Complete Robust Hybrid Systems Reachability

Noah Abou El Wafa   André Platzer

IJCAR 2026

Karlsruhe Institute of Technology
Karlsruhe, Germany

Complete Robust Hybrid Systems Reachability

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Complete Robust Hybrid Systems Reachability

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Complete Robust Hybrid Systems Reachability

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Complete Robust Hybrid Systems Reachability

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Complete Robust Hybrid Systems Reachability

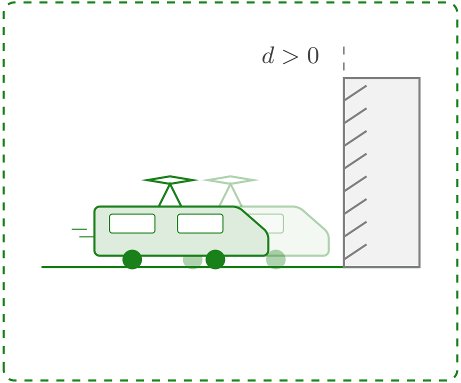
Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

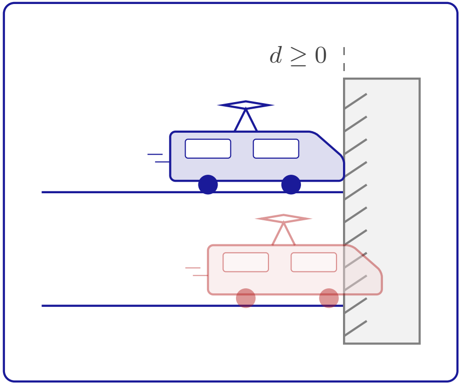
- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Robustness Topologically - Intuition

ROBUST



NON-ROBUST

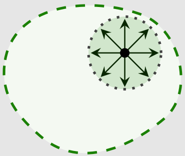


$$x' = f(x)$$

Intuition

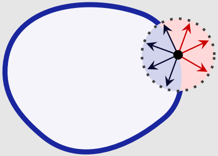
A property is robust if it is described by a topologically open set.

Open/Robust



Property remains true for sufficiently small perturbation.

Closed/Marginal



Property is false for arbitrarily small perturbation.

Robustness is not quantitative: Not for positive $\delta > 0$ but for infinitesimal δ.

Robustness Syntactically - Definition

Hybrid Programs:

$$\alpha, \gamma ::= x{:=}v \mid ?\varphi \mid \alpha \cup \gamma \mid \alpha;\gamma \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \varphi \qquad \text{dL programs}$$

where φ is a (robust) dL formula.

Definition (Differential Dynamic Logic)

The formulas of differential dynamic logic are

$$\varphi, \rho ::= v > w \mid \neg\varphi \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{dL formula}$$

$$\varphi, \rho ::= v > w \mid {\sim}\varphi \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{robust dL}_r \text{ formula}$$

Robust Negation:

$${\sim}(v > w) \equiv v < w \quad {\sim}(\varphi \vee \rho) \equiv {\sim}\varphi \wedge {\sim}\rho \quad {\sim}(\exists x\, \varphi) \equiv \forall x\, {\sim}\varphi \quad {\sim}(\langle\alpha\rangle\varphi) \equiv [\alpha]{\sim}\varphi$$

Robustness Syntactically - Definition

Hybrid Programs:

$$\alpha, \gamma ::= x{:=}v \mid ?\varphi \mid \alpha \cup \gamma \mid \alpha;\gamma \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \varphi \qquad \text{dL programs}$$

where φ is a (robust) dL formula.

Definition (Differential Dynamic Logic)

The formulas of differential dynamic logic are

$$\varphi, \rho ::= v > w \mid \neg\varphi \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{dL formula}$$

$$\varphi, \rho ::= v > w \mid {\sim}\varphi \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{robust } \mathsf{dL_r} \text{ formula}$$

Robust Negation:

$${\sim}(v > w) \equiv v < w \quad {\sim}(\varphi \vee \rho) \equiv {\sim}\varphi \wedge {\sim}\rho \quad {\sim}(\exists x\, \varphi) \equiv \forall x\, {\sim}\varphi \quad {\sim}(\langle\alpha\rangle\varphi) \equiv [\alpha]{\sim}\varphi$$

Robustness Syntactically - Definition

Hybrid Programs:

$$\alpha, \gamma ::= x{:=}v \mid ?\varphi \mid \alpha \cup \gamma \mid \alpha;\gamma \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \varphi \qquad \text{dL programs}$$

where φ is a (robust) dL formula.

Definition (Differential Dynamic Logic)

The formulas of differential dynamic logic are

$$\varphi, \rho ::= v > w \mid \boxed{\neg\varphi} \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{dL formula}$$

$$\varphi, \rho ::= v > w \mid \boxed{\sim\varphi} \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{robust } \mathsf{dL_r} \text{ formula}$$

Robust Negation:

$$\sim(v > w) \equiv v < w \quad \sim(\varphi \vee \rho) \equiv \sim\varphi \wedge \sim\rho \quad \sim(\exists x\, \varphi) \equiv \forall x \sim\varphi \quad \sim(\langle\alpha\rangle\varphi) \equiv [\alpha]\sim\varphi$$

Robustness Syntactically - Definition

Hybrid Programs:

$$\alpha, \gamma ::= x{:=}v \mid ?\varphi \mid \alpha \cup \gamma \mid \alpha;\gamma \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \varphi \qquad \text{dL programs}$$

where φ is a (robust) dL formula.

Definition (Differential Dynamic Logic)

The formulas of differential dynamic logic are

$$\varphi, \rho ::= v > w \mid \boxed{\neg\varphi} \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{dL formula}$$

$$\varphi, \rho ::= v > w \mid \boxed{\sim\varphi} \mid \varphi \vee \rho \mid \exists x\, \varphi \mid \langle\alpha\rangle\varphi \qquad \text{robust } \mathsf{dL_r} \text{ formula}$$

Robust Negation:

$$\boxed{\sim(v > w) \equiv v < w} \quad \sim(\varphi \vee \rho) \equiv \sim\varphi \wedge \sim\rho \quad \sim(\exists x\, \varphi) \equiv \forall x \sim\varphi \quad \sim(\langle\alpha\rangle\varphi) \equiv [\alpha]\sim\varphi$$

Complete Robust Hybrid Systems Reachability

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust $\boxed{\text{reachability}}$ properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Robust Reachability Properties

Definition (Robust Formulas)

Robust reachability ($\mathsf{dL}_{r\diamond}$) formulas are robust dL formulas where

- Universal quantifiers are bounded $\forall x\, |x| \leq v \to \varphi$
- Bounded (time and space) continuous evolution and iteration in boxes $[\alpha]\varphi$

Robust Reachability Formulas

admit *arbitrary* $\exists$-quantifiers and $\langle\alpha\rangle$-modalities.

Robust Reachability Properties

Definition (Robust Formulas)

Robust reachability ($\mathsf{dL}_{\mathsf{r}\diamond}$) formulas are robust dL formulas where

- Universal quantifiers are bounded $\forall x\, |x| \leq v \rightarrow \varphi$
- Bounded (time and space) continuous evolution and iteration in boxes $[\alpha]\varphi$

$$\varphi ::= v > w \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \exists x\, \varphi \mid \forall\ x\ \varphi \mid \langle\alpha\rangle\varphi \mid [\gamma]\varphi$$

in $\langle\rangle$: $\alpha ::= x{:=}v \mid ?\varphi \mid \alpha_1 \cup \alpha_2 \mid \alpha_1 ; \alpha_2 \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \varphi$

in $[]$: $\gamma ::= x{:=}v \mid ?\neg\varphi \mid \gamma_1 \cup \gamma_2 \mid \gamma_1 ; \gamma_2 \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \neg\varphi$

Robust Reachability Formulas

admit *arbitrary* $\exists$-quantifiers and $\langle\alpha\rangle$-modalities.

Definition (Robust Formulas)

Robust reachability ($\mathsf{dL}_{r\diamond}$) formulas are robust dL formulas where

- Universal quantifiers are bounded $\forall x\, |x| \leq v \rightarrow \varphi$
- Bounded (time and space) continuous evolution and iteration in boxes $[\alpha]\varphi$

$$\varphi ::= v > w \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \exists x\, \varphi \mid \boxed{\forall_I x\, \varphi} \mid \langle\alpha\rangle\varphi \mid [\gamma]\varphi$$

$$\text{in } \langle\rangle\text{:} \quad \alpha ::= x{:=}v \mid ?\varphi \mid \alpha_1 \cup \alpha_2 \mid \alpha_1;\alpha_2 \mid \alpha^* \mid \mathsf{x}' = \mathsf{v} \,\&\, \varphi$$

$$\text{in } []\text{:} \quad \gamma ::= x{:=}v \mid ?\neg\varphi \mid \gamma_1 \cup \gamma_2 \mid \gamma_1;\gamma_2 \mid \boxed{\cancel{\alpha^*}} \mid \mathsf{x}' = \mathsf{v} \,\&\, \neg\varphi \boxed{\wedge\, |\mathsf{x}| \leq k}$$

Robust Reachability Formulas

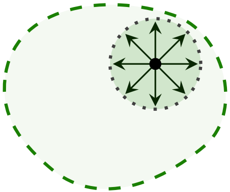
admit *arbitrary* $\exists$-quantifiers and $\langle\alpha\rangle$-modalities.

Robust Reachability $\mathsf{dL}_{r\diamond}$ Topologically

Example (Robust Reachability Formula $\mathsf{dL}_{r\diamond}$)

The plane can reach the desired altitude range:

$$\mathrm{pre} \rightarrow \langle(\mathrm{ctrl};\mathrm{flight\_dynamics})^*\rangle\big(H_{\mathrm{min}} < h < H_{\mathrm{max}}\big)$$



Theorem

A robust reachability $\mathsf{dL}_{r\diamond}$ formula semantically describes an open set in $\mathbb{R}^{\mathcal{V}}$.

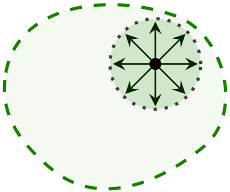
- Atomic formulas $v > w$ are open
- Universal quantification is bounded over a compact set $\forall x\, |x| \leq v \rightarrow \varphi$

Robust Reachability $\mathsf{dL}_{r\diamond}$ Topologically

Example (Robust Reachability Formula $\mathsf{dL}_{r\diamond}$)

The plane can reach the desired altitude range:

$$\mathrm{pre} \rightarrow \langle(\mathrm{ctrl}; \mathrm{flight\_dynamics})^*\rangle\left(H_{\mathrm{min}} < h < H_{\mathrm{max}}\right)$$



Theorem

A robust reachability $\mathsf{dL}_{r\diamond}$ formula semantically describes an open set in $\mathbb{R}^{\mathcal{V}}$.

- Atomic formulas $v > w$ are open
- Universal quantification is bounded over a compact set $\forall x\, |x| \leq v \rightarrow \varphi$

Complete Robust Hybrid Systems Reachability

Theorem: Robustness for Completeness

Differential dynamig logic (dL) for robust reachability properties of hybrid systems is complete.

- Verification *undecidable* in general
- Completeness *relative* to oracles
- Decidability up to $\delta > 0$ perturbations

Completeness for $\mathsf{dL}_{r\diamond}$

Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ *sentence is provable.*

Completeness for $\mathsf{dL}_{r\diamond}$

Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ *sentence is provable.*

Proof Calculus:

- Axioms and rules for classical first-order logic
- Hybrid program decomposition axioms, e.g.,

$$\langle \alpha \cup \gamma \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \vee \langle \gamma \rangle \varphi$$

$$\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi$$

- Differential equation discretization axiom $\langle x' = v \,\&\, \rho \rangle \varphi \leftrightarrow \langle \mathrm{step}^* \rangle \varphi$
- ODE box-diamond duality axiom

Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ *sentence is provable.*

Proof Calculus:

- Axioms and rules for classical first-order logic
- Hybrid program decomposition axioms, e.g.,

$$\langle \alpha \cup \gamma \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \lor \langle \gamma \rangle \varphi$$
$$\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \lor \langle \alpha \rangle \langle \alpha^* \rangle \varphi$$

- Differential equation discretization axiom $\langle x' = v \,\&\, \rho \rangle \varphi \leftrightarrow \langle \mathrm{step}^* \rangle \varphi$
- ODE box-diamond duality axiom

Completeness for $\mathsf{dL}_{r\diamond}$

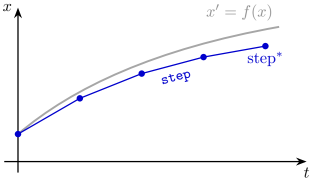
Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ sentence is provable.

Proof Calculus:

- Axioms and rules for classical first-order logic
- Hybrid program decomposition axioms, e.g.,

$$\langle \alpha \cup \gamma \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \vee \langle \gamma \rangle \varphi$$
$$\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi$$



- Differential equation discretization axiom $\langle x' = v \,\&\, \rho \rangle \varphi \leftrightarrow \langle \mathrm{step}^* \rangle \varphi$
- ODE box-diamond duality axiom

Completeness for $\mathsf{dL}_{r\diamond}$

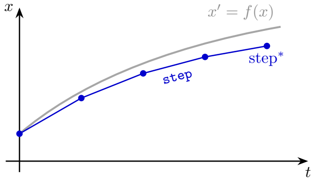
Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ *sentence is provable.*

Proof Calculus:

- Axioms and rules for classical first-order logic
- Hybrid program decomposition axioms, e.g.,

$$\langle \alpha \cup \gamma \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \lor \langle \gamma \rangle \varphi$$
$$\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \lor \langle \alpha \rangle \langle \alpha^* \rangle \varphi$$



- Differential equation discretization axiom $\langle x' = v \,\&\, \rho \rangle \varphi \leftrightarrow \langle \mathrm{step}^* \rangle \varphi$
- ODE box-diamond duality axiom

Completeness Proof Sketch

Theorem (Completeness of $\mathsf{dL}_{\mathsf{r}\diamond}$)

Every valid robust reachability $\mathsf{dL}_{\mathsf{r}\diamond}$ *sentence is provable.*

Proof Sketch

1. Define a well-founded order $\prec$ on sequents satisfying for example

$$\langle \alpha^n \rangle \varphi \prec \langle \alpha^* \rangle \varphi \quad \text{for all } n \in \mathbb{N}$$

$$\langle \mathrm{step}^* \rangle \varphi \prec \langle x' = v \,\&\, \rho \rangle \varphi$$

2. By induction on $\prec$ reduce to step-by-step simplification. For example:

   a) Case: existential quantifier

$$\exists x\, \varphi$$

$$\dfrac{\dfrac{\dfrac{\exists x\, \varphi \text{ valid}}{\vDash \varphi[x/q] \text{ for } q \in \mathbb{Q}}\ \varphi \text{ open + } \mathbb{Q} \text{ dense}}{\vdash \varphi[x/q] \text{ for } q \in \mathbb{Q}}\ \varphi[x/q] \prec \exists x\, \varphi}{\vdash \exists x\, \varphi}\ \text{Axiom}$$

Completeness Proof Sketch

Theorem (Completeness of $\mathsf{dL_{r\diamond}}$)

Every valid robust reachability $\mathsf{dL_{r\diamond}}$ *sentence is provable.*

Proof Sketch

1. Define a well-founded order $\prec$ on sequents satisfying for example

$$\langle \alpha^n \rangle \varphi \prec \langle \alpha^* \rangle \varphi \quad \text{for all } n \in \mathbb{N}$$

$$\langle \mathrm{step}^* \rangle \varphi \prec \langle x' = v \,\&\, \rho \rangle \varphi$$

2. By induction on $\prec$ reduce to step-by-step simplification. For example:

   a) Case: existential quantifier

$$\exists x\, \varphi$$

$$\cfrac{\cfrac{\cfrac{\exists x\, \varphi \text{ valid}}{\vDash \varphi[x/q] \text{ for } q \in \mathbb{Q}}\;\varphi \text{ open + } \mathbb{Q} \text{ dense}}{\vdash \varphi[x/q] \text{ for } q \in \mathbb{Q}}\;\varphi[x/q] \prec \exists x\, \varphi}{\vdash \exists x\, \varphi}\;\text{Axiom}$$

Completeness Proof Sketch

Theorem (Completeness of $\mathsf{dL}_{r\Diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\Diamond}$ *sentence is provable.*

Proof Sketch

1. Define a well-founded order $\prec$ on sequents satisfying for example

$$\langle \alpha^n \rangle \varphi \prec \langle \alpha^* \rangle \varphi \quad \text{for all } n \in \mathbb{N}$$

$$\langle \mathrm{step}^* \rangle \varphi \prec \langle x' = v \,\&\, \rho \rangle \varphi$$

2. By induction on $\prec$ reduce to step-by-step simplification. For example:

   a) Case: existential quantifier

$$\exists x \, \varphi$$

$$\cfrac{\cfrac{\cfrac{\exists x \, \varphi \text{ valid}}{\vDash \varphi[x/q] \text{ for } q \in \mathbb{Q}}\;\varphi \text{ open + } \mathbb{Q} \text{ dense}}{\vdash \varphi[x/q] \text{ for } q \in \mathbb{Q}}\;\varphi[x/q] \prec \exists x \, \varphi}{\vdash \exists x \, \varphi}\;\text{Axiom}$$

Completeness Proof Sketch

Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ *sentence is provable.*

Proof Sketch

1. Define a well-founded order $\prec$ on sequents satisfying for example

$$\langle \alpha^n \rangle \varphi \prec \langle \alpha^* \rangle \varphi \quad \text{for all } n \in \mathbb{N}$$

$$\langle \mathrm{step}^* \rangle \varphi \prec \langle x' = v \,\&\, \rho \rangle \varphi$$

2. By induction on $\prec$ reduce to step-by-step simplification. For example:

   b) Case: unbounded iteration

$$\langle \alpha^* \rangle \varphi$$

$$\begin{array}{cl}
C \to \langle \alpha^* \rangle \varphi \text{ valid} & \\
\hline
C \vDash \langle \alpha^{\leq n} \rangle \varphi \text{ for } n \in \mathbb{N} & \langle \alpha^{\leq n} \rangle \varphi \text{ open + } C \text{ compact} \\
\hline
C \vdash \langle \alpha^{\leq n} \rangle \varphi \text{ for } n \in \mathbb{N} & \langle \alpha^{n} \rangle \varphi \prec \langle \alpha^* \rangle \varphi \\
\hline
\vdash C \to \langle \alpha^* \rangle \varphi & \text{Axiom}
\end{array}$$

Completeness Proof Sketch

Theorem (Completeness of $\mathsf{dL}_{r\diamond}$)

Every valid robust reachability $\mathsf{dL}_{r\diamond}$ *sentence is provable.*

Proof Sketch

1. Define a well-founded order $\prec$ on sequents satisfying for example

$$\langle\alpha^n\rangle\varphi \prec \langle\alpha^*\rangle\varphi \quad \text{for all } n \in \mathbb{N}$$

$$\langle \mathrm{step}^*\rangle\varphi \prec \langle x' = v \,\&\, \rho\rangle\varphi$$

2. By induction on $\prec$ reduce to step-by-step simplification. For example:

b) Case: unbounded iteration

$$\langle\alpha^*\rangle\varphi$$

$$\cfrac{\cfrac{\cfrac{C \to \langle\alpha^*\rangle\varphi \text{ valid}}{C \vDash \langle\alpha^{\leq n}\rangle\varphi \text{ for } n \in \mathbb{N}}\ \langle\alpha^{\leq n}\rangle\varphi \text{ open + } C \text{ compact}}{C \vdash \langle\alpha^{\leq n}\rangle\varphi \text{ for } n \in \mathbb{N}}\ \langle\alpha^n\rangle\varphi \prec \langle\alpha^*\rangle\varphi}{\vdash C \to \langle\alpha^*\rangle\varphi}\ \text{Axiom}$$

Decisions for $\mathsf{dL}_{r\diamond}$

A sentence is robustly determined if $\varphi \vee {\sim}\varphi$ is true and bounded[1].

Then φ or ${\sim}\varphi$ is provable, since both are robust.

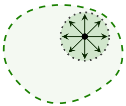
Theorem (Decidability)

Validity of every robustly determined $\mathsf{dL}_{r\diamond}$ *sentence is decidable.*

Proof search can decide robustly determined reachability properties of hybrid systems.

[1] i.e., all quantifiers and modalities are bounded

Conclusion



Robust = Open

Robust properties survive small perturbations.

$$\begin{aligned} \varphi^\diamond, \rho^\diamond &::= v > w \mid \varphi^\diamond \vee \rho^\diamond \mid \varphi^\diamond \wedge \rho^\diamond \mid \exists x\, \varphi^\diamond \\ \alpha^\diamond, \gamma^\diamond &::= x{:=}v \mid ?\varphi^\diamond \mid \alpha^\diamond \cup \gamma^\diamond \mid \alpha^\diamond ; \gamma^\diamond \mid {\alpha^\diamond}^* \\ \alpha^\blacklozenge, \gamma^\blacklozenge &::= x{:=}v \mid ?\neg\varphi^\diamond \mid \alpha^\blacklozenge \cup \gamma^\blacklozenge \mid \alpha^\blacklozenge ; \gamma^\blacklozenge \mid \mathsf{x}' \end{aligned}$$

$\mathrm{dL}_{r\diamond}$ Fragment

Robustness is captured syntactically.

$$\dfrac{\dfrac{\dfrac{\Delta_1 \vdash \Gamma_1^1, \Gamma_1^2, \Gamma, \Gamma' \quad \ldots}{\Delta_1 \vdash \Gamma_1^1, \varphi_2^\diamond, \Gamma, \Gamma'} \quad \ldots}{\Delta \vdash \varphi_1^\diamond, \varphi_2^\diamond, \Gamma, \Gamma'}}{\Delta \vdash \varphi_1^\diamond \vee \varphi_2^\diamond, \Gamma, \Gamma'}$$

Proof Calculus

Reachability formulas become provable.

Robust reachability properties can be characterized by robust formulas in dL with:
Completeness: true $\Leftrightarrow$ provable
Decidability: provable or disprovable if robustly determined

Appendix

ODE Duality Axiom

Theorem (ODE Duality Soundness)

The ODE duality axiom is sound

$$(\langle ODE \rangle) \quad [\mathsf{x}' = \mathsf{v} \,\&\, \rho^{\blacklozenge} \wedge |\mathsf{x}| \leq k]\varphi^{\diamond} \leftrightarrow \langle \mathsf{x}' = \mathsf{v} \,\&\, \varphi^{\diamond} \vee |\mathsf{x}| > k\rangle(\neg\rho^{\blacklozenge} \vee |\mathsf{x}| > k)$$

for a strict $\mathsf{dL}_{\mathsf{r}\diamond}$ *formula* $\varphi^{\diamond}$, *a non-strict* $\mathsf{dL}_{\mathsf{r}\diamond}$ *formula* $\rho^{\blacklozenge}$ *and a term* k *not mentioning variables from* x.

- The robustness as important. In particular, closedness of ρ ensures that the boundary is reached.
- The space and time boundedness are crucial to handle divergent evolutions.

Euler Axiom

Theorem (Euler Axiom Soundness)

The Euler axiom $\langle E \rangle$ *is sound*

$$(\langle E \rangle) \qquad \begin{array}{l} \langle \mathsf{x}' = \mathsf{v} \,\&\, \rho^\diamond \rangle \varphi^\diamond \\ \leftrightarrow \exists k, m, \ell{>}0 \,(\mathrm{bound}^\diamond \wedge \exists h{>}0 \,\langle \varepsilon{:=}0; (?\rho^\diamond_{\mathsf{x}\pm\varepsilon}; \mathrm{step}^\diamond)^* \rangle (\varphi^\diamond_{\mathsf{x}\pm\varepsilon} \wedge \varepsilon < 1)) \end{array}$$

where $\varphi^\diamond$ *is a strict* $\mathsf{dL}_{\mathsf{r}\diamond}$ *formula and* $h, k, m, \ell, \varepsilon \notin FV(\varphi^\diamond) \cup \{\mathsf{x}, \mathsf{v}\}$ *are fresh.*