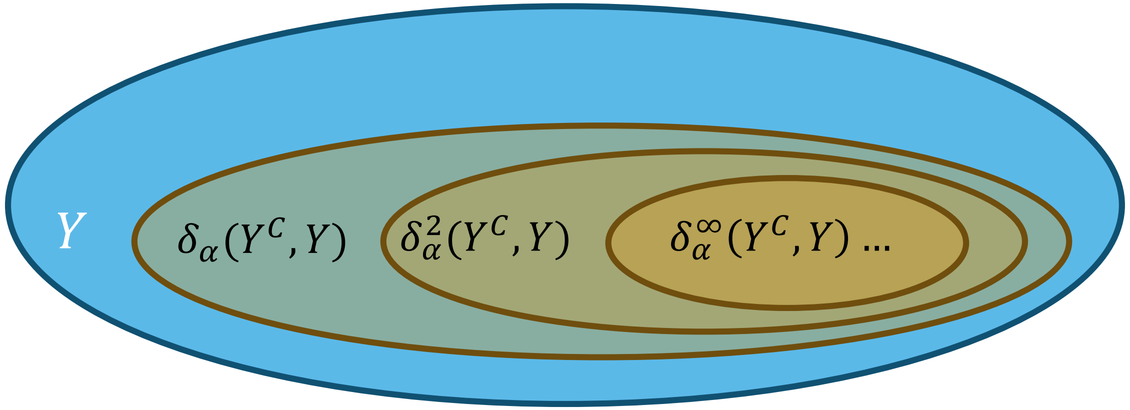


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha^*}(X,Y) = \bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z,Z^C) \subseteq Z\} \cup$
$\boxed{\bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\big) \subseteq Z\}}$

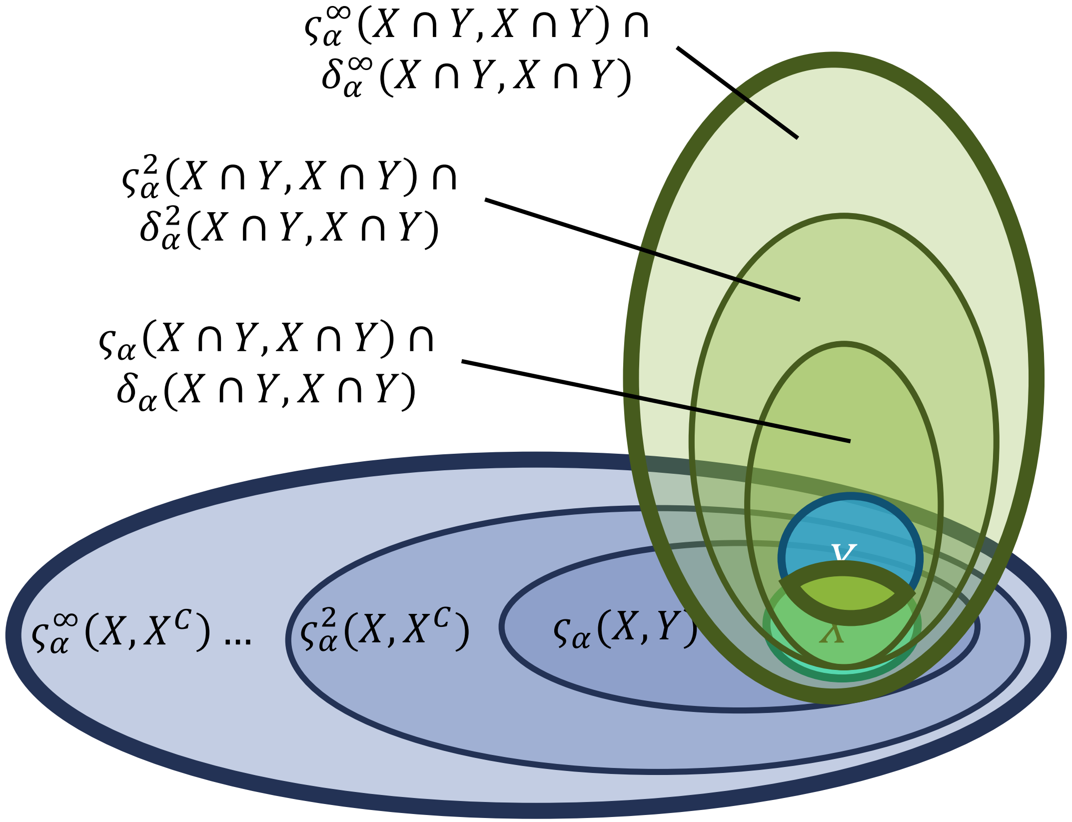



- $\delta_{\alpha^*}(X,Y) = \bigcup\{Z \subseteq \mathcal{S} \mid Z \subseteq Y \cap \delta_\alpha(Z^C,Z)\} \cup$
$\bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\big) \subseteq Z\}$

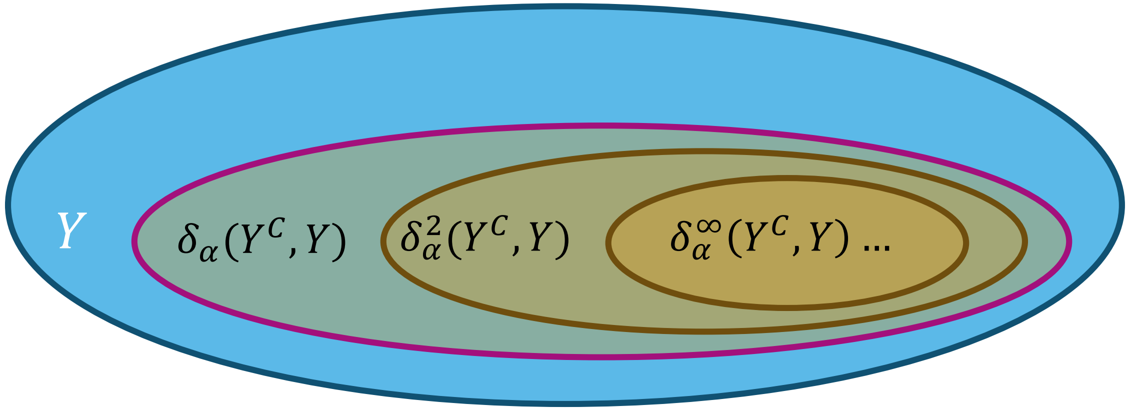


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha^*}(X,Y) = \bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z,Z^C) \subseteq Z\} \cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\big) \subseteq Z\}$

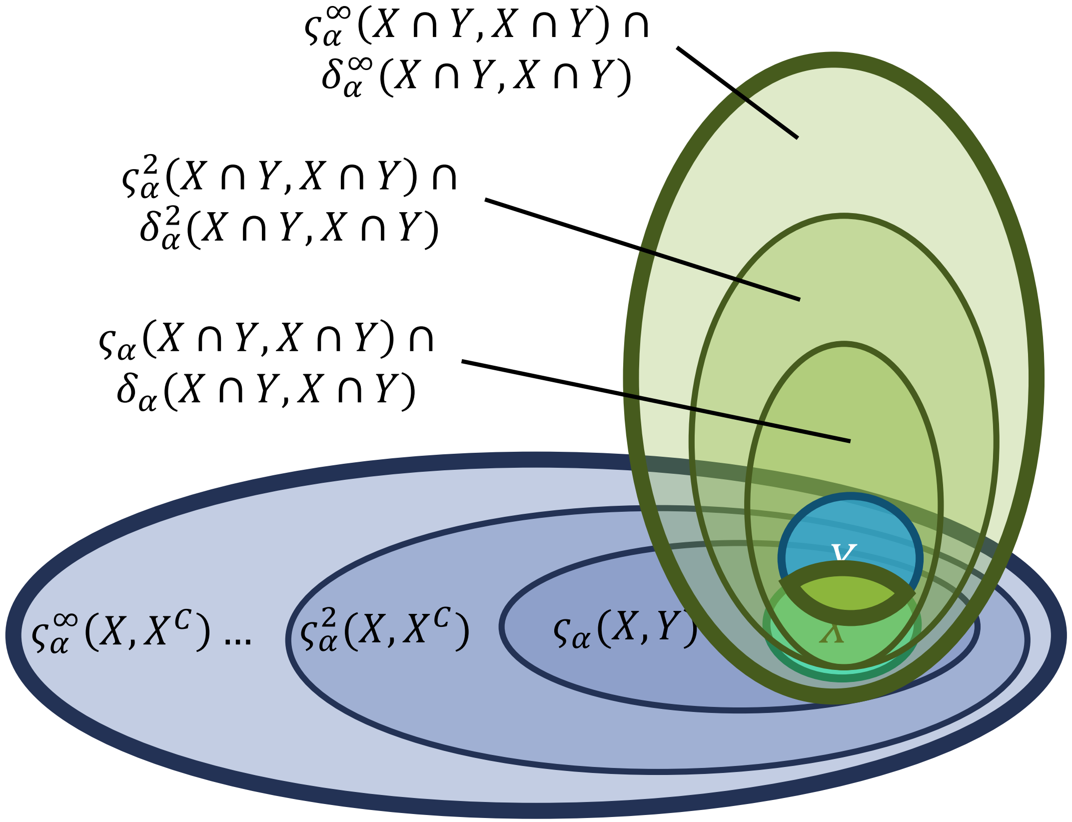



- $\delta_{\alpha^*}(X,Y) = \boxed{\bigcup\{Z \subseteq \mathcal{S} \mid Z \subseteq Y \cap \delta_\alpha(Z^C,Z)\}} \cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\big) \subseteq Z\}$

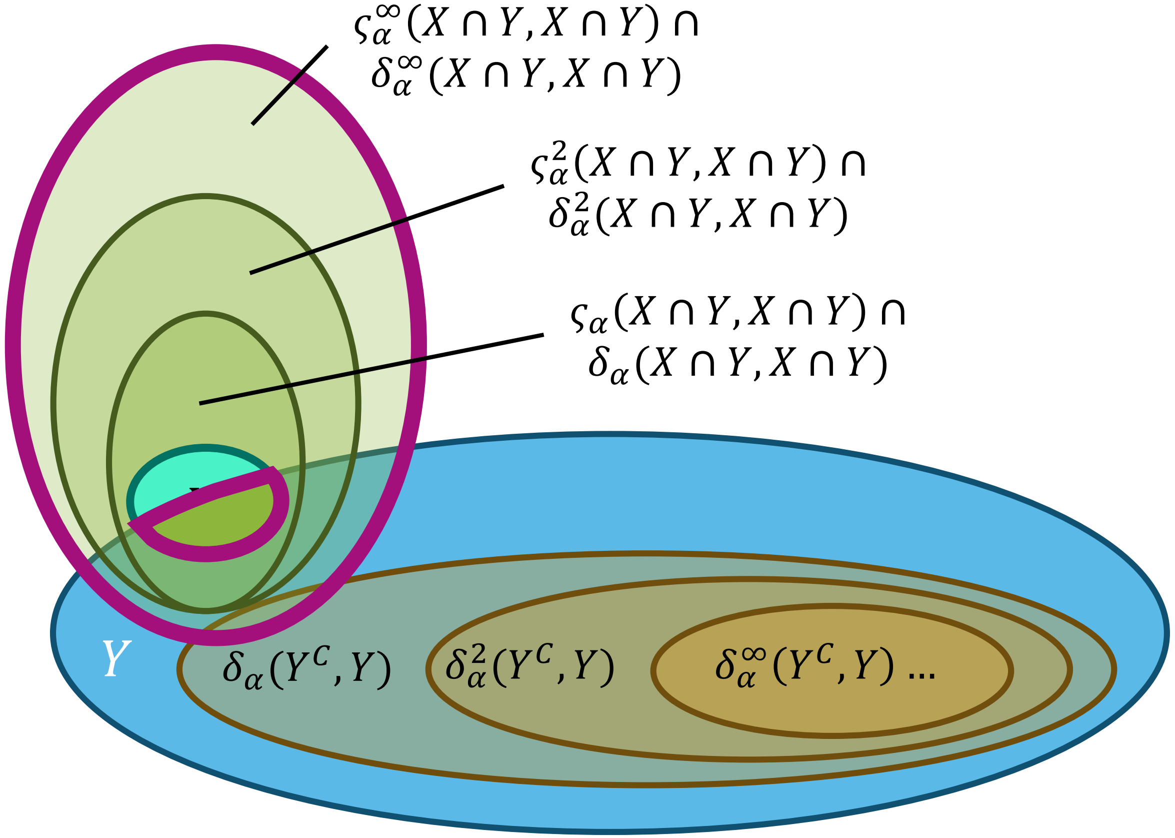


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha^*}(X,Y) = \bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z,Z^C) \subseteq Z\} \cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\big) \subseteq Z\}$




- $\delta_{\alpha^*}(X,Y) = \bigcup\{Z \subseteq \mathcal{S} \mid Z \subseteq Y \cap \delta_\alpha(Z^C,Z)\} \cup \boxed{\bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z,Z) \cap \delta_\alpha(Z,Z)\big) \subseteq Z\}}$



# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

  ⇨ Goals of players are opposing

- Important property: determinacy $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

➡ Goals of players are opposing

$$\langle\alpha\rangle(P, P^C)$$

- Important property: determinacy $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

Goals of players are opposing

$$\langle \alpha \rangle (P, \cancel{P^C})$$

- Important property: determinacy $\neg\langle \alpha \rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

⇨ Goals of players are opposing

$$\langle\alpha\rangle(P, \cancel{P^C}) \Rightarrow \langle\alpha\rangle P$$

- Important property: determinacy $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

➡ Goals of players are opposing

$$\langle\alpha\rangle(P, \cancel{P^C}) \Rightarrow \langle\alpha\rangle P \qquad\qquad [\alpha](Q^C, Q)$$

- Important property: determinacy $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

$\Rightarrow$ Goals of players are opposing

$$\langle\alpha\rangle(P, \cancel{P^C}) \Rightarrow \langle\alpha\rangle P \qquad\qquad [\alpha](\cancel{Q^C}, Q)$$

- Important property: determinacy $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Excursion: dGL

- Logic for reasoning about two-player hybrid games with ***adversarial*** players

  ⇒ Goals of players are opposing

$$\langle\alpha\rangle(P, \cancel{P^C}) \Rightarrow \langle\alpha\rangle P \qquad [\alpha](\cancel{Q^C}, Q) \Rightarrow [\alpha]Q$$

- Important property: determinacy $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$
- Has sound and relative complete proof calculus

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- Properties
  - Monotone
  - If goals are complementary, players behave adversarially
  - Reducible to dGL:

$$\varsigma_\alpha(X,Y) = \varsigma_{\alpha^{-d}}(X \cap Y) \cup \varsigma_\alpha(X) \quad and \quad \delta_\alpha(X,Y) = \varsigma_{\alpha^{-d}}(X \cap Y) \cup \delta_\alpha(Y)$$

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- Properties
  - Monotone
  - If goals are complementary, players behave adversarially
  - Reducible to dGL:

$$\varsigma_\alpha(X,Y) = \boxed{\varsigma_{\alpha^{-d}}(X \cap Y)} \cup \varsigma_\alpha(X) \quad and \quad \delta_\alpha(X,Y) = \varsigma_{\alpha^{-d}}(X \cap Y) \cup \delta_\alpha(Y)$$

Win by cooperation

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- Properties
  - Monotone
  - If goals are complementary, players behave adversarially
  - Reducible to dGL:

$$\varsigma_\alpha(X,Y) = \boxed{\varsigma_{\boxed{\alpha^{-d}}}(X \cap Y)} \cup \varsigma_\alpha(X) \quad and \quad \delta_\alpha(X,Y) = \varsigma_{\alpha^{-d}}(X \cap Y) \cup \delta_\alpha(Y)$$

Remove all
dual games
from $\alpha$

Win by cooperation

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- Properties
  - Monotone
  - If goals are complementary, players behave adversarially
  - Reducible to dGL:

$$\varsigma_\alpha(X,Y) = \boxed{\varsigma_{\boxed{\alpha^{-d}}}(X \cap Y)} \cup \boxed{\varsigma_\alpha(X)} \quad and \quad \delta_\alpha(X,Y) = \varsigma_{\alpha^{-d}}(X \cap Y) \cup \delta_\alpha(Y)$$

Remove all dual games from $\alpha$

Win by competition

Win by cooperation

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- Properties
  - Monotone
  - If goals are complementary, players behave adversarially
  - Reducible to dGL:

$$\varsigma_\alpha(X,Y) = \boxed{\varsigma_{\boxed{\alpha-d}}(X\cap Y)} \cup \boxed{\varsigma_\alpha(X)} \quad and \quad \delta_\alpha(X,Y) = \varsigma_{\alpha-d}(X\cap Y)\cup\delta_\alpha(Y)$$

Remove all
dual games
from $\alpha$

Win by competition

or

Win by cooperation

**Semi-competitive!**

# Proof Calculus

# Proof Calculus

$$
\begin{array}{ll}
\langle{:=}\rangle & \langle x := e\rangle(p(x), q(x)) \leftrightarrow p(e) \\
\langle{'}\rangle & \langle x' = f(x)\rangle(P, Q) \leftrightarrow \exists t{\geq}0\, \langle x := x(t)\rangle(P, Q) \qquad (x' = f(x)) \\
\langle ?\rangle & \langle ?R\rangle(P, Q) \leftrightarrow R \land P \\
\langle\cup\rangle & \langle \alpha \cup \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(P, Q) \lor \langle\beta\rangle(P, Q) \\
\langle ;\rangle & \langle \alpha; \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(\langle\beta\rangle(P, Q), [\beta](P, Q)) \\
\langle^{d}\rangle & \langle \alpha^{d}\rangle(P, Q) \leftrightarrow [\alpha](Q, P) \\
\langle^{*}\rangle & \langle \alpha^{*}\rangle(P, Q) \leftrightarrow P \lor \langle \alpha; \alpha^{*}\rangle(P, Q) \\
\text{FP} & \dfrac{P \lor \langle\alpha\rangle R_1 \to R_1 \qquad (P \land Q) \lor (\langle\alpha\rangle(R_2, R_2) \land [\alpha](R_2, R_2)) \to R_2}{\langle\alpha^{*}\rangle(P, Q) \to R_1 \lor R_2}
\end{array}
$$

Axioms for Angel

# Proof Calculus

Axioms for Angel

$$
\begin{array}{ll}
\langle:=\rangle & \langle x := e\rangle(p(x), q(x)) \leftrightarrow p(e) \\
\langle'\rangle & \langle x' = f(x)\rangle(P,Q) \leftrightarrow \exists t{\geq}0\, \langle x := x(t)\rangle(P,Q) \qquad (x' = f(x)) \\
\langle?\rangle & \langle ?R\rangle(P,Q) \leftrightarrow R \land P \\
\langle\cup\rangle & \langle \alpha \cup \beta\rangle(P,Q) \leftrightarrow \langle\alpha\rangle(P,Q) \lor \langle\beta\rangle(P,Q) \\
\langle;\rangle & \langle \alpha;\beta\rangle(P,Q) \leftrightarrow \langle\alpha\rangle(\langle\beta\rangle(P,Q), [\beta](P,Q)) \\
\langle^d\rangle & \langle \alpha^d\rangle(P,Q) \leftrightarrow [\alpha](Q,P) \\
\langle^*\rangle & \langle \alpha^*\rangle(P,Q) \leftrightarrow P \lor \langle\alpha;\alpha^*\rangle(P,Q) \\
\mathrm{FP} & \dfrac{P \lor \langle\alpha\rangle R_1 \to R_1 \qquad (P \land Q) \lor (\langle\alpha\rangle(R_2,R_2) \land [\alpha](R_2,R_2)) \to R_2}{\langle\alpha^*\rangle(P,Q) \to R_1 \lor R_2}
\end{array}
$$

Axioms for Demon

$$
\begin{array}{ll}
[:=] & [x := e](p(x), q(x)) \leftrightarrow q(e) \\
[\,'\,] & [x' = f(x)](P,Q) \leftrightarrow \forall t{\geq}0\, [x{:=}x(t)](P,Q) \lor \exists t{\geq}0\, \langle x{:=}x(t)\rangle(P \land Q, Q) \quad (x' = f(x)) \\
[?] & [?R](P,Q) \leftrightarrow \neg R \lor Q \\
[\cup] & [\alpha \cup \beta](P,Q) \leftrightarrow ([\alpha](P,Q) \land [\beta](P,Q)) \lor ([\alpha](P,Q) \land \langle\alpha\rangle(P,Q)) \lor ([\beta](P,Q) \land \\
 & \langle\beta\rangle(P,Q)) \\
[;] & [\alpha;\beta](P,Q) \leftrightarrow [\alpha](\langle\beta\rangle(P,Q), [\beta](P,Q)) \\
[^*] & [\alpha^*](P,Q) \leftrightarrow (P \land Q) \lor (\langle\alpha;\alpha^*\rangle(P,Q) \land [\alpha;\alpha^*](P,Q)) \lor (Q \land [\alpha;\alpha^*]Q)
\end{array}
$$

# Proof Calculus

Axioms for Angel

$\langle:=\rangle$ $\langle x := e\rangle(p(x), q(x)) \leftrightarrow p(e)$

$\langle'\rangle$ $\langle x' = f(x)\rangle(P, Q) \leftrightarrow \exists t{\geq}0\, \langle x := x(t)\rangle(P, Q)$ $\quad (x' = f(x))$

$\langle?\rangle$ $\langle ?R\rangle(P, Q) \leftrightarrow R \wedge P$

$\langle\cup\rangle$ $\langle \alpha \cup \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(P, Q) \vee \langle\beta\rangle(P, Q)$

$\langle;\rangle$ $\langle \alpha; \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(\langle\beta\rangle(P, Q), [\beta](P, Q))$

$\langle^d\rangle$ $\langle \alpha^d\rangle(P, Q) \leftrightarrow [\alpha](Q, P)$

$\langle^*\rangle$ $\langle \alpha^*\rangle(P, Q) \leftrightarrow P \vee \langle\alpha; \alpha^*\rangle(P, Q)$

$$\text{FP}\quad \frac{P \vee \langle\alpha\rangle R_1 \to R_1 \qquad (P \wedge Q) \vee (\langle\alpha\rangle(R_2, R_2) \wedge [\alpha](R_2, R_2)) \to R_2}{\langle\alpha^*\rangle(P, Q) \to R_1 \vee R_2}$$

Axioms for Demon

$[:=]$ $[x := e](p(x), q(x)) \leftrightarrow q(e)$

$[']$ $[x' = f(x)](P, Q) \leftrightarrow \forall t{\geq}0\, [x{:=}x(t)](P, Q) \vee \exists t{\geq}0\, \langle x{:=}x(t)\rangle(P \wedge Q, Q)$ $\quad (x' = f(x))$

$[?]$ $[?R](P, Q) \leftrightarrow \neg R \vee Q$

$[\cup]$ $[\alpha \cup \beta](P, Q) \leftrightarrow ([\alpha](P, Q) \wedge [\beta](P, Q)) \vee ([\alpha](P, Q) \wedge \langle\alpha\rangle(P, Q)) \vee ([\beta](P, Q) \wedge \langle\beta\rangle(P, Q))$

$[;]$ $[\alpha; \beta](P, Q) \leftrightarrow [\alpha](\langle\beta\rangle(P, Q), [\beta](P, Q))$

$[^*]$ $[\alpha^*](P, Q) \leftrightarrow (P \wedge Q) \vee (\langle\alpha; \alpha^*\rangle(P, Q) \wedge [\alpha; \alpha^*](P, Q)) \vee (Q \wedge [\alpha; \alpha^*]Q)$

Monotonicity

$$\text{M}\langle\rangle\quad \frac{P_1 \to P_2 \qquad Q_1 \to Q_2}{\langle\alpha\rangle(P_1, Q_1) \to \langle\alpha\rangle(P_2, Q_2)}$$

$$\text{M}_2\langle\rangle\quad \frac{P_1 \to P_2 \qquad P_1 \wedge Q_1 \to \bot}{\langle\alpha\rangle(P_1, Q_1) \to \langle\alpha\rangle(P_2, Q_2)}$$

# Proof Calculus

Axioms for Angel

$$
\begin{array}{ll}
\langle:=\rangle & \langle x := e\rangle(p(x), q(x)) \leftrightarrow p(e) \\
\langle'\rangle & \langle x' = f(x)\rangle(P, Q) \leftrightarrow \exists t{\ge}0\, \langle x := x(t)\rangle(P, Q) \qquad (x' = f(x)) \\
\langle?\rangle & \langle ?R\rangle(P, Q) \leftrightarrow R \land P \\
\langle\cup\rangle & \langle \alpha \cup \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(P, Q) \lor \langle\beta\rangle(P, Q) \\
\langle;\rangle & \langle \alpha; \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(\langle\beta\rangle(P, Q), [\beta](P, Q)) \\
\langle^d\rangle & \langle \alpha^d\rangle(P, Q) \leftrightarrow [\alpha](Q, P) \\
\langle^*\rangle & \langle \alpha^*\rangle(P, Q) \leftrightarrow P \lor \langle\alpha; \alpha^*\rangle(P, Q) \\
\text{FP} & \dfrac{P \lor \langle\alpha\rangle R_1 \to R_1 \qquad (P \land Q) \lor (\langle\alpha\rangle(R_2, R_2) \land [\alpha](R_2, R_2)) \to R_2}{\langle\alpha^*\rangle(P, Q) \to R_1 \lor R_2}
\end{array}
$$

Axioms for Demon

$$
\begin{array}{ll}
[:=] & [x := e](p(x), q(x)) \leftrightarrow q(e) \\
['] & [x' = f(x)](P, Q) \leftrightarrow \forall t{\ge}0\, [x{:=}x(t)](P, Q) \lor \exists t{\ge}0\, \langle x{:=}x(t)\rangle(P \land Q, Q) \quad (x' = f(x)) \\
[?] & [?R](P, Q) \leftrightarrow \neg R \lor Q \\
[\cup] & [\alpha \cup \beta](P, Q) \leftrightarrow ([\alpha](P, Q) \land [\beta](P, Q)) \lor ([\alpha](P, Q) \land \langle\alpha\rangle(P, Q)) \lor ([\beta](P, Q) \land \\
 & \langle\beta\rangle(P, Q)) \\
[;] & [\alpha; \beta](P, Q) \leftrightarrow [\alpha](\langle\beta\rangle(P, Q), [\beta](P, Q)) \\
[^*] & [\alpha^*](P, Q) \leftrightarrow (P \land Q) \lor (\langle\alpha; \alpha^*\rangle(P, Q) \land [\alpha; \alpha^*](P, Q)) \lor (Q \land [\alpha; \alpha^*]Q)
\end{array}
$$

Monotonicity

$$
\begin{array}{ll}
\text{M}\langle\rangle & \dfrac{P_1 \to P_2 \qquad Q_1 \to Q_2}{\langle\alpha\rangle(P_1, Q_1) \to \langle\alpha\rangle(P_2, Q_2)} \\
\text{M}_2\langle\rangle & \dfrac{P_1 \to P_2 \qquad P_1 \land Q_1 \to \bot}{\langle\alpha\rangle(P_1, Q_1) \to \langle\alpha\rangle(P_2, Q_2)}
\end{array}
$$

Determinacy

$$
\text{det} \quad \neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P
$$

# Proof Calculus

Axioms for Angel

| | |
|---|---|
| $\langle{:=}\rangle$ | $\langle x := e\rangle(p(x), q(x)) \leftrightarrow p(e)$ |
| $\langle{'}\rangle$ | $\langle x' = f(x)\rangle(P, Q) \leftrightarrow \exists t{\geq}0\, \langle x := x(t)\rangle(P, Q) \qquad (x' = f(x))$ |
| $\langle ?\rangle$ | $\langle ?R\rangle(P, Q) \leftrightarrow R \wedge P$ |
| $\langle\cup\rangle$ | $\langle \alpha \cup \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(P, Q) \vee \langle\beta\rangle(P, Q)$ |
| $\langle ;\rangle$ | $\langle \alpha; \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(\langle\beta\rangle(P, Q), [\beta](P, Q))$ |
| $\langle{}^d\rangle$ | $\langle \alpha^d\rangle(P, Q) \leftrightarrow [\alpha](Q, P)$ |
| $\langle{}^*\rangle$ | $\langle \alpha^*\rangle(P, Q) \leftrightarrow P \vee \langle\alpha; \alpha^*\rangle(P, Q)$ |
| FP | $\dfrac{P \vee \langle\alpha\rangle R_1 \rightarrow R_1 \qquad (P \wedge Q) \vee (\langle\alpha\rangle(R_2, R_2) \wedge [\alpha](R_2, R_2)) \rightarrow R_2}{\langle\alpha^*\rangle(P, Q) \rightarrow R_1 \vee R_2}$ |

Axioms for Demon

| | |
|---|---|
| $[{:=}]$ | $[x := e](p(x), q(x)) \leftrightarrow q(e)$ |
| $[{'}]$ | $[x' = f(x)](P, Q) \leftrightarrow \forall t{\geq}0\, [x{:=}x(t)](P, Q) \vee \exists t{\geq}0\, \langle x{:=}x(t)\rangle(P \wedge Q, Q) \quad (x' = f(x))$ |
| $[?]$ | $[?R](P, Q) \leftrightarrow \neg R \vee Q$ |
| $[\cup]$ | $[\alpha \cup \beta](P, Q) \leftrightarrow ([\alpha](P, Q) \wedge [\beta](P, Q)) \vee ([\alpha](P, Q) \wedge \langle\alpha\rangle(P, Q)) \vee ([\beta](P, Q) \wedge \langle\beta\rangle(P, Q))$ |
| $[;]$ | $[\alpha; \beta](P, Q) \leftrightarrow [\alpha](\langle\beta\rangle(P, Q), [\beta](P, Q))$ |
| $[{}^*]$ | $[\alpha^*](P, Q) \leftrightarrow (P \wedge Q) \vee (\langle\alpha; \alpha^*\rangle(P, Q) \wedge [\alpha; \alpha^*](P, Q)) \vee (Q \wedge [\alpha; \alpha^*]Q)$ |

Monotonicity

| | |
|---|---|
| $\mathrm{M}\langle\rangle$ | $\dfrac{P_1 \rightarrow P_2 \qquad Q_1 \rightarrow Q_2}{\langle\alpha\rangle(P_1, Q_1) \rightarrow \langle\alpha\rangle(P_2, Q_2)}$ |
| $\mathrm{M}_2\langle\rangle$ | $\dfrac{P_1 \rightarrow P_2 \qquad P_1 \wedge Q_1 \rightarrow \bot}{\langle\alpha\rangle(P_1, Q_1) \rightarrow \langle\alpha\rangle(P_2, Q_2)}$ |

Determinacy

| | |
|---|---|
| det | $\neg\langle\alpha\rangle P \leftrightarrow [\alpha]\neg P$ |

**+ all FOL rules**

# Example

$$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(P, Q) \vee \langle\beta\rangle(P, Q)$$

$$\varsigma_{\alpha\cup\beta}(X, Y) = \varsigma_{\alpha}(X, Y) \cup \varsigma_{\beta}(X, Y)$$

# Example

$$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle(P, Q) \leftrightarrow \boxed{\langle\alpha\rangle(P, Q)} \vee \langle\beta\rangle(P, Q)$$

$$\varsigma_{\alpha\cup\beta}(X, Y) = \boxed{\varsigma_\alpha(X, Y)} \cup \varsigma_\beta(X, Y)$$

# Example

$$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle(P,Q) \leftrightarrow \langle\alpha\rangle(P,Q) \vee \boxed{\langle\beta\rangle(P,Q)}$$

$$\varsigma_{\alpha\cup\beta}(X,Y) = \varsigma_{\alpha}(X,Y) \cup \boxed{\varsigma_{\beta}(X,Y)}$$

# Example

$$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle(P, Q) \leftrightarrow \langle\alpha\rangle(P, Q) \vee \langle\beta\rangle(P, Q)$$

$$\varsigma_{\alpha\cup\beta}(X, Y) = \varsigma_{\alpha}(X, Y) \cup \varsigma_{\beta}(X, Y)$$

# Proof Calculus

- **Properties**
  - Sound
  - Relatively complete
    - Proof by reduction to dGL via admissible axioms:

$$\textit{Complementarization axiom 1: } \langle\alpha\rangle(P, Q) \leftrightarrow \langle\alpha^{-d}\rangle(P \wedge Q) \vee \langle\alpha\rangle P$$
$$\textit{Complementarization axiom 2: } [\alpha](P, Q) \leftrightarrow \langle\alpha^{-d}\rangle(P \wedge Q) \vee [\alpha]Q$$

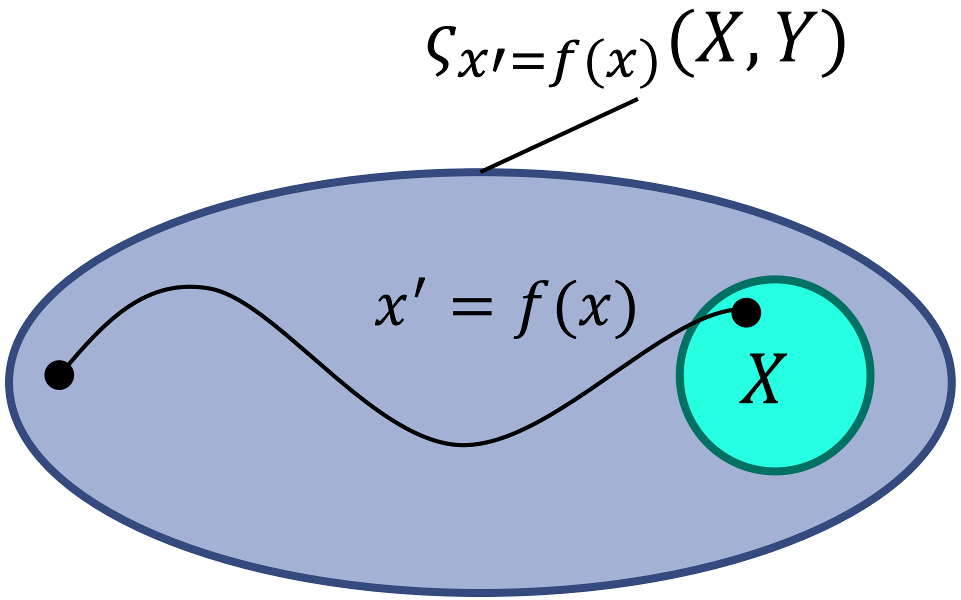
# Summary

- $dGL_{sc}$ is a logic for two players, each with an individual goal
  - Who behave semi-competitively, i.e. players cooperate where possible and compete where necessary
  - Developed syntax and semantics
  - Sound and relatively complete proof calculus

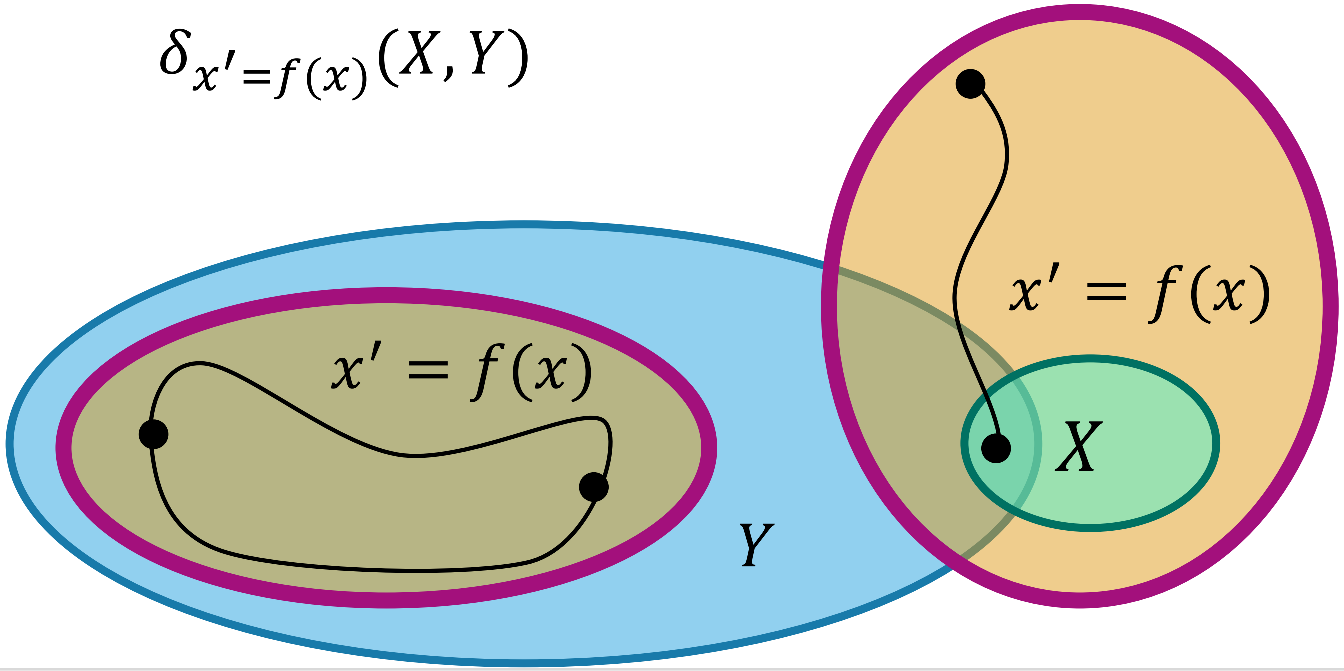
➡ Incorporating goals for all players enables more safety proofs

# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{x'=f(x)\&Q}(X,Y) = \{\varphi(0) \in \mathcal{S} \mid \varphi(r) \in X$ for some $r$ with $\varphi \vDash x' = f(x) \wedge Q\}$

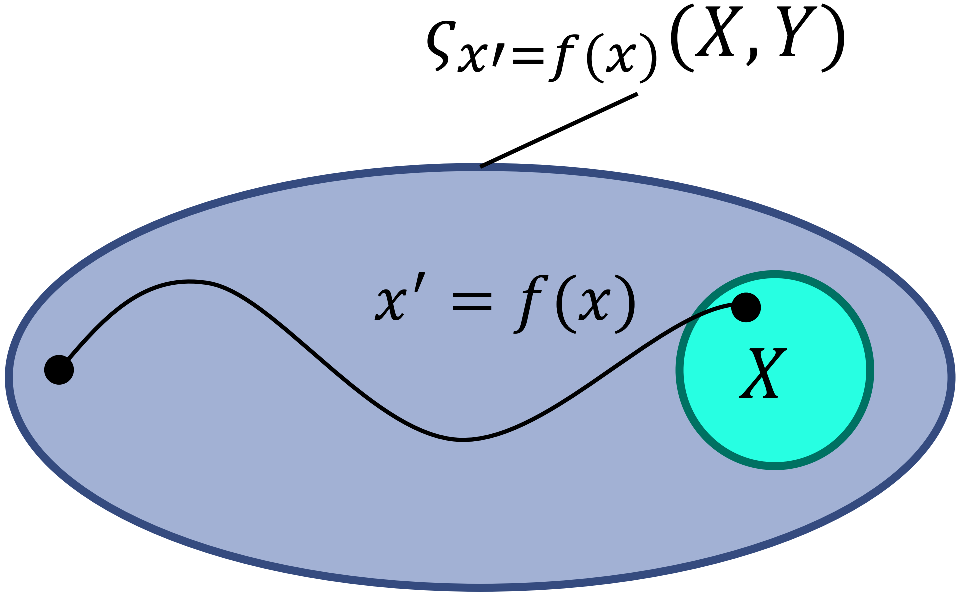



- $\delta_{x'=f(x)\&Q}(X,Y) = \{\varphi(0) \in \mathcal{S} \mid \varphi(r) \in Y$ for all $r$ with $\varphi \vDash x' = f(x) \wedge Q\}$ $\cup \{\varphi(0) \in \mathcal{S} \mid \varphi(0) \in X \cap Y$ for some $r$ with $\varphi \vDash x' = f(x) \wedge Q\}$

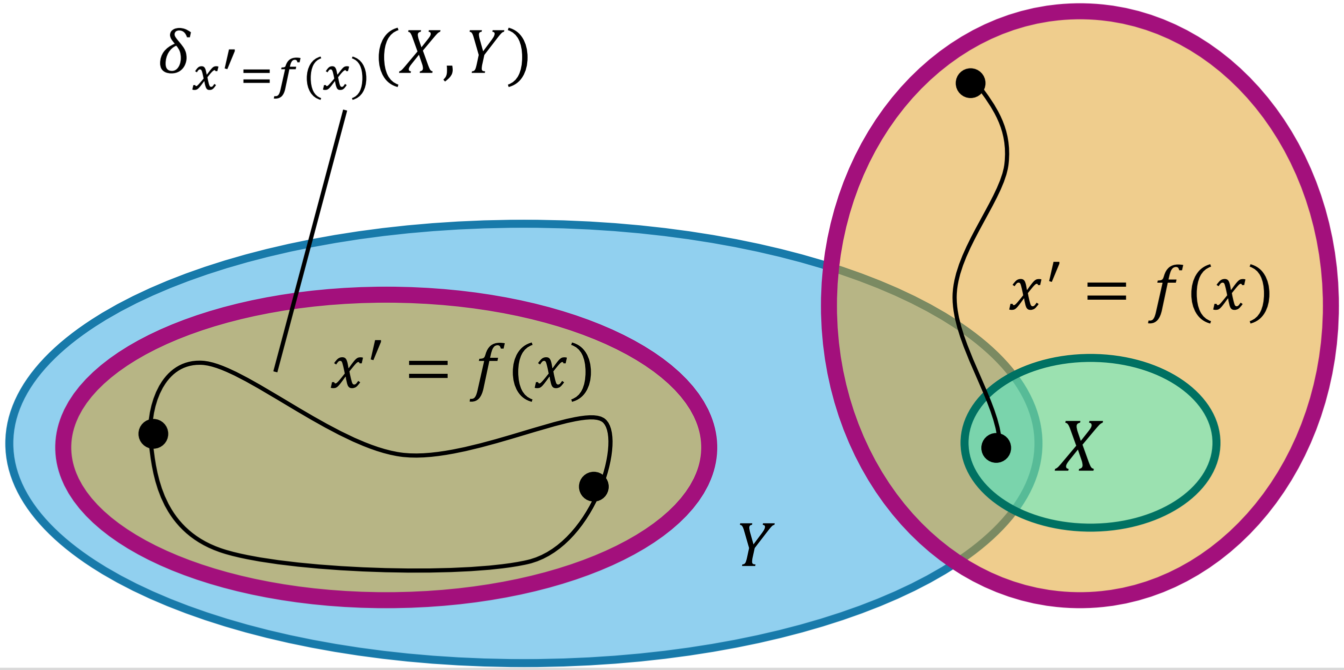


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{x'=f(x)\&Q}(X,Y) = \{\varphi(0) \in \mathcal{S} \mid \varphi(r) \in X$ for some $r$ with $\varphi \vDash x' = f(x) \wedge Q\}$

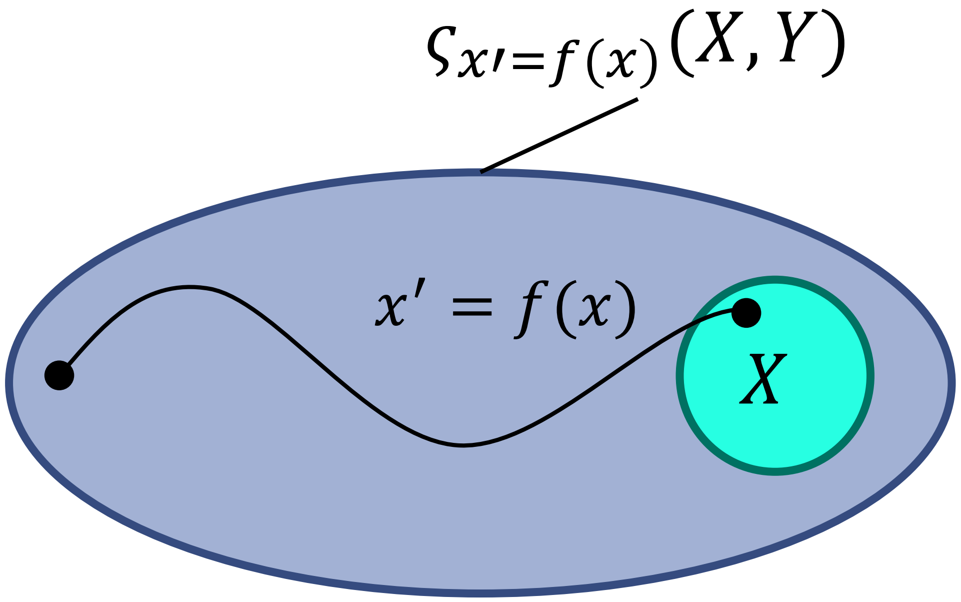



- $\delta_{x'=f(x)\&Q}(X,Y) = \boxed{\{\varphi(0) \in \mathcal{S} \mid \varphi(r) \in Y \text{ for all } r \text{ with } \varphi \vDash x' = f(x) \wedge Q\}}$ $\cup \{\varphi(0) \in \mathcal{S} \mid \varphi(0) \in X \cap Y$ for some $r$ with $\varphi \vDash x' = f(x) \wedge Q\}$

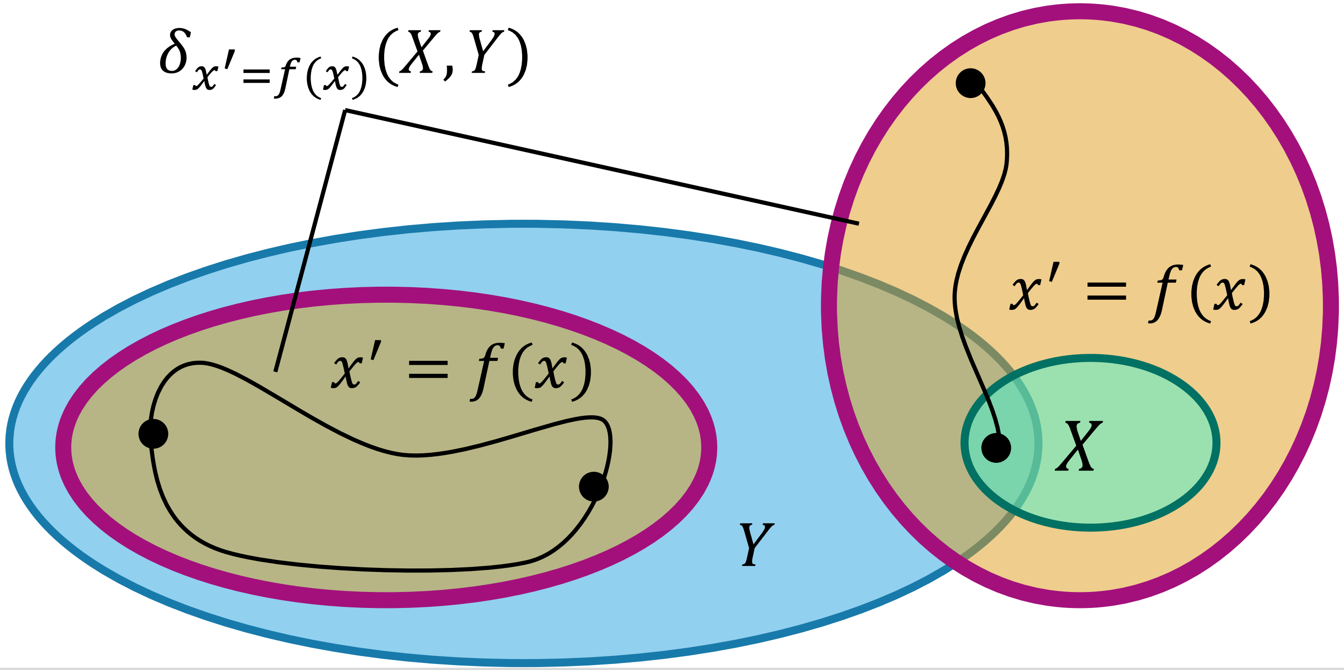


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{x'=f(x)\&Q}(X,Y) = \{\varphi(0) \in \mathcal{S} \mid \varphi(r) \in X \text{ for some } r \text{ with } \varphi \vDash x' = f(x) \wedge Q\}$

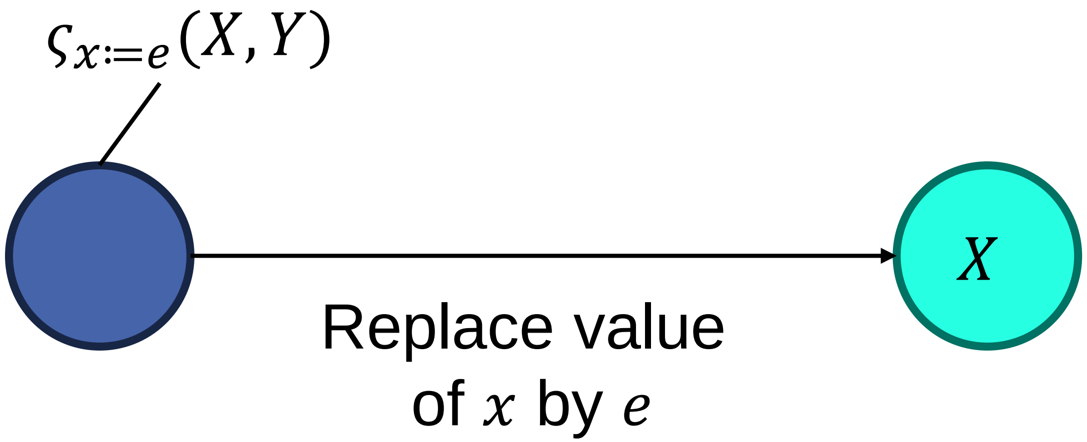



- $\delta_{x'=f(x)\&Q}(X,Y) = \{\varphi(0) \in \mathcal{S} \mid \varphi(r) \in Y \text{ for all } r \text{ with } \varphi \vDash x' = f(x) \wedge Q\} \cup \boxed{\{\varphi(0) \in \mathcal{S} \mid \varphi(0) \in X \cap Y \text{ for some } r \text{ with } \varphi \vDash x' = f(x) \wedge Q\}}$

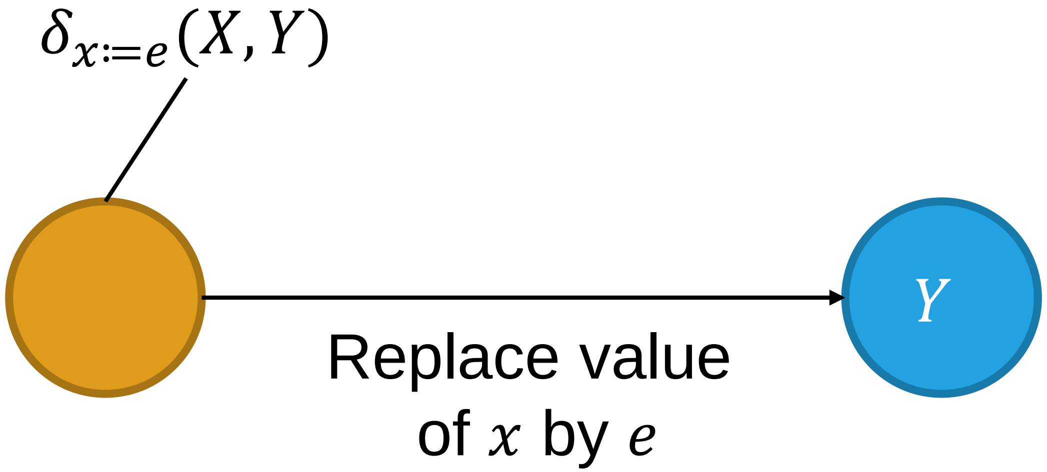


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{x:=e}(X,Y) = \left\{\omega \in \mathcal{S} \,\middle|\, \omega_x^{\omega[\![e]\!]} \in Y\right\}$

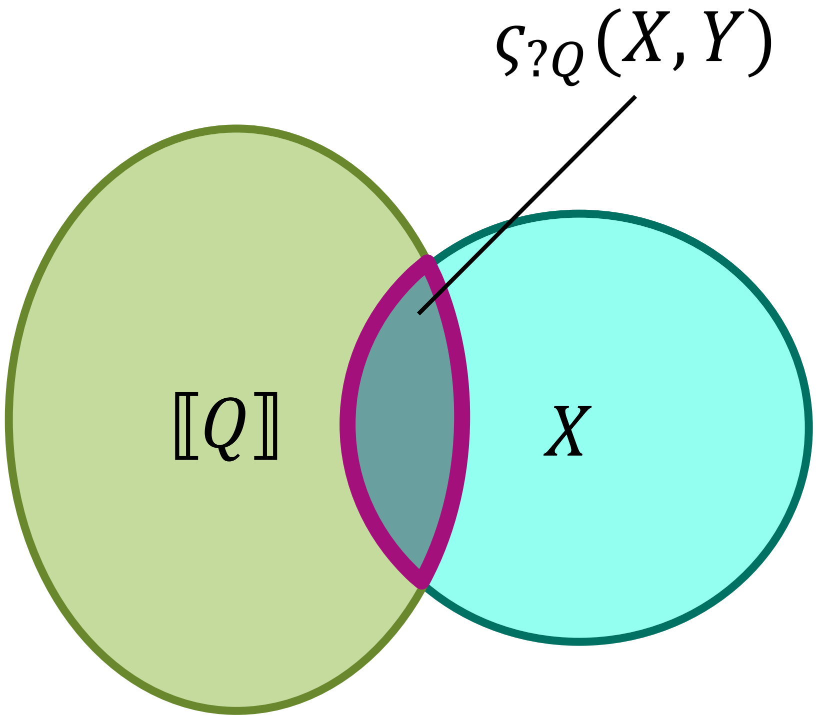



- $\delta_{x:=e}(X,Y) = \left\{\omega \in \mathcal{S} \,\middle|\, \omega_x^{\omega[\![e]\!]} \in Y\right\}$

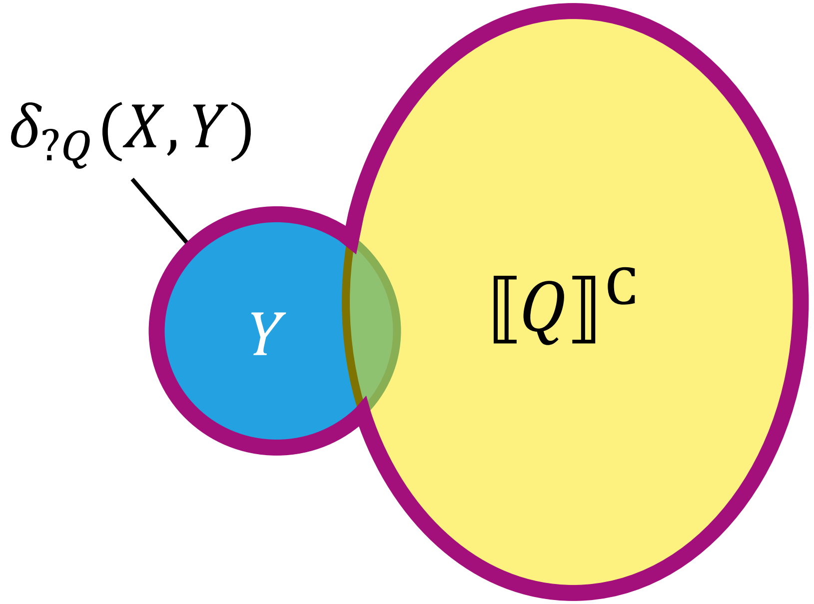


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{?Q}(X,Y) = [\![Q]\!] \cap X$

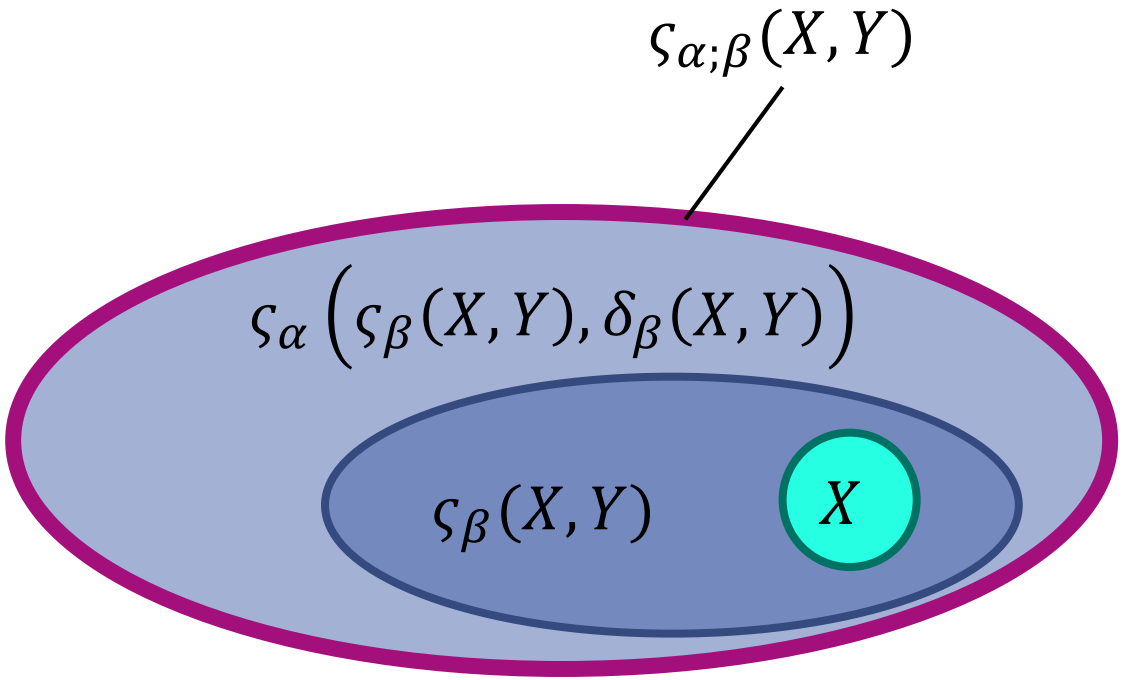



- $\delta_{?Q}(X,Y) = [\![Q]\!]^C \cup Y$

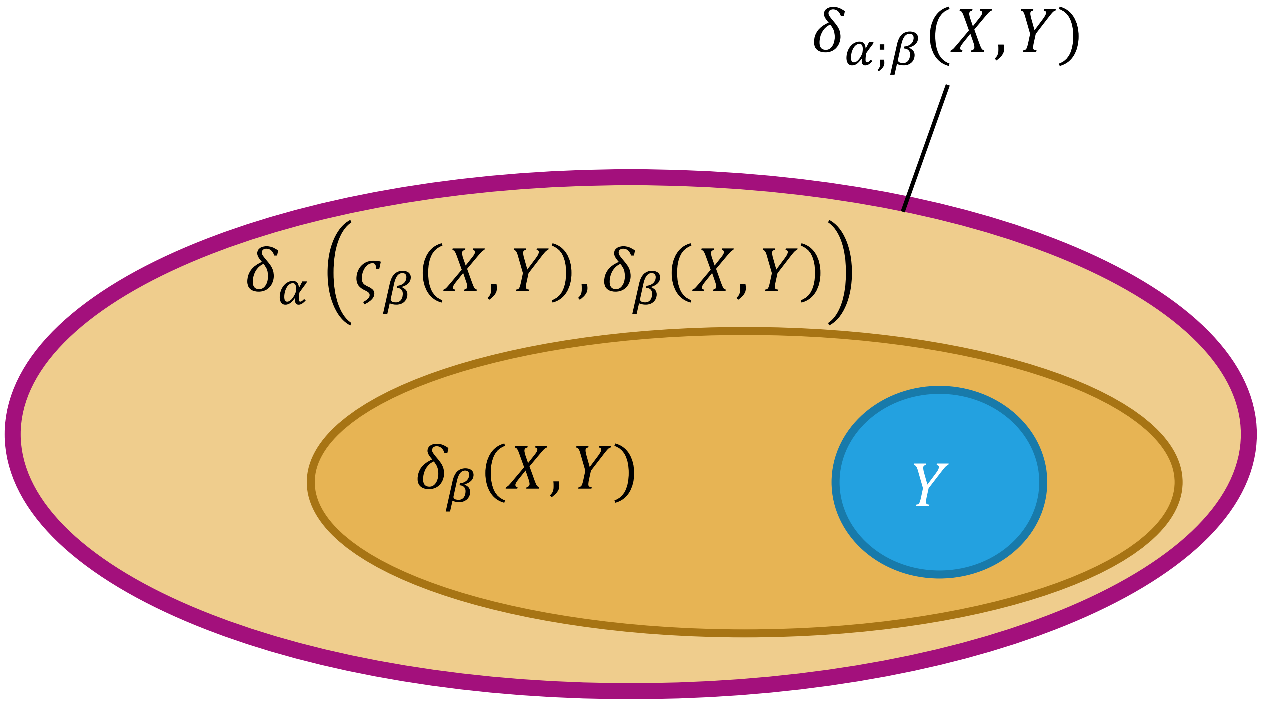


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha;\beta}(X,Y) = \varsigma_\alpha\left(\varsigma_\beta(X,Y), \delta_\beta(X,Y)\right)$

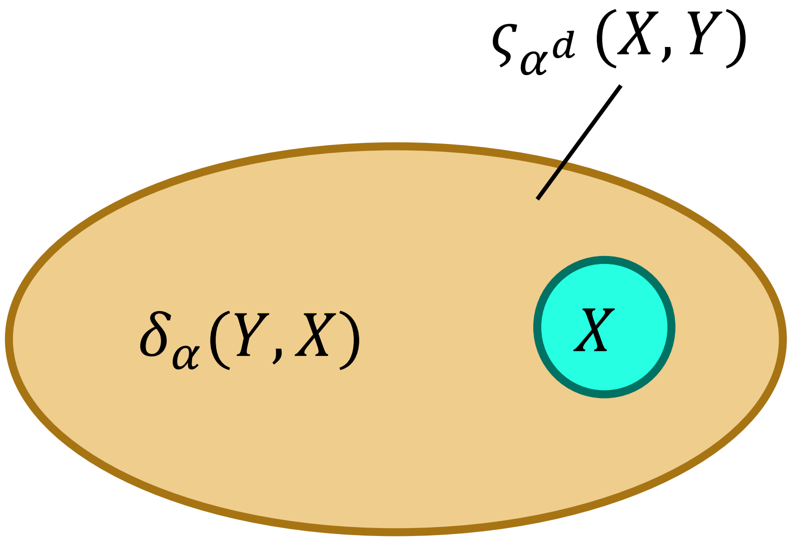



- $\delta_{\alpha;\beta}(X,Y) = \delta_\alpha\left(\varsigma_\beta(X,Y), \delta_\beta(X,Y)\right)$

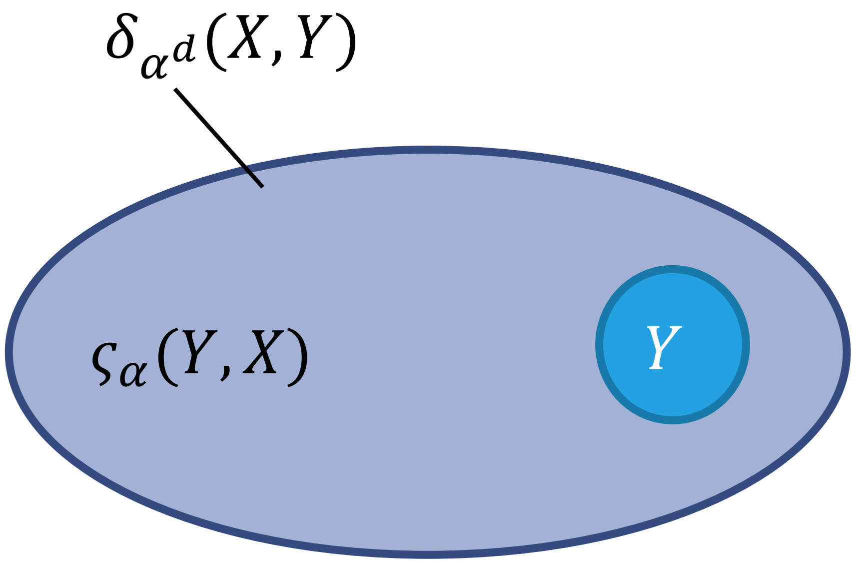


# Semi-Competitive Differential Game Logic ($dGL_{sc}$)

- $\varsigma_{\alpha^d}(X,Y) = \delta_\alpha(Y,X)$




- $\delta_{\alpha^d}(X,Y) = \varsigma_\alpha(Y,X)$



# Example

- FP $\dfrac{P \vee \langle\alpha\rangle R_1 \to R_1 \qquad (P \wedge Q) \vee (\langle\alpha\rangle(R_2, R_2) \wedge [\alpha](R_2, R_2)) \to R_2}{\langle\alpha^*\rangle(P, Q) \to R_1 \vee R_2}$

$$\varsigma_{\alpha^*}(X, Y) = \bigcap \{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z, Z^C) \subseteq Z\}$$

$$\cup \bigcap \{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z, Z) \cap \delta_\alpha(Z, Z)\big) \subseteq Z\}$$

# Example

- FP $$\frac{\boxed{P \vee \langle\alpha\rangle R_1 \to R_1} \qquad (P \wedge Q) \vee (\langle\alpha\rangle(R_2, R_2) \wedge [\alpha](R_2, R_2)) \to R_2}{\langle\alpha^*\rangle(P, Q) \to R_1 \vee R_2}$$

$$\varsigma_{\alpha^*}(X, Y) = \boxed{\bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z, Z^C) \subseteq Z\}}$$

$$\cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z, Z) \cap \delta_\alpha(Z, Z)\big) \subseteq Z\}$$

## Example

- FP $$\frac{P \vee \langle\alpha\rangle R_1 \to R_1 \qquad \boxed{(P \wedge Q) \vee (\langle\alpha\rangle(R_2, R_2) \wedge [\alpha](R_2, R_2)) \to R_2}}{\langle\alpha^*\rangle(P, Q) \to R_1 \vee R_2}$$

$$\varsigma_{\alpha^*}(X, Y) = \bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z, Z^C) \subseteq Z\}$$

$$\boxed{\cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \big(\varsigma_\alpha(Z, Z) \cap \delta_\alpha(Z, Z)\big) \subseteq Z\}}$$

# Example

- FP $$\frac{P \vee \langle\alpha\rangle R_1 \to R_1 \qquad (P \wedge Q) \vee (\langle\alpha\rangle(R_2, R_2) \wedge [\alpha](R_2, R_2)) \to R_2}{\langle\alpha^*\rangle(P, Q) \to R_1 \vee R_2}$$

$$\varsigma_{\alpha^*}(X, Y) = \bigcap\{Z \subseteq \mathcal{S} \mid X \cup \varsigma_\alpha(Z, Z^C) \subseteq Z\}$$

$$\cup \bigcap\{Z \subseteq \mathcal{S} \mid (X \cap Y) \cup \left(\varsigma_\alpha(Z, Z) \cap \delta_\alpha(Z, Z)\right) \subseteq Z\}$$

# Example

$$
\begin{array}{rl}
 & \qquad\qquad\qquad\qquad *\\ \cline{2-2}
\mathbb{R} & o=0,t=0 \vdash t+5=5 \land o+3=3 \\ \cline{2-2}
\lor\mathrm{R},\ \mathrm{wR},\ \langle:=\rangle,\ \exists\mathrm{R} & o=0,t=0 \vdash \forall s{\geq}0[o:=o+s](t+5=5, o=3) \\
 & \qquad\qquad\qquad \lor\exists s{\geq}0\langle o=o+s\rangle(t+5=5\land o=3, o=3) \\ \cline{2-2}
[{}'] & o=0,t=0 \vdash [x'=1](t+5=5, o=3) \\ \cline{2-2}
\exists\mathrm{R},\ \langle:=\rangle & o=0,t=0 \vdash \exists s{\geq}0\langle t:=t+s\rangle([x'=1](t=5,o=3), \\
 & \qquad\qquad\qquad\qquad\qquad \langle x'=1\rangle(t=5,o=3)) \\ \cline{2-2}
\langle{}'\rangle & o=0,t=0 \vdash \langle y'=1\rangle([x'=1](t=5,o=3), \\
 & \qquad\qquad\qquad\qquad\qquad \langle x'=1\rangle(t=5,o=3)) \\ \cline{2-2}
\langle;\rangle,\ \langle^d\rangle,\ [^d] & o=0,t=0 \vdash \langle\{t'=1\};\{o'=1\}^d\rangle(o=3, t=5)
\end{array}
$$

# Example

$$
\begin{array}{rl}
 & \qquad\qquad * \\ \cline{2-2}
\mathbb{R} & o=0, t=0 \ \vdash \ t+5=5 \land o+3=3 \\ \cline{2-2}
\lor\mathrm{R},\ \mathrm{wR},\ \langle :=\rangle,\ \exists\mathrm{R} & o=0, t=0 \ \vdash \ \forall s{\geq}0[o := o+s](t+5=5, o=3) \\
 & \qquad\qquad \lor \exists s{\geq}0\langle o = o+s\rangle(t+5=5 \land o=3, o=3) \\ \cline{2-2}
[{}'] & o=0, t=0 \ \vdash \ [x'=1](t+5=5, o=3) \\ \cline{2-2}
\exists\mathrm{R},\ \langle :=\rangle & o=0, t=0 \ \vdash \ \exists s{\geq}0\langle t := t+s\rangle([x'=1](t=5, o=3), \\
 & \qquad\qquad \langle x'=1\rangle(t=5, o=3)) \\ \cline{2-2}
\langle{}'\rangle & o=0, t=0 \ \vdash \ \langle y'=1\rangle([x'=1](t=5, o=3), \\
 & \qquad\qquad \langle x'=1\rangle(t=5, o=3)) \\ \cline{2-2}
\langle ;\rangle,\ \langle^d\rangle,\ [^d] & o=0, t=0 \ \vdash \ \langle \{t'=1\};\{o'=1\}^d\rangle(o=3, t=5)
\end{array}
$$

Formula breakdown

# Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\ast}{o=0,t=0 \ \vdash \ t+5=5 \land o+3=3}\ \mathbb{R}}{\boxed{\begin{array}{l} o=0,t=0 \ \vdash \ \forall s{\geq}0[o:=o+s](t+5=5, o=3) \\ \qquad\qquad \lor \exists s{\geq}0\langle o=o+s\rangle(t+5=5 \land o=3, o=3) \end{array}}}\ \lor\mathrm{R},\ \mathrm{wR},\ \langle:=\rangle,\ \exists\mathrm{R}}{o=0,t=0 \ \vdash \ [x'=1](t+5=5, o=3)}\ [']}{\begin{array}{l} o=0,t=0 \ \vdash \ \exists s{\geq}0\langle t:=t+s\rangle([x'=1](t=5, o=3), \\ \qquad\qquad \langle x'=1\rangle(t=5, o=3)) \end{array}}\ \exists\mathrm{R},\ \langle:=\rangle}{\begin{array}{l} o=0,t=0 \ \vdash \ \langle y'=1\rangle([x'=1](t=5, o=3), \\ \qquad\qquad \langle x'=1\rangle(t=5, o=3)) \end{array}}\ \langle'\rangle}{o=0,t=0 \ \vdash \ \langle\{t'=1\};\{o'=1\}^d\rangle(o=3, t=5)}\ \langle;\rangle,\ \langle^d\rangle,\ [^d]
$$

Case distinction