Refactoring-as-Propositions: Proved Refactoring of Hybrid Systems via Proved Refinements

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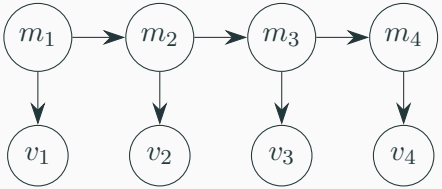
Karlsruhe Institute of Technology

Cyber-Physical Systems and Refactoring-as-Propositions

Complex interactions of discrete and continuous behaviors



$\rightarrow$ iterative design

$\rightarrow$ repeated verification

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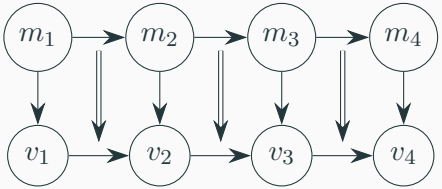
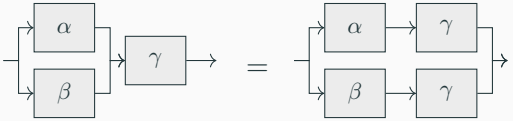
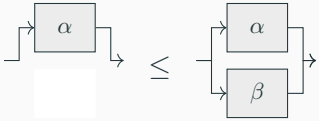


$\rightarrow$ iterative design

$\rightarrow$ repeated verification

$\Rightarrow$ safe refactoring techniques





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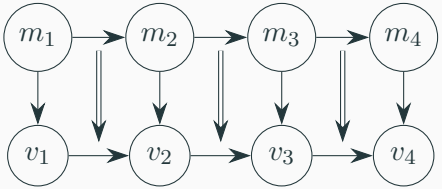

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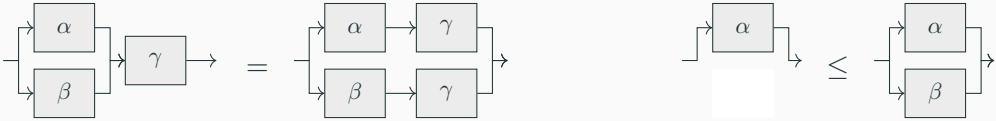
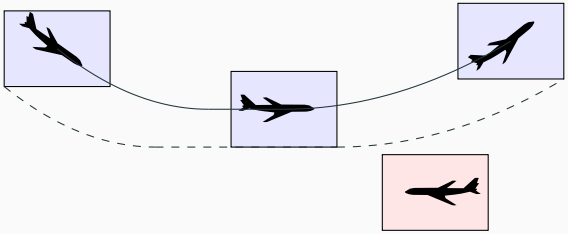
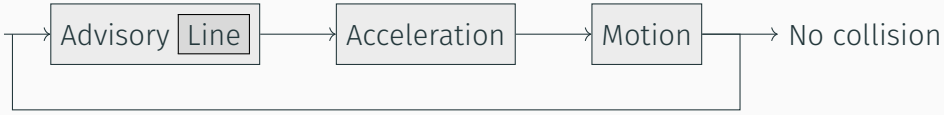


Refactoring-as-propositions are refinements: $\alpha \leq \beta$

Example: Airborne Collision Avoidance System X[1]





[1] Jeannin et al. A formally verified hybrid system for safe advisories in the next-generation airborne collision avoidance system. STTT 2017

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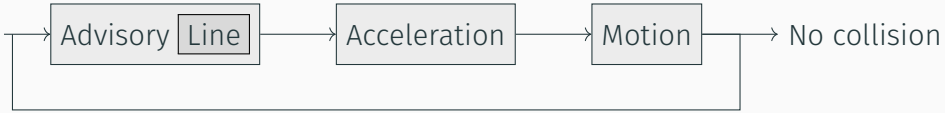

Two versions of Line:

```
Bool case1(Real r, Real v, Real w, Real vlo, Real ru)  <-> -rp <= r & r < -rp - rv + minI(v,w)/alo;
Bool bound1(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> w = rv^2 + h < alo/2 * (r + rp)^2 - w * rv = v * (r+rp) - rv^2 + hp;
Bool case2(Real r, Real v, Real w, Real vlo, Real ru)  <-> -rp - rv = minI(v,w)/alo <= r & r <= rp - rv = minI(v,w)/alo;
Bool bound2(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> w = h < (-(minI(v,w)^2))/(2*alo) - hp;
Bool case3(Real r, Real v, Real w, Real vlo, Real ru)  <-> rp - rv = minI(v,w)/alo < r & r <= rp + rv = maxI(v,w,vlo)/alo;
Bool bound3(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> w = rv^2 + h < alo/2 * (r - rp)^2 + w = rv + v * (r - rp) - rv^2 + hp;
Bool case4(Real r, Real v, Real w, Real vlo, Real ru)  <-> rp + rv = maxI(v,w,vlo)/alo < r;
Bool bounds(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> ru < 0 | w < rv + h  :  w = vlo + (r - rp) - rv = maxI(v,w,vlo)^2/(2*alo) - rv + hp;
Bool case5(Real r, Real v, Real w, Real vlo, Real ru)  <-> -rp <= r & r < -rp + rv = maxI(v,w,vlo)/alo;
Bool bound5(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> w = rv^2 + h < alo/2 * (r + rp)^2 + w = rv + v * (r + rp) - rv^2 + hp;
Bool case6(Real r, Real v, Real w, Real vlo, Real ru)  <-> -rp < rv - rv = maxI(v,w,vlo)/alo <= r;
Bool bound6(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> ( (rv = 0 & r > rp)
                    | w = rv + h :  w = vlo + (r - rp) - rv = maxI(v,w,vlo)^2/(2*alo) - rv + hp );

Bool Lexp1(Real r, Real h, Real v, Real w, Real vlo, Real ru) <-> (
  (w + vlo >= 0 ->
     (case1(r,v,w,vlo,ru) -> bound1(r,h,v,w,vlo,ru))
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  )
  &
  (w + vlo < 0 ->
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  )
);
```

```
Bool Limp1(Real r, Real h, Real v, Real w, Real vlo, Real rv) <->
  \forall t. \forall ru !\forall ru (
    (0 <= t & t < maxI(v,w,vlo)/alo & ru = rv + t & hu = (w + alo)/2 * t^2 + v * t)
    | (t >= maxI(v,w,vlo)/alo & ru = rv + t & hu = vlo * t - w * maxI(v,w,vlo)^2/(2*alo))
    -> (abs(r - ru) > rp | w * h < w * hu - hp)
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```

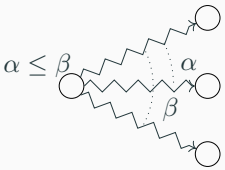
Safety transfer using refinements:

⇒ Manual steps reduced from 200 to 20

Refinements

$$\alpha \leq \beta$$

Meaning: β can reach all states that α can.



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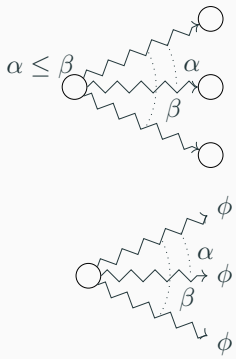
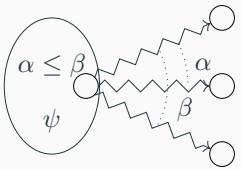
$$[\beta]\phi \to [\alpha]\phi$$



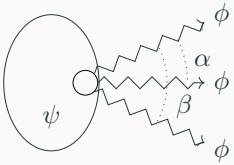
Refinements

$$\psi \to \alpha \leq \beta$$

Meaning: Starting from ψ, β can reach all states that α can.



$$\psi \to ([\beta]\phi \to [\alpha]\phi)$$



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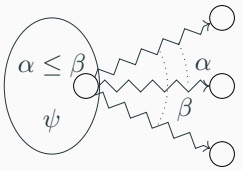

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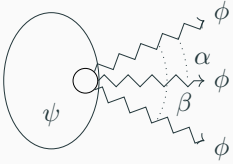

What about subsystems? Line

What about intermediate variables?

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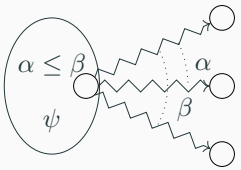

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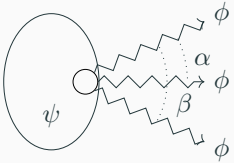

What about subsystems? Line

What about intermediate variables?

⇒ Automation solves both.

Differential Refinement Logic

Formulas and programs defined by mutual induction:

$$\phi, \psi ::= \theta \sim \eta \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid [\alpha]\phi \mid \alpha \leq \beta$$

$$\alpha, \beta ::= \underbrace{?\psi \mid x := \theta \mid x := *}_{\text{discrete}} \mid \underbrace{x' = \theta \,\&\, \psi}_{\text{continuous}} \mid \underbrace{\alpha \cup \beta \mid \alpha; \beta \mid \alpha^*}_{\text{composition}}$$

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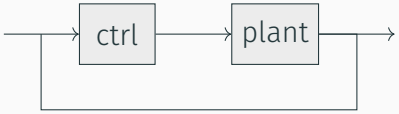
$(\text{ctrl}; \text{plant})^*$



Implemented in the KeYmaera X[1] theorem prover

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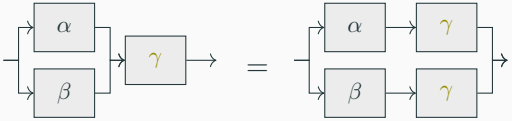
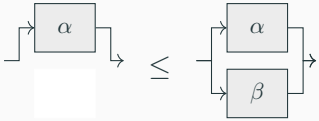
Monotony for Global Refinements



$$(\alpha \cup \beta); \gamma \quad = \quad (\alpha; \gamma) \cup (\beta; \gamma)$$

$\Rightarrow$ original axiom [2]



$$\alpha \quad \leq \quad \alpha \cup \beta$$

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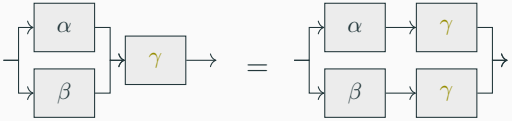

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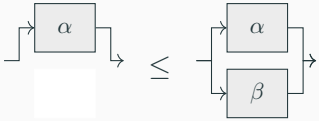
$$\frac{\alpha = \beta}{C(\alpha) = C(\beta)}$$



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Scalable: inference derivations automated

[2] Prebet, Platzer. Uniform substitution for differential refinement logic. IJCAR 2024

Monotony for Local Refinement

Context-aware refactoring: requires local information

$$\phi \to \alpha \leq \beta$$

Expected congruence principle:

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$$\frac{\widetilde{C}_\alpha(\phi) \quad \phi \to \alpha \leq \beta}{C(\alpha) \leq C(\beta)}$$

$\Rightarrow \widetilde{C}_\alpha$ defined inductively over C

$$C(\sqcup) = (\gamma; \sqcup)^* \rightsquigarrow \widetilde{C}_\alpha(\sqcup) = [(\gamma; \alpha)^*][\gamma]\sqcup$$

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Result: Local refinement automatically ensures safety of whole systems ✓

Partial Refinement

Refactorings can introduce variables.

$\alpha \leq \beta$ captures all variables.

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$\alpha; x := * \leq \beta;$ $\underbrace{x := *}_{\substack{\text{nondet.} \\ \text{assignment}}}$ captures all variables but x.

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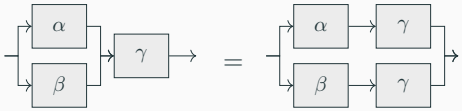
Result: equivalence derivation automated

Conclusion

Refinements are refactoring-as-propositions.

⇒ local (precondition)

⇒ partial (dismiss variables)



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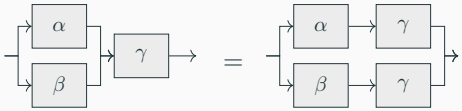

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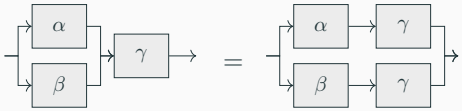

Thank you for listening!

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