Logic & Proofs for Cyber-Physical Systems

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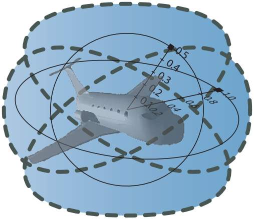


Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
   - Stochastic Hybrid Systems
   - Distributed Hybrid Systems
2. Dynamic Logic of Multi-Dynamical Systems
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Axioms
   - Example: Elementary Differential Invariants
5. Applications
6. Summary

Cyber-Physical Systems Analysis: Aircraft Example

Which control decisions are safe for aircraft collision avoidance?

Cyber-Physical Systems

CPSs combine cyber capabilities with physical capabilities
to solve problems that neither part could solve alone.

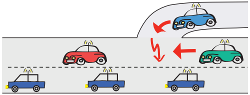
CPSs Promise Transformative Impact!

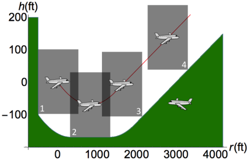
Prospects: Safe & Efficient

Driver assistance
Autonomous cars

Pilot decision support
Autopilots / UAVs

Train protection
Robots near humans





Prerequisite: CPSs need to be safe

How do we make sure CPSs make the world a better place?

Can you trust a computer to control physics?

Can you trust a computer to control physics?

1. Depends on how it has been programmed
2. And on what will happen if it malfunctions

Rationale

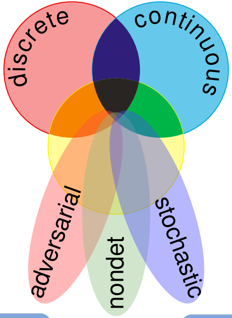
1. Safety guarantees require analytic foundations.
2. A common foundational core helps all application domains.
3. Foundations revolutionized digital computer science & our society.
4. Need even stronger foundations when software reaches out into our physical world.

CPSs deserve proofs as safety evidence!

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

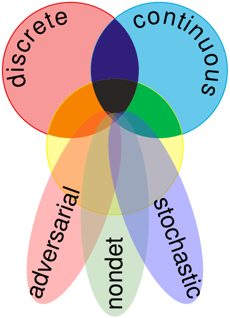
Exploiting compositionality tames CPS complexity.

Analytic simplification

CPSs are Multi-Dynamical Systems

hybrid systems

HS = discrete + ODE

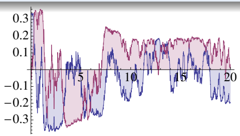


hybrid games

HG = HS + adversary

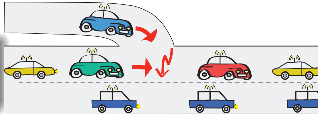
stochastic hybrid sys.

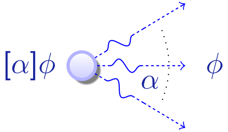
SHS = HS + stochastics



distributed hybrid sys.

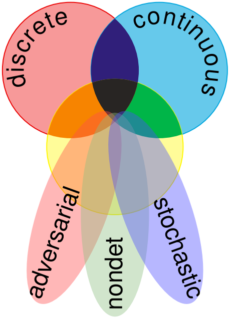
DHS = HS + distributed





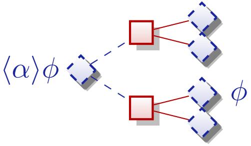
differential dynamic logic

$d\mathcal{L} = \mathrm{DL} + \mathrm{HP}$



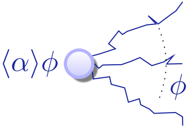
differential game logic

$dG\mathcal{L} = \mathrm{GL} + \mathrm{HG}$



stochastic differential DL

$Sd\mathcal{L} = \mathrm{DL} + \mathrm{SHP}$



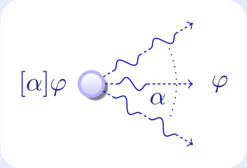
quantified differential DL

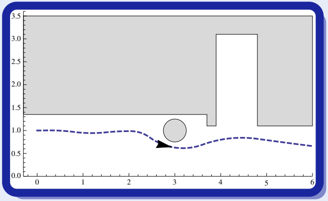
$Qd\mathcal{L} = \mathrm{FOL} + \mathrm{DL} + \mathrm{QHP}$

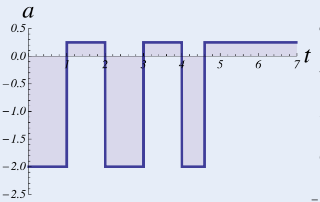
JAR'08,CADE'11,LMCS'12,LICS'12

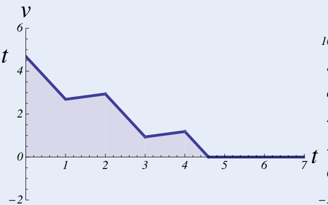
LICS'12,CADE'15,TOCL'15

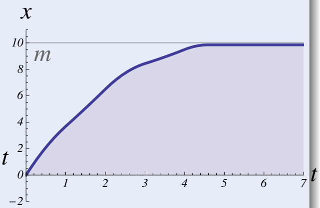
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)





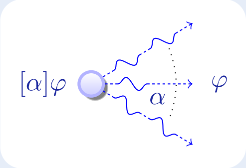


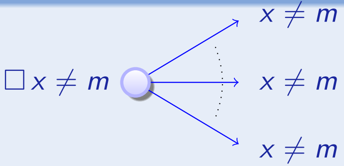


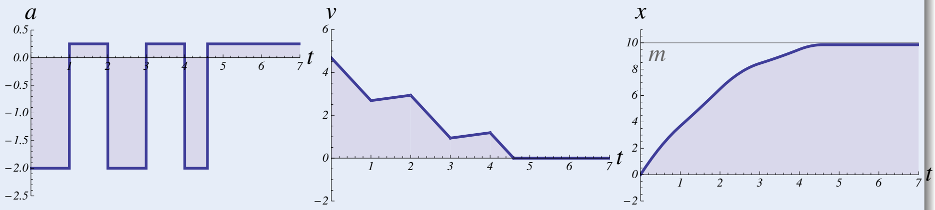


CPS Analysis

Concept (Differential Dynamic Logic) (JAR'08,LICS'12)







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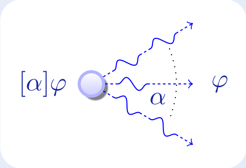

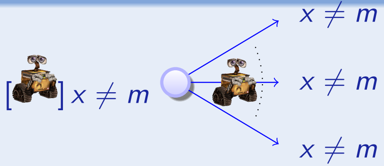

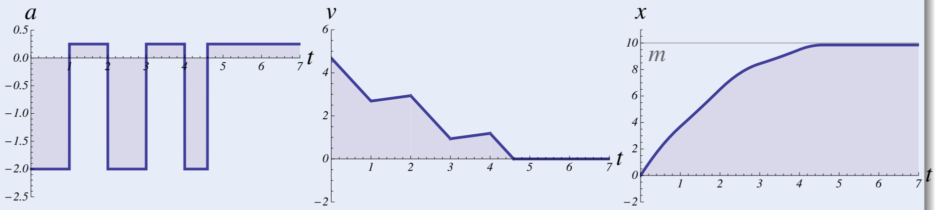

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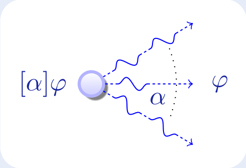
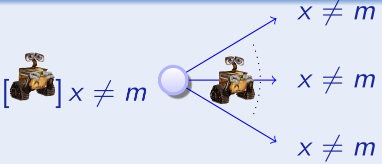
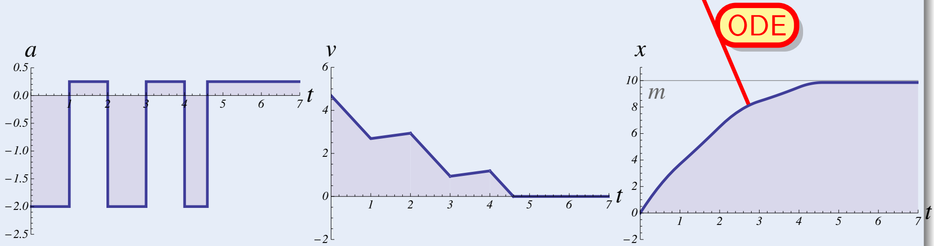




$x' = v, v' = a$



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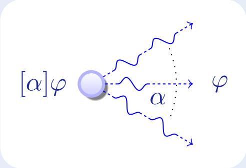
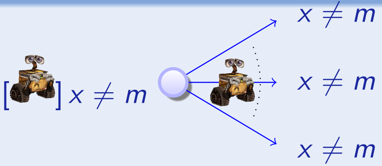
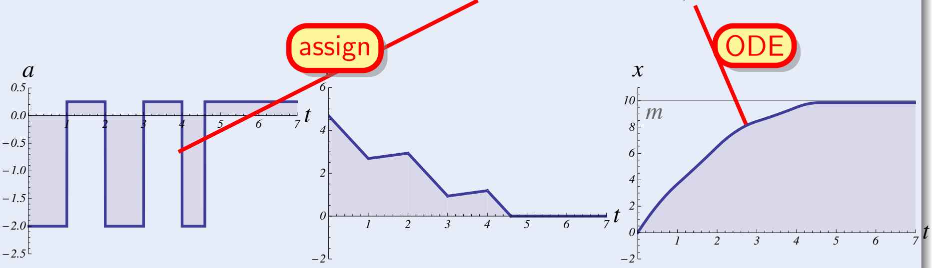




$a := -b$ $\quad x' = v, v' = a$



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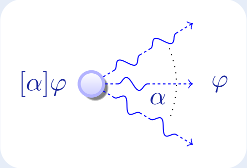
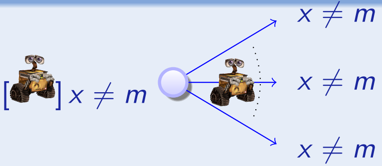
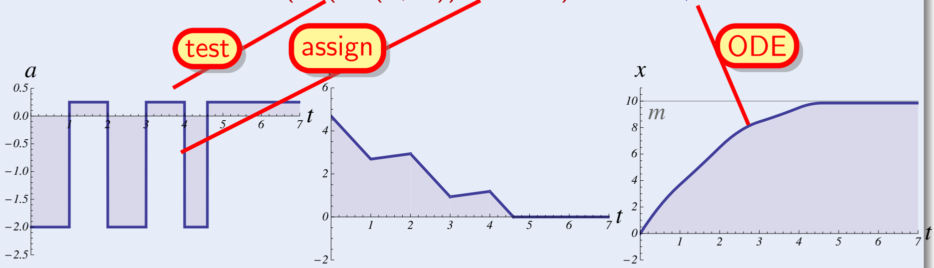




$(\texttt{if}(\mathrm{SB}(x, m))\ a := -b) \quad x' = v, v' = a$



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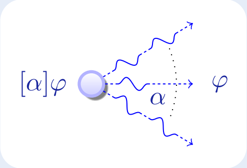


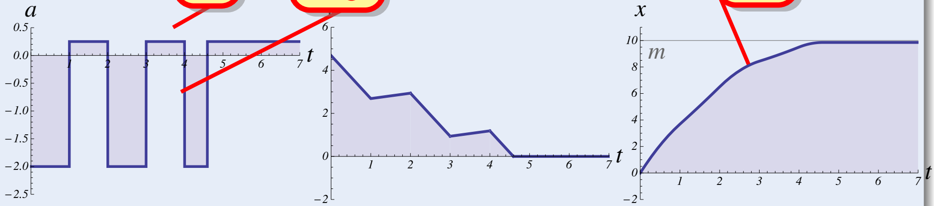
seq. compose

$(\texttt{if}(SB(x,m))\ a := -b)\ ;\ x' = v, v' = a$

test

assign

ODE



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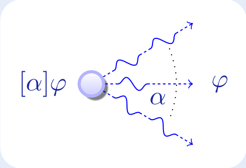


$$\left(\left(\mathtt{if}(SB(x,m))\ a := -b\right)\ ;\ x' = v, v' = a\right)^*$$

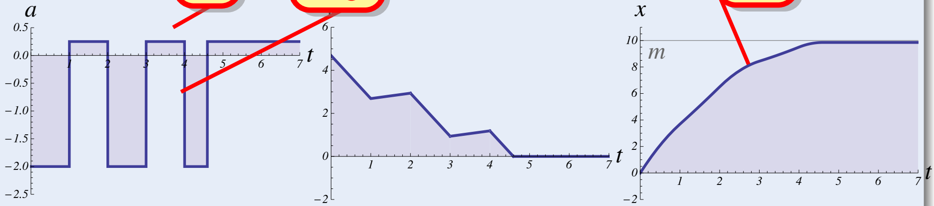
seq. compose

nondet. repeat

test

assign

ODE



CPS Analysis

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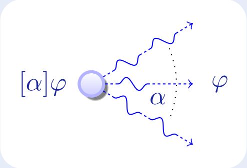
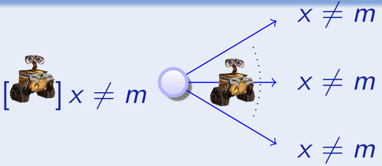
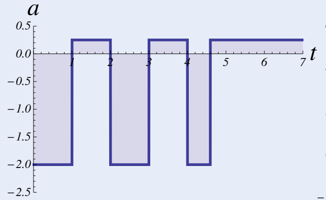
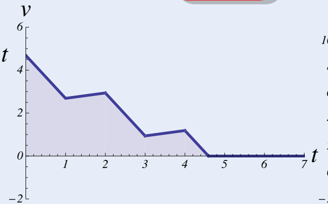


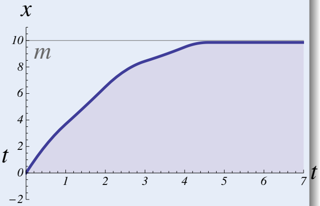


$$\left[\left(\left(\mathtt{if}(SB(x,m))\ a := -b\right)\ ;\ x' = v, v' = a\right)^*\right] \underbrace{x \neq m}_{\text{post}}$$

all runs







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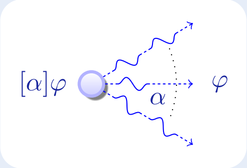
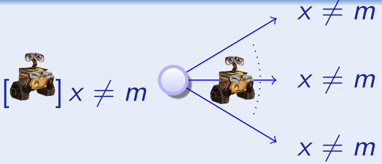
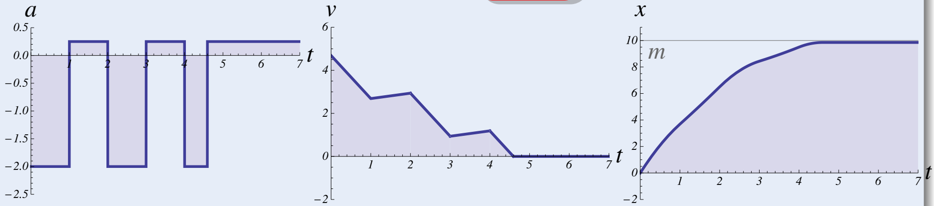




$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \big[\big((\mathtt{if}(SB(x, m))\ a := -b)\ ;\ x' = v, v' = a\big)^*\big]\underbrace{x \neq m}_{\text{post}}$$

all runs



Differential Dynamic Logic d$\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := f(x) \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Tableaux'07,JAutomReas'08,LICS'12

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

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All Reals | Some Reals | All Runs | Some Runs

Tableaux'07,JAutomReas'08,LICS'12

Differential Dynamic Logic: Axiomatization

equations of truth

[:=]	$[x := e]P(x) \leftrightarrow P(e)$
[?]	$[?Q]P \leftrightarrow (Q \rightarrow P)$
[']	$[x' = f(x)]P \leftrightarrow \forall t{\geq}0\, [x := y(t)]P \qquad (y'(t) = f(y))$
[∪]	$[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$
[;]	$[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$
[*]	$[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$
K	$[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$
I	$[\alpha^*](P \rightarrow [\alpha]P) \rightarrow (P \rightarrow [\alpha^*]P)$
C	$[\alpha^*]\forall v{>}0\, (P(v) \rightarrow \langle\alpha\rangle P(v-1)) \rightarrow \forall v\, (P(v) \rightarrow \langle\alpha^*\rangle \exists v{\leq}0\, P(v))$

LICS'12,CADE'15

Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (J.Autom.Reas. 2008, LICS'12)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* **or** *to discrete dynamics.* Proof 25pp

Corollary (Complete Proof-theoretical Bridge)

proving continuous = proving hybrid = proving discrete

JAutomReas'08,LICS'12

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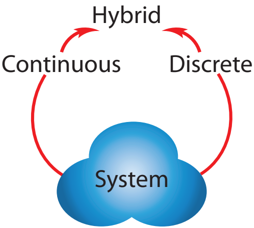

JAutomReas'08,LICS'12

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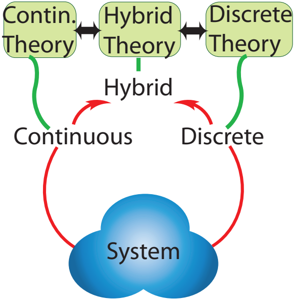
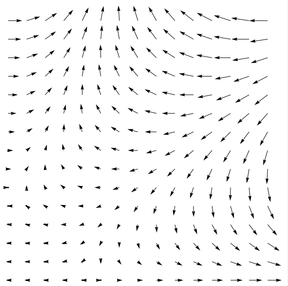


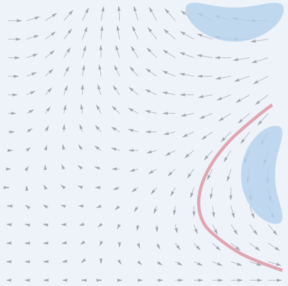
JAutomReas'08,LICS'12

Differential Invariants for Differential Equations

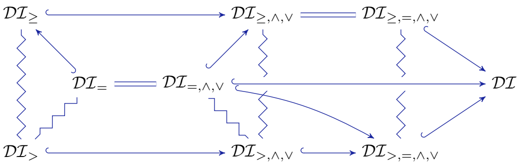
Differential Invariant



Differential Cut



Differential Ghost



Logic

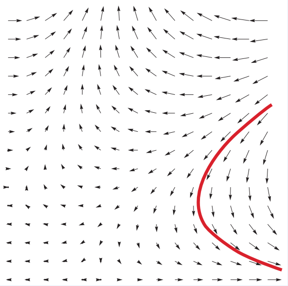
Provability theory

Math

Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,LICS'12,ITP'12,CADE'15

Differential Invariant



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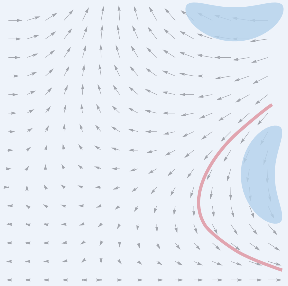

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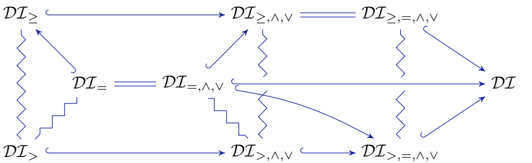

Logic

Provability theory

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Differential Invariants for Differential Equations

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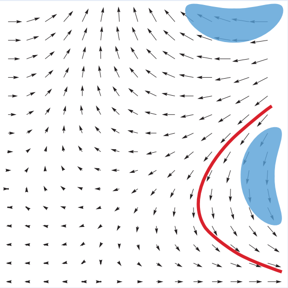

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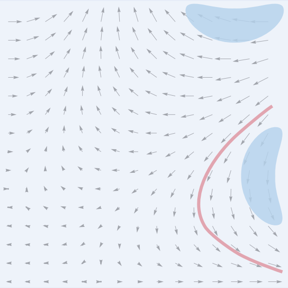
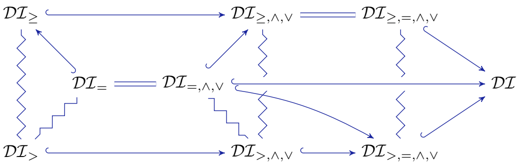


Differential Ghost



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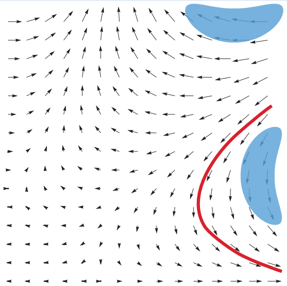
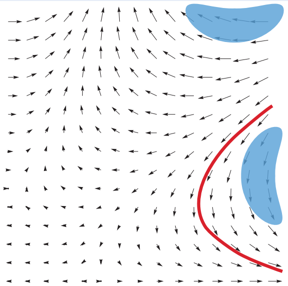
Differential Invariants for Differential Equations

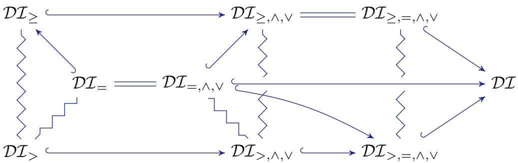
Differential Invariant



Differential Cut



Differential Ghost



Logic

Provability theory

Math

Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,LICS'12,ITP'12,CADE'15

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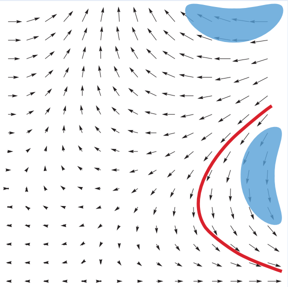

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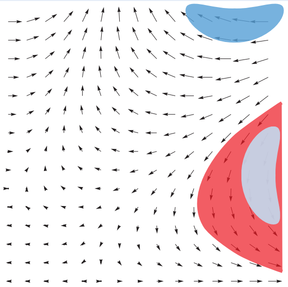

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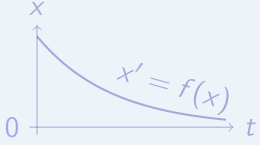

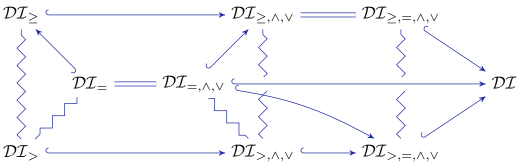

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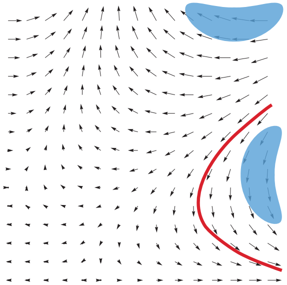

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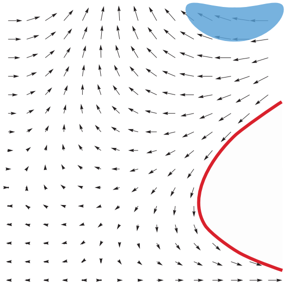

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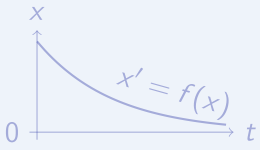

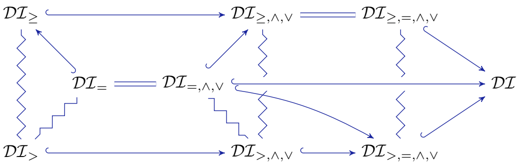

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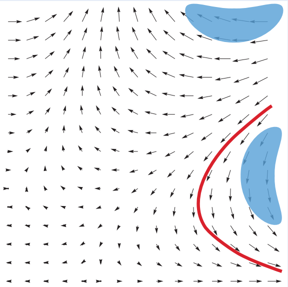

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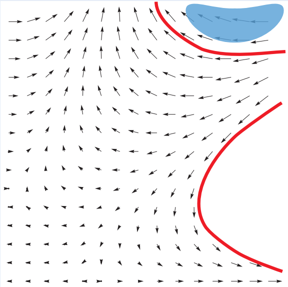

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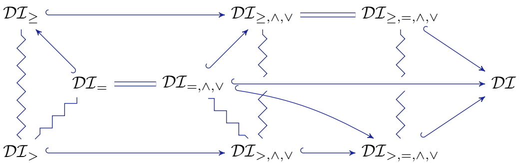

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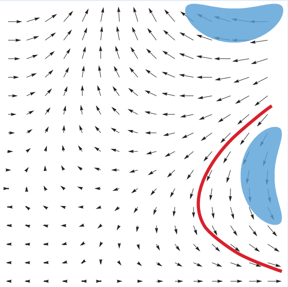

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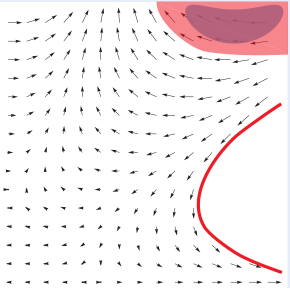

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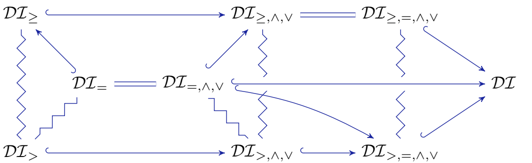

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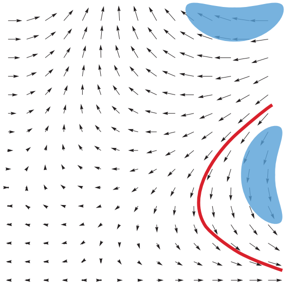

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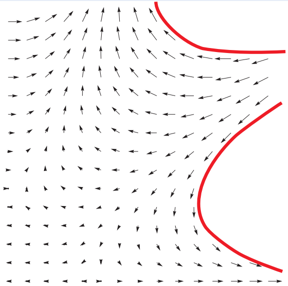

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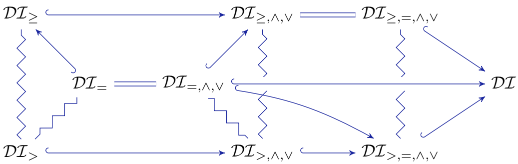

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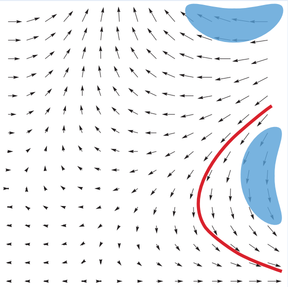

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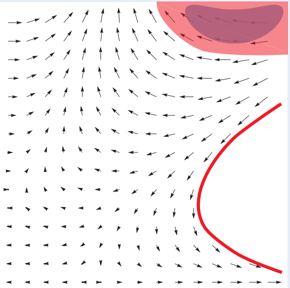

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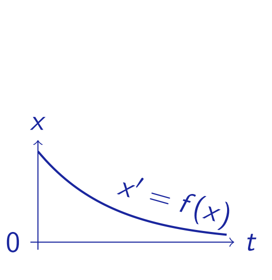

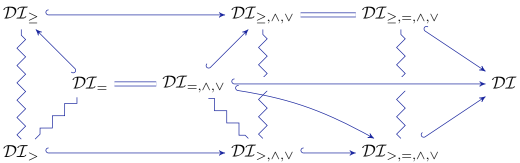

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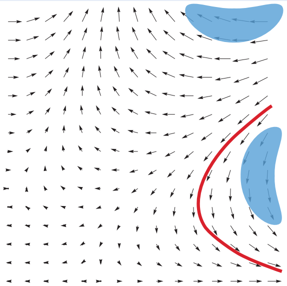

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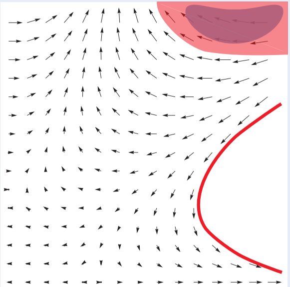

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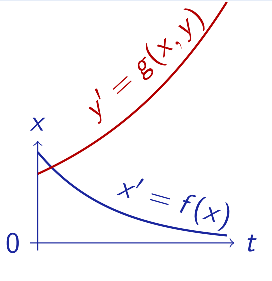

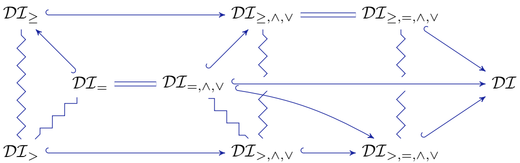

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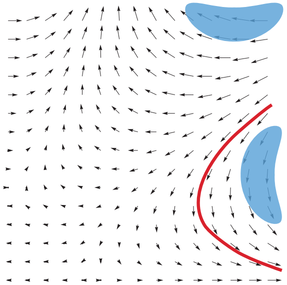

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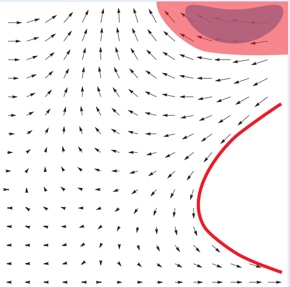

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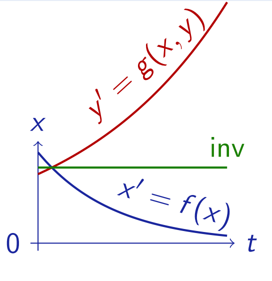

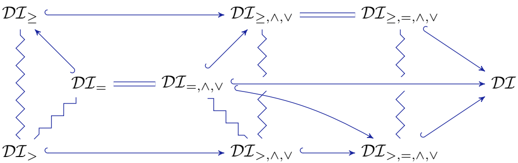

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Differential Equation Axioms & Differential Axioms

DW $[x' = f(x) \,\&\, Q]Q$

DC $\big([x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q \wedge R]P\big) \leftarrow [x' = f(x) \,\&\, Q]R$

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

DI $\big([x' = f(x) \,\&\, Q]P \leftrightarrow [?Q]P\big) \leftarrow [x' = f(x) \,\&\, Q](P)'$

DG $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$

DS $[x' = c() \,\&\, Q]P \leftrightarrow \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(x{+}c()s)) \rightarrow [x := x{+}c()t]P\big)$

$[':=]$ $[x' := e]p(x') \leftrightarrow p(e)$

$+'$ $(e + k)' = (e)' + (k)'$

$\cdot'$ $(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$

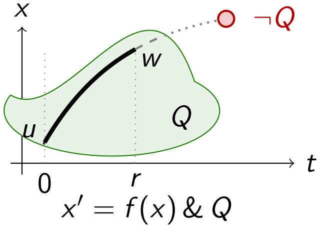
$\circ'$ $[y := g(x)][y' := 1]\big((f(g(x)))' = (f(y))' \cdot (g(x))'\big)$

CADE'15

Differential Equation Axioms

Axiom (Differential Weakening) (CADE'15)

DW $[x' = f(x) \,\&\, Q]Q$



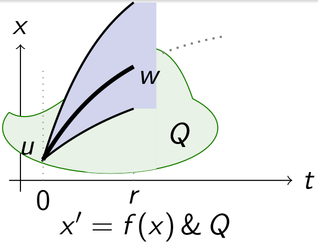
Differential equations cannot leave their evolution domains. Implies:

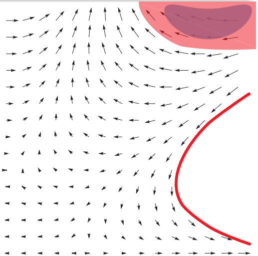
$$[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q]\big(Q \rightarrow P\big)$$

Differential Equation Axioms

Axiom (Differential Cut) (CADE'15)

DC $\big([x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q \land R]P\big) \leftarrow [x' = f(x)\,\&\,Q]R$

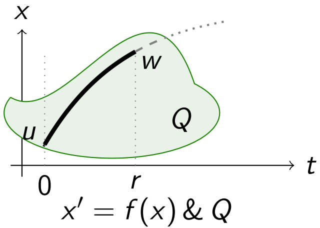


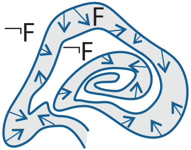


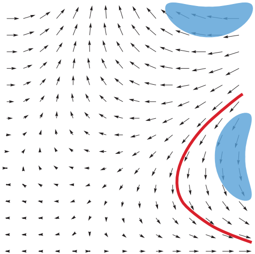
DC is a cut for differential equations.
DC is a differential modal modus ponens K.
Can't leave R, then might as well restrict state space to R.

Axiom (Differential Invariant) (CADE'15)

DI $([x' = f(x) \,\&\, Q]P \leftrightarrow [?Q]P) \leftarrow [x' = f(x) \,\&\, Q](P)'$





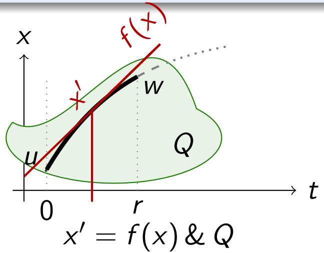


Differential invariant: if P true now and if differential $(P)'$ true always
What's the differential of a formula???
What's the meaning of a differential term . . . in a state???

Differential Equation Axioms

Axiom (Differential Effect) (CADE'15)

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$



Effect of differential equation on differential symbol x'

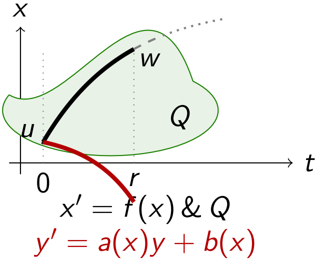
$[x' := f(x)]$ instantly mimics continuous effect $[x' = f(x)]$ on x'

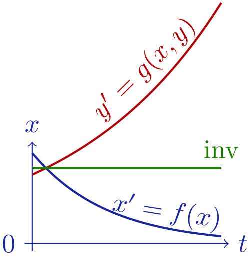
$[x' := f(x)]$ selects vector field $x' = f(x)$ for subsequent differentials

Differential Equation Axioms

Axiom (Differential Ghost) (CADE'15)

DG $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$





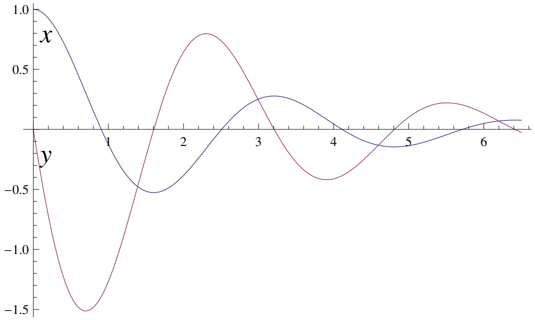
Differential ghost/auxiliaries: extra differential equations that exist

Can cause new invariants

"Dark matter" counterweight to balance conserved quantities

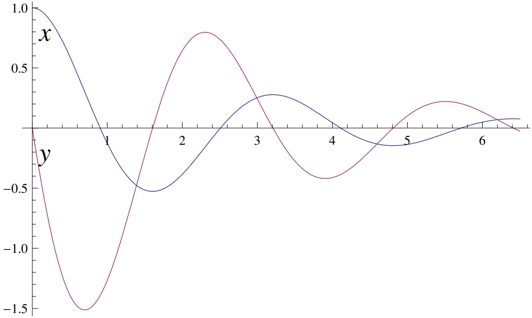
Differential Invariants for Differential Equations

$$\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \land d\geq 0)]\, \omega^2x^2+y^2\leq c^2}$$

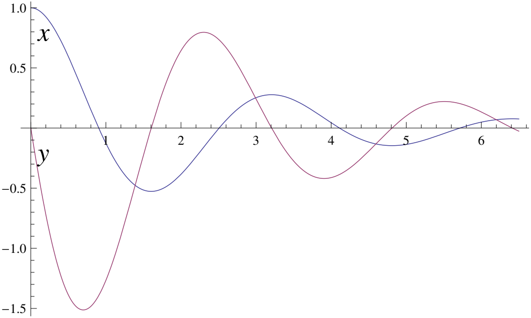


Differential Invariants for Differential Equations

$$\frac{\omega \geq 0 \land d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 x x' + 2 y y' \leq 0}{\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega \geq 0 \land d \geq 0)]\, \omega^2 x^2 + y^2 \leq c^2}$$

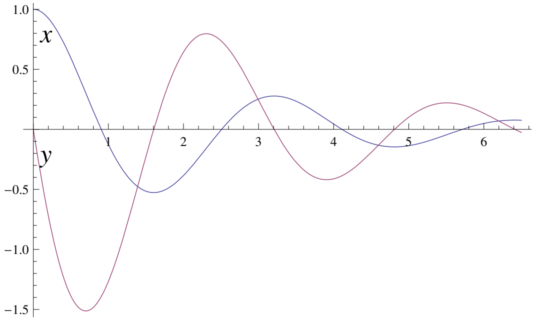


$$\dfrac{\dfrac{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]2\omega^2 x x' + 2y y' \leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \land d\geq 0)]\, \omega^2x^2+y^2\leq c^2}$$



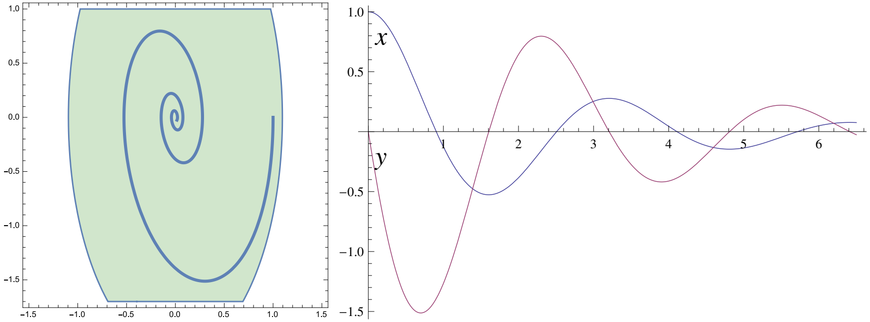
Differential Invariants for Differential Equations

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y] 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \land d\geq 0)]\, \omega^2x^2+y^2\leq c^2}}$$



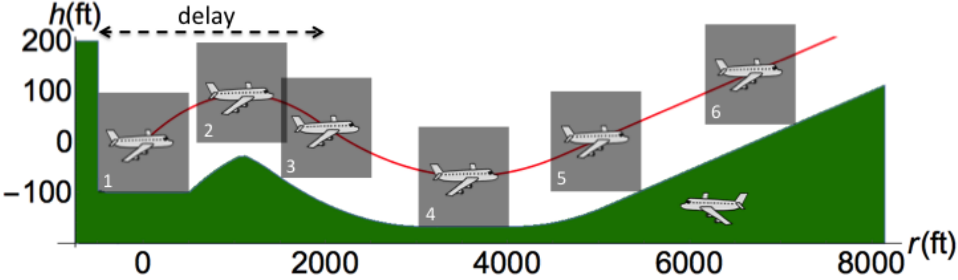
Differential Invariants for Differential Equations

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \wedge d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0 \wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]2\omega^2 x x' + 2yy' \leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, (\omega\geq 0 \wedge d\geq 0)]\, \omega^2x^2+y^2\leq c^2}$$



Airborne Collision Avoidance System ACAS X: Verify

- Developed by the FAA to replace current TCAS in aircraft
- Approximately optimizes Markov Decision Process on a grid
- Advisory from lookup tables with numerous 5D interpolation regions

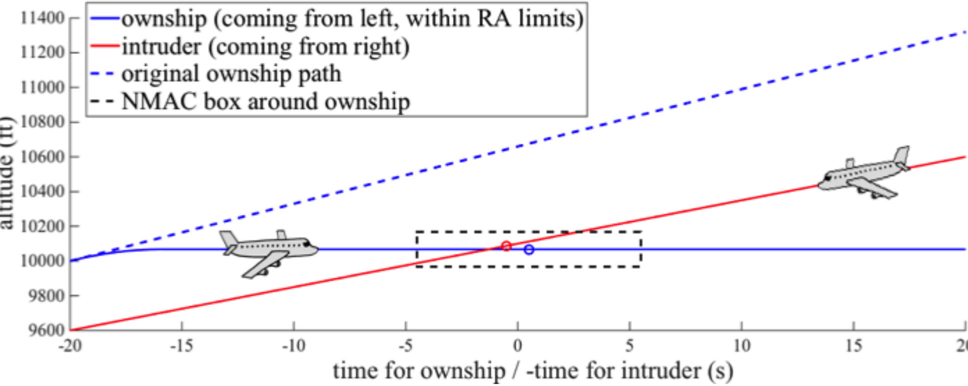


1. Identified safe region for each advisory symbolically
2. Proved safety for hybrid systems flight model in KeYmaera X

TACAS'15,EMSOFT'15

Airborne Collision Avoidance System ACAS X: Compare

ACAS X table comparison shows safe advisory in 97.7% of the 648,591,384,375 states compared (15,160,434,734 counterexamples).

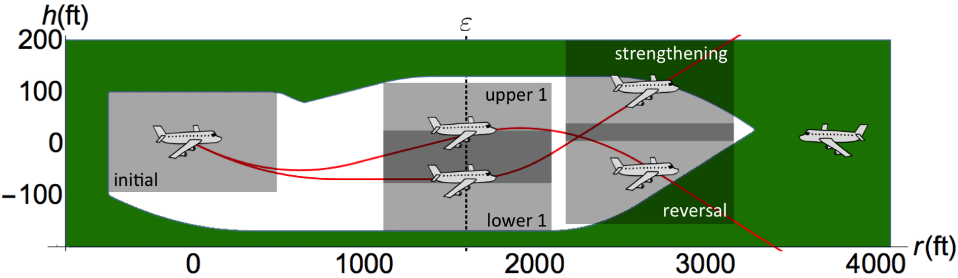


ACAS X issues DNC advisory, which induces collision unless corrected

TACAS'15,EMSOFT'15

Airborne Collision Avoidance System ACAS X: Refine

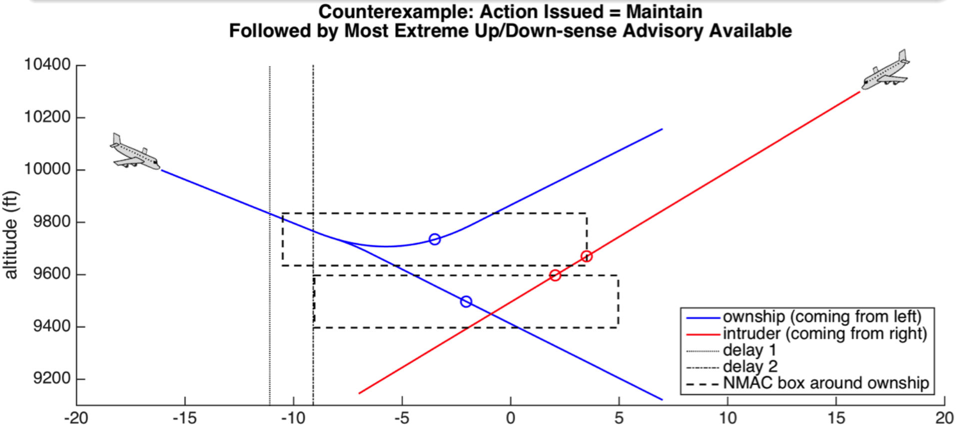
- Conservative, so too many counterexamples
- Settle for: safe for a little while with safe possible future
- Safeable advisory: a subsequent advisory can safely avoid collision



1. Identified safeable region for each advisory symbolically
2. Proved safety for hybrid systems flight model in KeYmaera X

STTT

ACAS X table comparison shows safeable advisory in more of the 648,591,384,375 states compared ($\approx$31.6 to 898.7 10^6 counterexamples).



ACAS X issues Maintain advisory instead of CL1500

STTT

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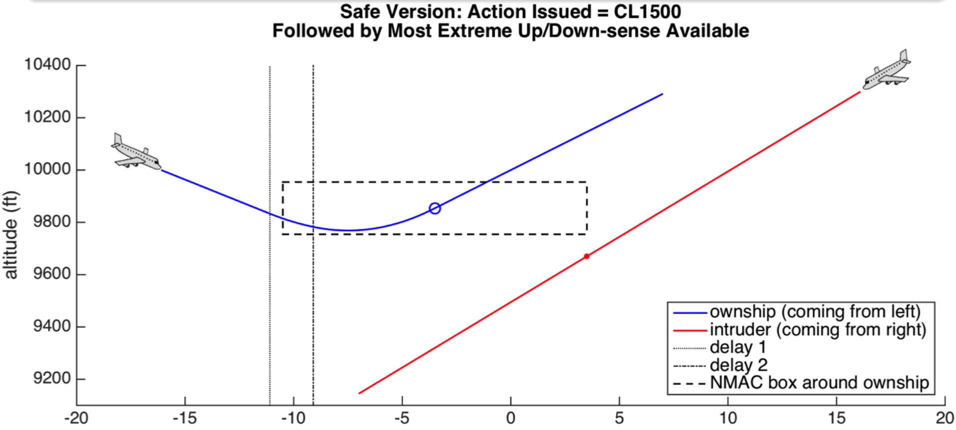
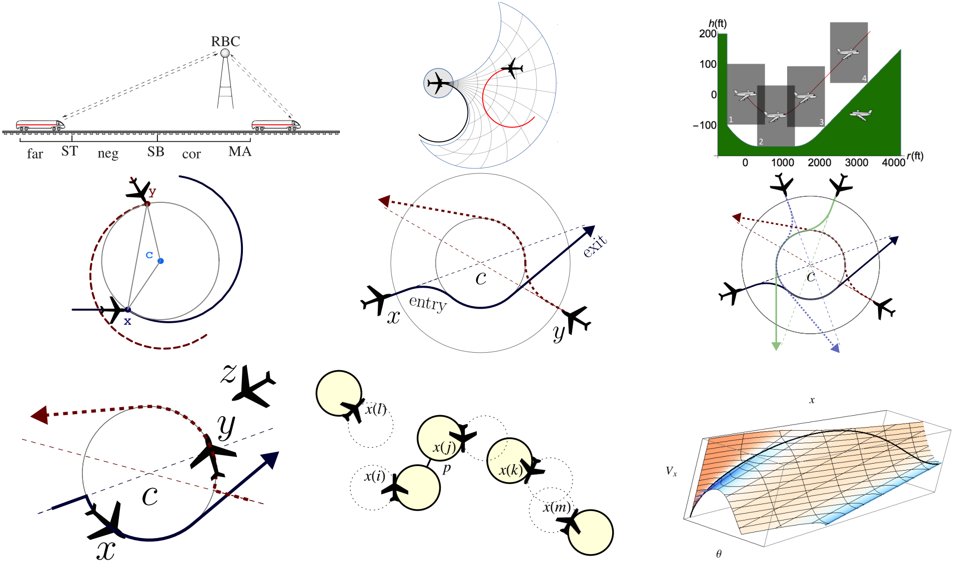


ACAS X issues Maintain advisory instead of CL1500

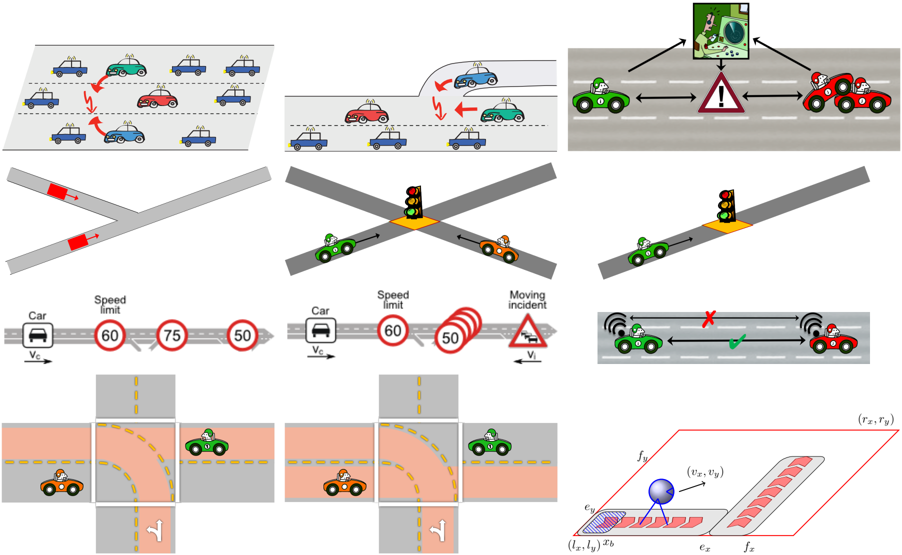
STTT

Verified CPS Applications



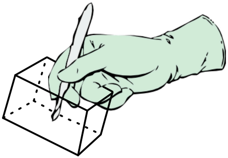
FEM'09,JAIS'14,TACAS'15,EMSOFT'15,CAV'08,FM'09,HSCC'11,HSCC'13,TACAS'14

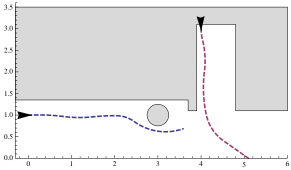
Verified CPS Applications

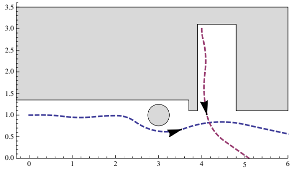


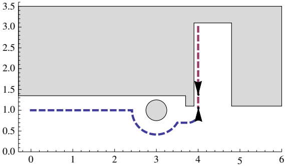
FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

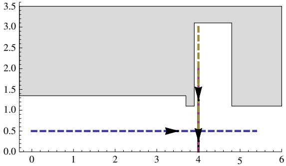
Verified CPS Applications

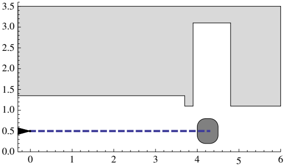


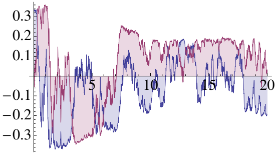


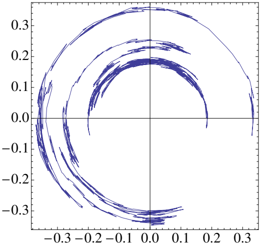


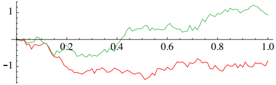




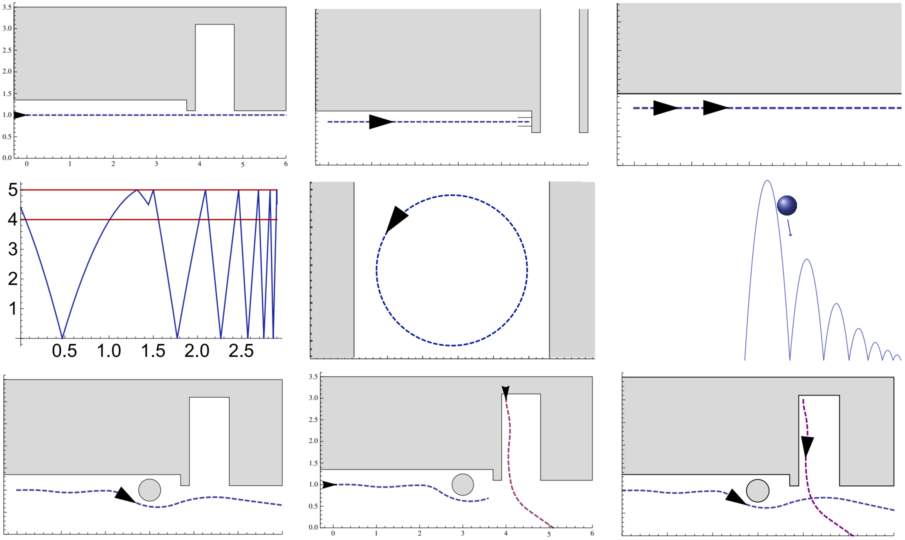






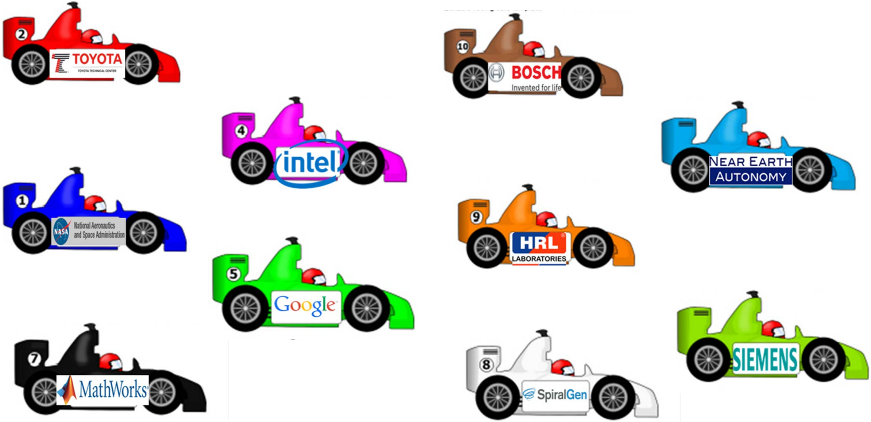


HSCC'13,RSS'13,CADE'12

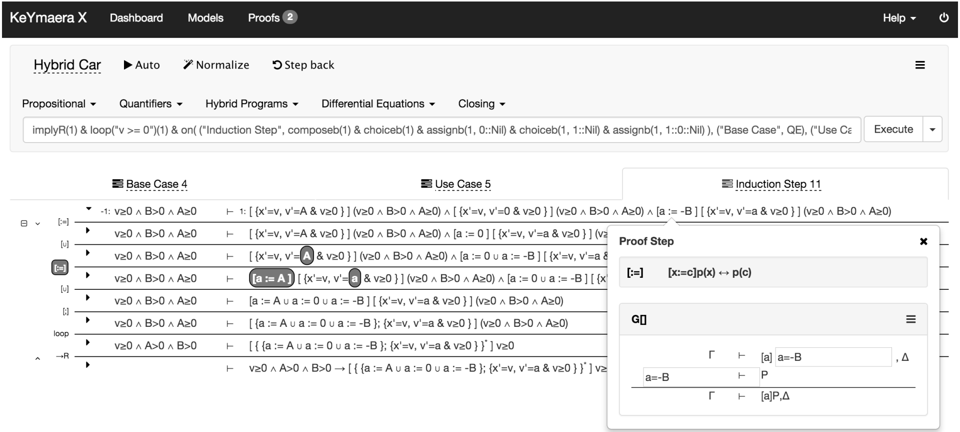


15-424/624/824 *Foundations of Cyber-Physical Systems* students

CPS V&V Grand Prix: Undergrad Course Competition



KeYmaera X http://keymaeraX.org/



CADE'15

An aXiomatic Tactical Theorem Prover for CPS

KeYmaera X http://keymaeraX.org/

Small Core	Increases trust, modularity, enables experimentation	(1652)
Tactics	Bridging between small core and powerful reasoning steps	(Hilbert) (Sequent)
Separation	Tactics can make courageous inferences Core establishes soundness	
Search&Do	Search-based tactics follow proof search strategies Constructive tactics directly build a proof	
Interaction	Interactive proofs mixed with tactical proofs and proof search	
Extensible	Flexible for new algorithms, new tactics, new logics, new proof rules, new axioms, ...	
Customize	Modular user interface, API	

CADE'15

KeYmaera X Microkernel for Soundness

	≈LOC		
KeYmaera X	1 652		hybrid prover
KeYmaera	65 989		hybrid prover
KeY	51 328		Java
Nuprl	15 000	+ 50 000	general math
MetaPRL	8 196		general math
Isabelle/Pure	8 113		general math
Coq	20 000		general math
HOL Light	396		general math
PHAVer	30 000		hybrid verifier
HSolver	20 000		hybrid verifier
SpaceEx	100 000		hybrid verifier
Flow*	25 000		hybrid verifier
dReal	50 000	+ millions	hybrid verifier
HyCreate2	6 081	+ user model analysis	hybrid verifier

Disclaimer: Self-reported estimates of the soundness-critical lines of code + rules

Uniform Substitution

Theorem (Soundness) replace all occurrences of $p(\cdot)$

$$(US)\ \frac{\phi}{\sigma(\phi)}$$

provided $FV(\sigma|_{\Sigma(\theta)}) \cap BV(\otimes(\cdot)) = \emptyset$ *for each operation* $\otimes(\theta)$ *in* ϕ

i.e. bound variables $U = \mathrm{BV}(\otimes(\cdot))$ of operator $\otimes$
are not free in the substitution on its argument θ (U-admissible)

$$\text{US}\ \frac{[a \cup b]p(\bar{x}) \leftrightarrow [a]p(\bar{x}) \land [b]p(\bar{x})}{[x := x+1 \cup x' = 1]x \geq 0 \leftrightarrow [x := x+1]x \geq 0 \land [x' = 1]x \geq 0}$$

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Students and postdocs of the Logical Systems Lab at Carnegie Mellon
Brandon Bohrer, Nathan Fulton, David Henriques, Sarah Loos, João Martins
Erik Zawadzki, Khalil Ghorbal, Jean-Baptiste Jeannin, Stefan Mitsch

NSF DARPA DEPARTMENT OF TRANSPORTATION UNITED STATES OF AMERICA FEDERAL AVIATION ADMINISTRATION

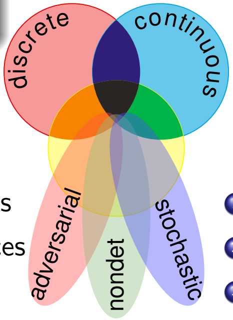
BOSCH Invented for life TOYOTA TOYOTA TECHNICAL CENTER APL JOHNS HOPKINS APPLIED PHYSICS LABORATORY

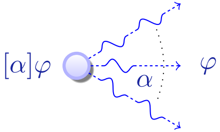
Logic & Proofs for Cyber-Physical Systems

Logical foundations make a big difference for CPS, and vice versa

differential dynamic logic

$$d\mathcal{L} = DL + HP$$





- Strong analytic foundations
- Practical reasoning advances
- Significant applications
- Catalyze many science areas

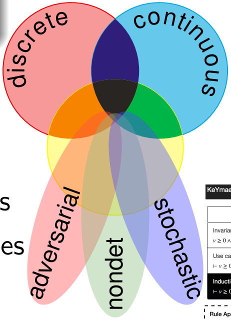
1. Multi-dynamical systems
2. Combine simple dynamics
3. Tame complexity
4. Complete axiomatization

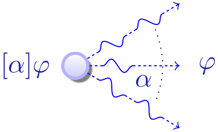
Numerous wonders remain to be discovered

Logical foundations make a big difference for CPS, and vice versa

differential dynamic logic

$$d\mathcal{L} = \text{DL} + \text{HP}$$





- Strong analytic foundations
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- Significant applications
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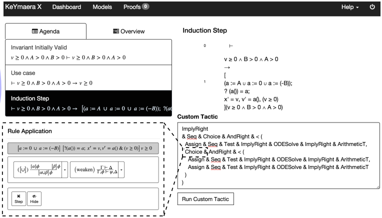
KeYmaera X

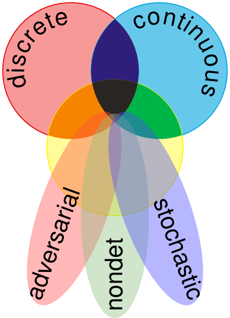


Numerous wonders remain to be discovered

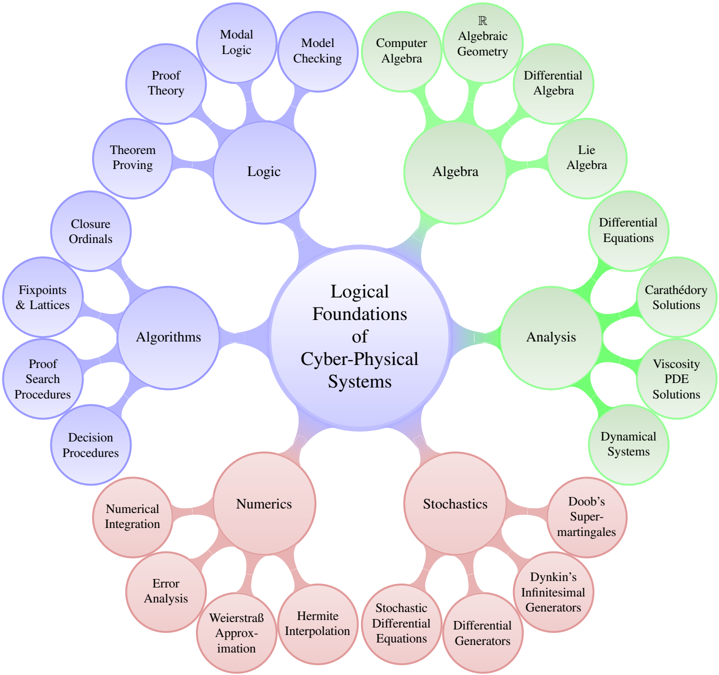
Future CPS Challenges

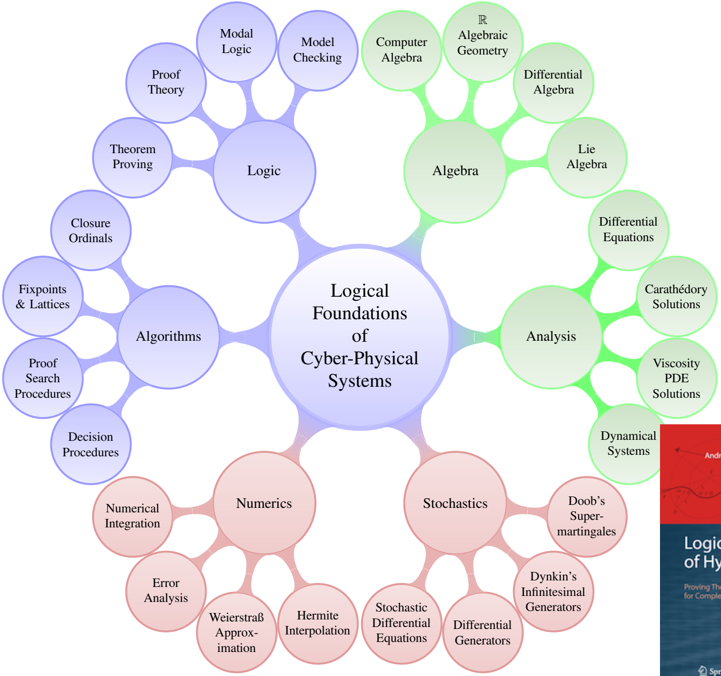
Numerous wonders remain to be discovered

- Scalable continuous stochastics — CADE'11
- Concurrent CPS
- Real arithmetic: Scalable and verified — CADE'09
- Verified CPS implementations, ModelPlex — FMSD'16
- Correct CPS execution
- CPS-conducive tactic languages+libraries
- Tactics exploiting CPS structure/linearity/...
- Invariant generation
- Tactics & proofs for reachable set computations
- Parallel proof search & disprovers
- Correct model transformation
- Inspiring applications



CPSs deserve proofs as safety evidence!


Logical Foundations of Cyber-Physical Systems
Logic
Theorem Proving
Proof Theory
Modal Logic
Model Checking
Algebra
Computer Algebra
ℝ Algebraic Geometry
Differential Algebra
Lie Algebra
Analysis
Differential Equations
Carathédory Solutions
Viscosity PDE Solutions
Dynamical Systems
Stochastics
Doob's Super-martingales
Dynkin's Infinitesimal Generators
Differential Generators
Stochastic Differential Equations
Numerics
Hermite Interpolation
Weierstraß Approx-imation
Error Analysis
Numerical Integration
Algorithms
Decision Procedures
Proof Search Procedures
Fixpoints & Lattices
Closure Ordinals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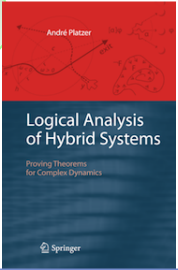
Stochastic Differential Equations


André Platzer
Logical Analysis of Hybrid Systems
Proving Theorems for Complex Dynamics
Springer

Differential Dynamic Logic d$\mathcal{L}$: Semantics

Definition (Hybrid program semantics) ($[\![\cdot]\!] : \mathrm{HP} \to \wp(\mathcal{S} \times \mathcal{S})$)

$$
\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) \;:\; \nu = \omega \text{ except } [\![x]\!]\nu = [\![e]\!]\omega\} \\
[\![?Q]\!] &= \{(\omega, \omega) \;:\; \omega \in [\![Q]\!]\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) \;:\; \varphi \models x' = f(x) \text{ for some duration } r\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha; \beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] \\
[\![\alpha^*]\!] &= \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!]
\end{aligned}
$$

Definition (d$\mathcal{L}$ semantics) ($[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$)

$$
\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega \;:\; [\![e]\!]\omega \geq [\![\tilde{e}]\!]\omega\} \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= [\![\alpha]\!] \circ [\![P]\!] = \{\omega \;:\; \nu \in [\![P]\!] \text{ for some } \nu :\; (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega \;:\; \nu \in [\![P]\!] \text{ for all } \nu :\; (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![\exists x\, P]\!] &= \{\omega \;:\; \omega_x^r \in [\![P]\!] \text{ for some } r \in \mathbb{R}\}
\end{aligned}
$$

André Platzer.
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